The Laplacian spectral radius for bicyclic graphs with given independence number*

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Abstract. In this paper we study the Laplacian spectral radius of bicyclic graphs with given independence number and characterize the extremal graphs completely.

Key words: Bicyclic graphs; Laplacian spectral radius; Independent number

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1. Introduction

Suppose that G is a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Let $d_G(v_i)$ be the degree of vertex v_i. Then $D(G) = diag(d_G(v_1), \cdots, d_G(v_n))$ is a diagonal matrix of the vertex degrees of G. If $A(G)$ is the adjacency matrix of G, then the matrix $L(G) = D(G) - A(G)$ is the *Laplacian matrix* of G. It is well known that $L(G)$ is a positive semi-definite singular matrix, and so we can write its eigenvalues as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0$, where $\lambda_1(G)$ is also the *Laplacian spectral radius* of G.

The Laplacian spectral radius of a graph G is related to some important graph invariants (algebraic connectivity of the complement of G, diameter, average distance), and used in theoretical chemistry [7], combinatorial optimization [10], communication networks [11]. For the background on the Laplacian eigenvalues of a graph, the reader is referred to [8, 9] and the references therein. The *independence number* of G is the size of a maximum independent set of G. Zhang [13] studied the Laplacian spectral radius

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of trees with given independence number. Feng, Yu and Ilić [1] studied the Laplacian spectral radius of unicyclic graphs with given independence number.

Connected graph in which the number of edges equals the number of vertices plus one is *bicyclic graph.* In this paper we first obtain the range of independence number of bicyclic graphs and then study the Laplacian spectral radius of bicyclic graphs with given independence number and characterize the extremal graphs completely.

2. Preliminaries

In this section we give five lemmas which are used in the next section to prove our main results. Throughout this paper we write $\lambda(G)$ for $\lambda_1(G)$.

Lemma 2.1 [3]. *Let G be a graph on n vertices with at least one edge. Then $\lambda(G) \geq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of graph G. Moreover, the equality holds if and only if $\Delta(G)=n-1$.*

Suppose that F is a semiregular bipartite graph with bipartition (U, W). Denote by F^+ the supergraph of F with the following property: if $uv \in E(F^+)$, then either $uv \in E(F)$ or $u, v \in U$ (respectively, W) with $N_W(u) = N_W(v)$ (respectively, $N_U(u) = N_U(v)$), where $N_W(u)$ is the set of neighbors of u in W. Then we have

Lemma 2.2 [12]. *Let G be a connected graph. Then $\lambda(G) \leq max\{d_G(u)+d_G(v) - |N_G(u) \cap N_G(v)| : uv \in E(G)\}$, with equality if and only if $G \in \Psi$, where $\Psi = \{F^+$: F is a semiregular bipartite graph$\}$.*

Lemma 2.3 [5]. *If G is a graph, then*

$$\lambda(G) \leq \max\{\frac{d_G(v_i)(d_G(v_i)+m_i)+d_G(v_j)(d_G(v_j)+m_j)}{d_G(v_i)+d_G(v_j)} : v_iv_j \in E(G)\}$$

where $m_i = \frac{\sum_{v_iv_j \in E(G)} d_G(v_j)}{d_G(v_i)}$. Moreover, the equality holds if and only if G is a semiregular bipartite graph.

Suppose that G is a graph with the vertex set $V(G) = \{v_1, \cdots, v_n\}$. In what follows we denote the characteristic vector of $L(G)$ corresponding to $\lambda(G)$ by $x(G) = (x_{v_1}(G), \cdots, x_{v_n}(G))^T$, where $x_{v_i}(G)$ corresponds to v_i $(1 \leq i \leq n)$.

Lemma 2.4 [5]. *Suppose that u and v are two vertices of a connected graph G. Let G_v be the graph obtained from G by attaching t new paths $vv_{i1}v_{i2}\cdots v_{iq_i}(i = 1, 2, \ldots, t)$ at v. Let $G_u = G_v - vv_{11} - vv_{21} - \cdots - vv_{t1} + uv_{11} + uv_{21} + \cdots + uv_{t1}$. If $x_u(G_v) \geq x_v(G_v)$, then $\lambda(G_u) \geq \lambda(G_v)$. Furthermore, if $x_u(G_v) > x_v(G_v)$, then $\lambda(G_u) > \lambda(G_v)$.*

Lemma 2.5 [4]. *Let u be a vertex of a simple connected graph G. For nonnegative integers k and l, let $G(k,l)$ denote the graph obtained from G by adding pendent paths of length k and l attached at u. If $k \geq l \geq 1$, then $\lambda(G(k,l)) \geq \lambda(G(k+1,l-1))$.*

3. Bicyclic graphs with given independence number

In this section we denote by $\alpha(G)$ and $k(G)$ the independence number and the number of pendent vertices of a graph G, respectively.

Let C_n and P_n denote the cycle and path of length n, respectively. Then we define $b(p,\ell,q)$ to be a graph consisting two vertex-disjoint cycles C_p and C_q and a path P_ℓ joining them having only its end-vertices in common with the cycles, $b(p,0,q)$ to be a graph consisting two cycles C_p and C_q with exactly one vertex in common, and $\theta(p,\ell,q)$ to be a graph consisting of two given vertices joined by three paths P_p, P_ℓ and P_q with any two of these paths having only the given vertices in common. Obviously, a bicyclic graph G is $b(p,\ell,q)$, $b(p,0,q)$ or $\theta(p,\ell,q)$ with trees attached. Let $\mathcal{R}(G)$ be the number of those vertices of $b(p,\ell,q)$, $b(p,0,q)$ or $\theta(p,\ell,q)$ which are not the roots of any attached tree.

Lemma 3.1. *Suppose that G is a bicyclic graph on n vertices. Then $\lambda(G) \leq k(G) + \mathcal{R}(G) + 2$.*

Proof. We only discuss the case that G is $b(p,0,q)$ with trees attached. For other two cases that G is $b(p,\ell,q)$ or $\theta(p,\ell,q)$ with trees attached, the argument is similar. Let $V_{\mathcal{R}}$ be the set of $\mathcal{R}(G)$ vertices on cycles where no tree is attached. If each vertex in $V(G)\backslash V(b(p,0,q))$ is of degree at most 2, then every tree attached on $b(p,0,q)$ is composed of pendant paths. If two vertices v_1 and v_2 of $b(p,0,q)$ with trees attached satisfy $x_{v_1}(G) \geq x_{v_2}(G)$, then transfer all but one pendant path from v_2 to v_1. Continuing this process we will get the bicyclic graph G_1 so that all vertices in $V(b(p,0,q))\backslash V_{\mathcal{R}}$ but y are attached with only one pendant path. It follows from Lemma 2.4 that $\lambda(G) \leq \lambda(G_1)$. By Lemma 2.2 we know that

$$\lambda(G_1) \leq d_{G_1}(y) + max_{y' \in N_{G_1}(y)}\{d_{G_1}(y') - |N_{G_1}(y) \cap N_{G_1}(y')|\}.$$

We can observe that if $d_{b(p,0,q)}(y) = 4$ then $d_{G_1}(y) \leq k(G) - ((p+q-1) - (\mathcal{R}(G)+1)) + 4$ and $d_{G_1}(y') \leq 3$; and otherwise $d_{G_1}(y) \leq k(G) - ((p+q-1) - (\mathcal{R}(G)+1)) + 2$ and $d_{G_1}(y') \leq 5$. Note that $|N_{G_1}(y) \cap N_{G_1}(y')| = 1$ if $p+q=6$. Thus, $\lambda(G_1) \leq k(G) + \mathcal{R}(G) + 2$.

Now suppose that there is a vertex $u \in V(G)\backslash V(b(p,0,q))$ which is of degree at least 3, and that T is the tree containing u. Let $\mathcal{P}(T)$ be the number of pendant vertices that T contains. If u_1u_2 is an edge on T, then we know that

$$\mathcal{P}(T) \geq \begin{cases} d_G(u_1) + d_G(u_2) - 6, & \text{if } u_1 \in b(p,0,q) \text{ and } d_{b(p,0,q)}(u_1) = 4; \\ d_G(u_1) + d_G(u_2) - 4, & \text{if } u_1 \in b(p,0,q) \text{ and } d_{b(p,0,q)}(u_1) = 2; \\ d_G(u_1) + d_G(u_2) - 3, & \text{otherwise.} \end{cases}$$

Note that the tree T contains at most $k(G) - ((p+q-1) - (\mathcal{R}(G)+1))$ pendant vertices. We have $d_G(u_1) + d_G(u_2) \leq k(G) - (p+q) + \mathcal{R}(G) + 8$. Therefore, while $p+q \geq 6$, we have $d_G(u_1) + d_G(u_2) \leq k(G) + \mathcal{R}(G) + 2$.

Suppose that u_1u_2 is an edge on $b(p,0,q)$ and that T_1 and T_2 are two trees attached at u_1 and u_2, respectively. Then we have

$$\mathcal{P}(T_1) + \mathcal{P}(T_2) \geq \begin{cases} (d_G(u_1) - 4) + (d_G(u_2) - 2), & \text{if } d_{b(p,0,q)}(u_1) = 4; \\ (d_G(u_1) - 2) + (d_G(u_2) - 2), & \text{otherwise.} \end{cases}$$

Note that T_1 and T_2 contain at most $k(G) - ((p+q-1) - (\mathcal{R}(G)+2))$ pendant vertices. We have $d_G(u_1) + d_G(u_2) \leq k(G) - (p+q) + \mathcal{R}(G) + 9$. If $p+q = 6$, then $|N_G(u_1) \cap N_G(u_2)| = 1$, and so by Lemma 2.2, $\lambda(G) \leq k(G) + \mathcal{R}(G) + 2$. □

Since for any graph G, there always exists a maximum independent set of G where all pendant vertices are contained, $k(G) \leq \alpha(G)$.

Lemma 3.2. *Suppose that G is a bicyclic graph on n vertices such that $k(G) \geq \alpha(G) - 1$. Then $\lambda(G) \leq \alpha(G) + 3$.*

Proof. If $\alpha(G) = k(G)$, then we can observe that $\mathcal{R}(G) = 0$, and so, by Lemma 3.1, $\lambda(G) \leq \alpha(G) + 2$. The case that $\alpha(G) = k(G) + 1$ implies that $\mathcal{R}(G) \leq 3$. If $\mathcal{R}(G) \leq 2$, then by Lemma 3.1 we have $\lambda(G) \leq \alpha(G) + 3$. So we assume $\mathcal{R}(G) = 3$, which implies that every tree attached on $b(p,\ell,q)$, $b(p,0,q)$ or $\theta(p,\ell,q)$ is composed of pendant paths of length 1 or 2. Next we only argue the case that G is $b(p,0,q)$ with trees attached. For other two cases that G is $b(p,\ell,q)$ or $\theta(p,\ell,q)$ with trees attached, the proof is similar. In this case, at least one of p and q must equal 3, say $p = 3$, and no tree is attached on the cycle C_3. As in the proof of Lemma 3.1, we can verify that $\lambda(G) \leq k(G) + 3 + 1$, that is, $\lambda(G) \leq \alpha(G) + 3$. □

Lemma 3.3. *Suppose that G is a bicyclic graph on n vertices such that $k(G) = \alpha(G) - 2$. Then $\lambda(G) \leq \alpha(G) + 3$.*

Proof. If $k(G) = 1$ then the result is clearly true. So assume $k(G) \geq 2$. We only argue the case that G is $b(p,0,q)$ with trees attached. For other two cases that G is $b(p,\ell,q)$ or $\theta(p,\ell,q)$ with trees attached, the proof is similar. The case that $k(G) = \alpha(G) - 2$ implies that $\mathcal{R}(G) \leq 5$. If $\mathcal{R}(G) \leq 3$, then by Lemma 3.1 we have $\lambda(G) \leq \alpha(G) + 3$. So we assume $\mathcal{R}(G) = 4$ or 5. As in the proof of Lemma 3.1, we can obtain that $\lambda(G) \leq k(G) - (p+q) + \mathcal{R}(G) + 9$.

Case 1. Suppose that $\mathcal{R}(G) = 4$. If $p + q \geq 8$, then we have $\lambda(G) \leq \alpha(G) + 3$. So we assume $p + q = 6$ or 7. In this case, G can only be isomorphic to one of the following seven graphs shown in Fig. 1.

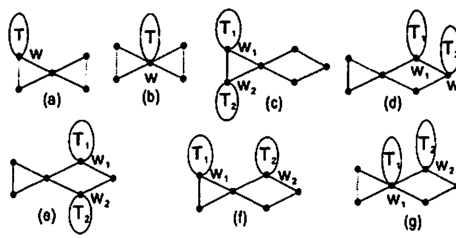

Fig. 1.

Suppose that G is isomorphic to the graph (g). For $i = 1$ or 2, if there is a vertex $y \in V(T_i)\backslash\{w_1, w_2\}$ such that $d_{T_i}(y) \geq 3$, then $d_{T_1}(w_1)+d_{T_2}(w_2) < k(G)$, and so by Lemma 2.2 we have $\lambda(G) \leq k(G)+5 = \alpha(G)+3$; otherwise G is isomorphic to the graph H shown in Fig. 2, where $H \cong H_1$ if $r = 1$ and $H \cong H_2$ or $H \cong H_3$ if $r = \alpha(G) - 3$.

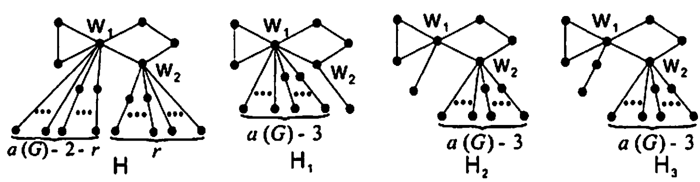

Fig. 2.

By Lemma 2.3 we obtain that
$\lambda(H_1) \leq \max\{\frac{\alpha(G)^2+5\alpha(G)+7}{\alpha(G)+4}, \frac{\alpha(G)^2+5\alpha(G)+11}{\alpha(G)+3}, \frac{\alpha(G)^2+5\alpha(G)+5}{\alpha(G)+2}\} < \alpha(G) + 3 = \alpha(H_1) + 3$.

Similarly, we have $\lambda(H_i) < \alpha(H_i) + 3$ ($i = 2, 3$). By Lemma 2.4, we know that $\lambda(H) \leq \max\{\lambda(H_1), \lambda(H_2), \lambda(H_3)\}$. Note that $\alpha(H) = \alpha(H_1) = \alpha(H_2) = \alpha(H_3)$. Therefore, $\lambda(H) \leq \alpha(H) + 3$.

If G is isomorphic to any other graph than (g) shown in Fig. 1, then we easily know by Lemma 2.2 that $\lambda(G) \leq k(G) + 5 = \alpha(G) + 3$.

Case 2. Suppose that $\mathcal{R}(G) = 5$. Then $p + q \geq 7$. If $p + q \geq 9$, then we have $\lambda(G) \leq \alpha(G) + 3$. So we assume $p + q = 7$ or 8. In this case, G can only be isomorphic to some one of the following four graphs shown in Fig. 3.

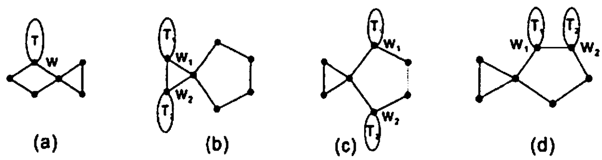

Fig. 3.

If G is isomorphic to any other graph than (a) shown in Fig. 3, then we easily know by Lemma 2.2 that $\lambda(G) \le k(G) + 5 = \alpha(G) + 3$.

Suppose that G is isomorphic to the graph (a). If there is a vertex $y \in V(T)\backslash\{w\}$ such that $d_T(y) \ge 3$, then $d_T(w) < k(G)$, and so by Lemma 2.2 we have $\lambda(G) \le k(G) + 5 = \alpha(G) + 3$; and otherwise by Lemma 2.3, we also have $\lambda(G) \le k(G) + 5 = \alpha(G) + 3$. □

Paths $P_{\ell_1}, P_{\ell_2}, \ldots, P_{\ell_t}$ are *almost equal* if $|\ell_i - \ell_j| \le 1$ for $1 \le i, j \le t$. If w is a vertex on $b(p, 0, q)$, then let $b_w^t(p, 0, q)$ be the bicyclic graph on n vertices obtained by attaching t almost equal pendent paths at w.

Lemma 3.4 [6]. *Suppose that G is a bicyclic graph on n vertices. Let v denote the vertex of degree* 4 *on $b(4,0,3)$ and $b(4,0,4)$, respectively. Then we have the following*

(1) *if $k(G) = n - 6$, then $\lambda(G) \le \lambda(b_v^{n-6}(4,0,3))$, with equality if and only if $G \cong b_v^{n-6}(4,0,3)$;*

(2) *if $k(G) \le n - 7$, then $\lambda(G) \le \lambda(b_v^{k(G)}(4,0,4))$, with equality if and only if $G \cong b_v^{k(G)}(4,0,4)$.*

Lemma 3.5. *Let v denote the vertex of degree* 4 *on $b(3,0,3)$, $b(4,0,3)$ and $b(4,0,4)$, respectively. If $t > \ell$, then*

$$\lambda(b_v^t(3,0,3)) \ge max\{\lambda(b_v^\ell(4,0,3)), \lambda(b_v^\ell(4,0,4))\}.$$

Proof. By Lemma 2.2, we know that $max\{\lambda(b_v^\ell(4,0,3)), \lambda(b_v^\ell(4,0,4))\} \le \ell + 6$. By Lemma 2.1, we have $\lambda(b_v^t(3,0,3)) \ge t + 5$. □

Suppose that G is a bicyclic graph. Now we discuss what the range of $\alpha(G)$ is.

Lemma 3.6. *Suppose that G is $b(p, \ell, q)$ with trees attached on n vertices. Then $\lceil \frac{n}{2} \rceil - 1 \le \alpha(G) \le n - 2$.*

Proof. It is clear that $\alpha(G) \le n - 2$. Let $e_1 = u_1u_1'$ and $e_2 = u_2u_2'$ be two edges so that $G\backslash\{e_1, e_2\}$ is bipartite graph with bipartition (X, Y).

Then $|X| + |Y| = n$. Suppose that $|X| = |Y|$. Then n is even and $|X| = |Y| = \frac{n}{2}$. If ends of e_1 belong to one bipartition and ends of e_2 belong to another bipartition, say $\{u_1, u_1'\} \subseteq X$ and $\{u_2, u_2'\} \subseteq Y$, then $\alpha(G) = \frac{n}{2} - 1$; and otherwise $\alpha(G) = \frac{n}{2}$.

If $|X| \neq |Y|$, then as in the proof above we can verify that $\alpha(G) \geq \lceil \frac{n}{2} \rceil - 1$. □

Lemma 3.7. *Suppose that G is $b(p, 0, q)$ or $\theta(p, \ell, q)$ with trees attached on n vertices. Then $\lceil \frac{n-1}{2} \rceil \leq \alpha(G) \leq n - 2$.*

Proof. It is clear that $\alpha(G) \leq n - 2$. Let v be the vertex so that $G - v$ is bipartite graph. Since $\alpha(G - v) \geq \lceil \frac{n-1}{2} \rceil$, $\alpha(G) \geq \lceil \frac{n-1}{2} \rceil$. □

According to the range of $\alpha(G)$ obtained from Lemmas 3.6 and 3.7, now we give our main results.

Theorem 3.8. *Suppose that G is a bicyclic graph on n vertices. Then we have the following*

(i) *if $\alpha(G) = n - 2$, then $\lambda(G) \leq \lambda(\theta_v^{n-4}(2,1,2))$, with equality if and only if $G \cong \theta_v^{n-4}(2,1,2)$, where $\theta_v^{n-4}(2,1,2)$ is as in Fig. 4;*

(ii) *if $\lceil \frac{n-1}{2} \rceil \leq \alpha(G) \leq n - 3$ and $d_{b(3,0,3)}(v) = 4$, then*

$$\lambda(G) \leq \lambda(b_v^{\alpha(G)-2}(3,0,3))$$

with equality if and only if $G \cong b_v^{\alpha(G)-2}(3,0,3)$.

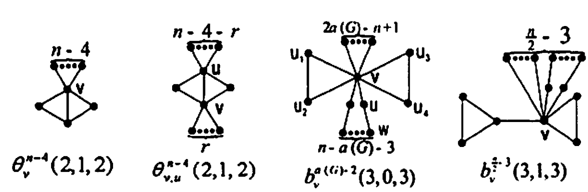

Fig. 4.

Proof. If $\alpha(G) = n - 2$, it is easy to see that $G \cong \theta_v^{n-4}(2,1,2)$ or $G \cong \theta_{v,u}^{n-4}(2,1,2)$ shown in Fig. 4. Suppose that $G \cong \theta_{v,u}^{n-4}(2,1,2)$. If $x_{\theta_{v,u}^{n-4}(2,1,2)}(v) \geq x_{\theta_{v,u}^{n-4}(2,1,2)}(u)$, then transferring all pendant paths from u to v, and otherwise transferring all pendant paths from v to u, we will get a bicyclic graph which is isomorphic to $\theta_v^{n-4}(2,1,2)$. By Lemma 2.4, we know that $\lambda(\theta_{v,u}^{n-4}(2,1,2)) \leq \lambda(\theta_v^{n-4}(2,1,2))$.

So we assume that $\lceil \frac{n-1}{2} \rceil \leq \alpha(G) \leq n - 3$. If $k(G) \geq \alpha(G) - 2$, then, by Lemmas 3.2 and 3.3, we know that $\lambda(G) \leq \alpha(G) + 3$. By Lemma 2.1, we know that $\alpha(G) + 3 \leq \lambda(b_v^{\alpha(G)-2}(3,0,3))$, and so $\lambda(G) \leq \lambda(b_v^{\alpha(G)-2}(3,0,3))$.

Next we suppose that $k(G) \leq \alpha(G) - 3$. Then $k(G) \leq n - 6$. By Lemma 3.4 we obtain that $\lambda(G) \leq \max\{\lambda(b_v^{k(G)}(4,0,3)), \lambda(b_v^{k(G)}(4,0,4))\}$. Furthermore, by Lemma 3.5, we have $\max\{\lambda(b_v^{k(G)}(4,0,3)), \lambda(b_v^{k(G)}(4,0,4))\} \leq \lambda(b_v^{k(G)+1}(3,0,3))$. Thus, $\lambda(G) \leq \lambda(b_v^{k(G)+1}(3,0,3))$.

If $k(G) = \alpha(G) - 3$, then $\lambda(G) \leq \lambda(b_v^{\alpha(G)-2}(3,0,3))$. Now suppose that $k(G) < \alpha(G) - 3$. We easily know from Lemma 2.5 that $\lambda(b_v^{k(G)+1}(3,0,3)) \leq \lambda(b_v^{\alpha(G)-2}(3,0,3))$, and so $\lambda(G) \leq \lambda(b_v^{\alpha(G)-2}(3,0,3))$. □

Lemma 3.9 [6]. *Let v be a vertex in a connected graph G and suppose that two new paths $P_i = vv_{i1}v_{i2}\cdots v_{ik}$ ($i = 1, 2$) are attached to G at v, respectively, to form a graph G'. Let $G'' = G' + v_{1k}v_{2k}$. Then $\lambda(G') = \lambda(G'')$.*

Theorem 3.10. *Suppose that G is $b(p,\ell,q)$ with trees attached, and that $n = |V(G)| \geq 6$ is even. If $\alpha(G) = \frac{n}{2} - 1$, then $\lambda(G) \leq \lambda(b_v^{\frac{n}{2}-3}(3,1,3))$, with equality if and only if $G \cong b_v^{\frac{n}{2}-3}(3,1,3)$, where $b_v^{\frac{n}{2}-3}(3,1,3)$ is shown in Fig. 4.*

Proof. If $k(G) \leq 2$, then it is easy to verify that $\lambda(G) \leq \lambda(b_v^{\frac{n}{2}-3}(3,1,3))$. So we assume that $k(G) \geq 3$. If $k(G) = \alpha(G)$, then every vertex in G is either a pendant vertex or its neighbor, and so $\alpha(G) \geq \frac{n}{2}$, a contradiction. If $k(G) = \alpha(G) - 1$, then there are at most three vertices which are neither pendant vertices nor their neighbors, that is, there are at least $n - 3$ vertices which are either pendant vertices or their neighbors, and so $\alpha(G) \geq \lceil \frac{n-3}{2} \rceil + 1 = \frac{n}{2}$, a contradiction. Therefore, $k(G) \leq \alpha(G) - 2$. By Lemma 2.2, it is easy to see that $\lambda(G) \leq k(G) + 6$. Further, if $k(G) \leq \alpha(G) - 4$, then $\lambda(G) \leq \alpha(G) + 2$. Next we discuss the remaining two cases.

Case 1. Suppose that $k(G) = \alpha(G) - 2$. Then $\mathcal{R}(G) \leq 6$.

If $\mathcal{R}(G) \leq 2$, then by Lemma 3.1, we have $\lambda(G) \leq \alpha(G) + 2$ and so we assume that $3 \leq \mathcal{R}(G) \leq 6$. In this case we can find a maximum independent set S_1 which consists of all pendant vertices and one or two vertices of $V(b(p,\ell,q))$ where no tree is attached.

If there is exactly one vertex of $S_1 \cap V(b(p,\ell,q))$ where no tree is attached, then we can observe that for each edge $u_1u_2 \in E(G)$, $d_G(u_1) + d_G(u_2) - |N_G(u_1) \cap N_G(u_2)| \leq k(G) + 4 = \alpha(G) + 2$. By Lemma 2.2, we have $\lambda(G) \leq \alpha(G) + 2$. Thus, we assume that there are exactly two vertices of $S_1 \cap V(b(p,\ell,q))$ where no tree is attached.

If each vertex in $V(G) \setminus V(b(p,\ell,q))$ is of degree at most 2, then every tree attached on $b(p,\ell,q)$ is composed of pendant paths of length 1 or 2.

When $\ell = 1$, we can notice that all G but those graphs isomorphic to B_i ($1 \leq i \leq 10$) as in Fig. 5 satisfy $d_G(u_1) + d_G(u_2) - |N_G(u_1) \cap N_G(u_2)| \leq k(G) + 4 = \alpha(G) + 2$ for each edge $u_1u_2 \in E(G)$. It follows from Lemma 2.2 that $\lambda(G) \leq \alpha(G) + 2$.

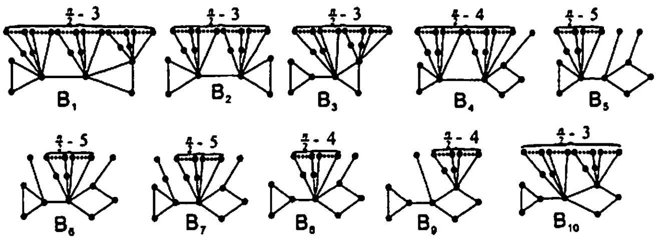

Fig. 5. The graphs B_i $(1 \le i \le 10)$

Now we observe the graph B_i $(1 \le i \le 10)$. Suppose that $y_i \in V(B_1)$ is a vertex on $b(3,1,3)$ where trees are attached $(i = 1,2)$. If $x_{y_1}(B_1) \ge x_{y_2}(B_1)$, then transfer all pendant paths from y_2 to y_1, and otherwise transfer all pendant paths from y_1 to y_2. Continuing this process we will get a bicyclic graph G_1 so that all pendant paths are attached to vertex y on $b(3,1,3)$. It follows from Lemma 2.4 that $\lambda(B_1) \le \lambda(G_1)$. If $d_{b(3,1,3)}(y) = 3$, then $G_1 \cong b_v^{\alpha(G)-2}(3,1,3)$, and otherwise we can observe that for each edge $u_1u_2 \in E(G_1)$, $d_{G_1}(u_1) + d_{G_1}(u_2) - |N_{G_1}(u_1) \cap N_{G_1}(u_2)| \le k(G_1) + 4 = \alpha(G_1) + 2$. By Lemma 2.2, $\lambda(G_1) \le \alpha(G_1) + 2$. Note that $\alpha(G_1) = \alpha(B_1)$. Thus, $\lambda(B_1) \le \alpha(B_1) + 2$. Similarly, we can verify that $\lambda(B_i) \le \alpha(B_i) + 2$ $(i = 2, 3)$.

In this case note that $n \ge 12$. By Lemma 2.3, we have
$\lambda(B_5) \le \max\{\frac{n^2-2n+88}{2n+8}, \frac{n^2-2n+20}{2n}, \frac{n^2-2n}{2n-4}, \frac{n+8}{4}\} \le \frac{n}{2} + 1 = \alpha(G) + 2 = \alpha(B_5) + 2$.

Similarly, we can verify that $\lambda(B_i) \le \alpha(B_i) + 2$ $(6 \le i \le 9)$. By Lemma 2.4, we have $\lambda(B_4) \le \max\{\lambda(B_5), \lambda(B_6), \lambda(B_7)\}$ and $\lambda(B_{10}) \le \max\{\lambda(B_8), \lambda(B_9)\}$. Note that $\alpha(G) = \alpha(B_i)$ $(4 \le i \le 10)$, we have $\lambda(B_4) \le \alpha(B_4) + 2$ and $\lambda(B_{10}) \le \alpha(B_{10}) + 2$.

When $\ell \ge 2$, we only observe that B_{11}, B_{12} and B_{13} shown in Fig. 6.

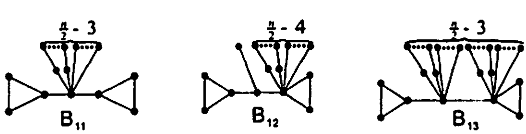

Fig. 6. The graphs B_i $(11 \le i \le 13)$

By Lemma 2.3, we have
$\lambda(B_{11}) \le \max\{\frac{n^2+2n+48}{2n+8}, \frac{n^2+2n+16}{2n+4}, \frac{n^2+2n}{2n}, \frac{n+42}{10}\} \le \frac{n}{2} + 1 = \alpha(G) + 2 = \alpha(B_{11}) + 2$.

Similarly, we have $\lambda(B_{12}) \leq \alpha(B_{12}) + 2$.

By Lemma 2.4, we have $\lambda(B_{13}) \leq \max\{\lambda(B_{11}), \lambda(B_{12})\}$. Note that $\alpha(B_{11}) = \alpha(B_{12}) = \alpha(B_{13}) = \alpha(G)$. Therefore, $\lambda(B_{13}) \leq \alpha(B_{13}) + 2$.

For all G but those three graphs isomorphic to B_i ($i = 11, 12, 13$), we can observe that for each edge $uu' \in E(G)$, $d_G(u) + d_G(u') \leq k(G) + 4 = \alpha(G) + 2$, and so by Lemma 2.2, $\lambda(G) \leq \alpha(G) + 2$.

Now suppose that there is a vertex in $V(G)\backslash V(b(3, \ell, 3))$ which is of degree at least 3. If G is as in Fig. 7, then by Lemma 2.3, we have

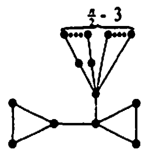

Fig. 7.

$\lambda(G) \leq \max\{\frac{n^2-2n+88}{2n+8}, \frac{n^2-2n+16}{2n}, \frac{n^2-2n}{2n-4}, \frac{n+76}{14}, \frac{n+62}{12}\} \leq \frac{n}{2}+1 = \alpha(G)+2$. Otherwise we can observe that for each edge $vv' \in E(G)$, $d_G(v) + d_G(v') \leq k(G) + 4 = \alpha(G) + 2$. By Lemma 2.2, we have $\lambda(G) \leq \alpha(G) + 2$.

Case 2. Suppose that $k(G) = \alpha(G) - 3$. Then $\mathcal{R}(G) \leq 8$.

If $\mathcal{R}(G) \leq 3$, then by Lemma 3.1, we have $\lambda(G) \leq \alpha(G) + 2$. Thus, we assume that $4 \leq \mathcal{R}(G) \leq 8$. In this case we can find a maximum independent set S_2 which consists of all pendant vertices and two or three vertices of $V(b(p, \ell, q))$ where no tree is attached.

If there exist exactly two vertices of $S_2 \cap V(b(p, \ell, q))$ where no tree is attached, then for all G but those graphs isomorphic to A_1 and A_2 shown in Fig. 8, we can observe that for each edge $u_1u_2 \in E(G)$, $d_G(u_1) + d_G(u_2) - |N_G(u_1) \cap N_G(u_2)| \leq k(G) + 5 = \alpha(G) + 2$. By Lemma 2.2, $\lambda(G) \leq \alpha(G) + 2$.

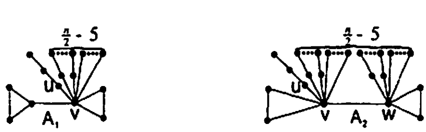

Fig. 8. The graphs A_1 and A_2

Suppose that $G \cong A_1$. Then we can obtain the graph $b_v^{\frac{n}{2}-3}(3, 1, 3)$ from A_1 by contracting the edge vu into one vertex v and then adding a pendant edge at v. By Lemma 2.5, we have $\lambda(A_1) \leq \lambda(b_v^{\frac{n}{2}-3}(3, 1, 3))$. Suppose that $G \cong A_2$. If $x_v(A_2) \geq x_w(A_2)$, then transferring all pendant paths in A_2

from w to v, and otherwise transferring all pendant paths from v to w, we get a bicyclic graph which is isomorphic to A_1. By Lemma 2.4, we have $\lambda(A_2) \leq \lambda(A_1)$ and so $\lambda(A_2) \leq \lambda(b_v^{\frac{n}{2}-3}(3,1,3))$.

If there exist exactly three vertices of $S_2 \cap V(b(p,\ell,q))$ where no tree is attached, then we distinguish the following two cases.

When $\ell = 1$, for all G but those graphs isomorphic to A_i $(3 \leq i \leq 8)$ shown in Fig. 9, we can observe that for each edge $u_1u_2 \in E(G)$, $d_G(u_1) + d_G(u_2) - |N_G(u_1) \cap N_G(u_2)| \leq k(G) + 5 = \alpha(G) + 2$. By Lemma 2.2, $\lambda(G) \leq \alpha(G) + 2$.

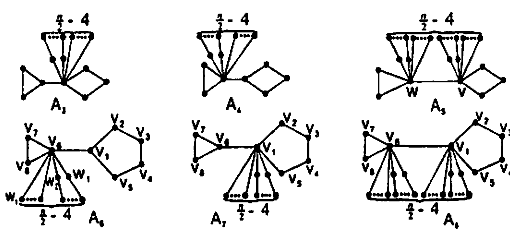

Fig. 9. The graphs A_i $(3 \leq i \leq 8)$

In this case note that $n \geq 14$. By Lemma 2.3, we have $\lambda(A_3) \leq \max\{\frac{n^2+2n+48}{2n+8}, \frac{n^2+2n+20}{2n+4}, \frac{n^2+2n-4}{2n}, \frac{n+26}{8}\} \leq \frac{n}{2} + 1 = \alpha(G) + 2 = \alpha(A_3) + 2$. Similarly, we have $\lambda(A_4) \leq \alpha(A_4)$

By Lemma 2.4, we have $\lambda(A_5) \leq \max\{\lambda(A_3), \lambda(A_4)\}$. Note that $\alpha(A_5) = \alpha(A_4) = \alpha(A_3) = \alpha(G)$. We have $\lambda(A_5) \leq \alpha(A_5) + 2$.

Let $H = A_6 - v_3v_4 - v_7v_8$ and $H' = H - v_2v_3 + v_1v_3$. By Lemma 2.5, we have $\lambda(H) \leq \lambda(H')$. If $x_{v_1}(H') \geq x_{v_6}(H')$, then let $H'' = H' - v_6w_1 - v_6w_2 - \cdots - v_6w_t + v_1w_1 + v_1w_2 + \cdots + v_1w_t$, and otherwise let $H'' = H' - v_1v_5 + v_6v_5$. It follows from Lemma 2.4 that $\lambda(H') \leq \lambda(H'')$. It is easy to observe that $H'' + v_2v_3 + v_7v_8$ is isomorphic to $b_v^{\frac{n}{2}-3}(3,1,3)$. By Lemma 3.9, we have $\lambda(A_6) = \lambda(H)$ and $\lambda(H'') = \lambda(b_v^{\frac{n}{2}-3}(3,1,3))$. Therefore, $\lambda(A_6) \leq \lambda(b_v^{\frac{n}{2}-3}(3,1,3))$. We can similarly prove that $\lambda(A_7) \leq \lambda(b_v^{\frac{n}{2}-3}(3,1,3))$. By Lemma 2.4, we know that $\lambda(A_8) \leq \max\{\lambda(A_6), \lambda(A_7)\}$, and so $\lambda(A_8) \leq \lambda(b_v^{\frac{n}{2}-3}(3,1,3))$.

When $\ell \geq 2$, we can observe that for each edge $ww' \in E(G)$, $d_G(w) + d_G(w') \leq k(G) + 5 = \alpha(G) + 2$. By Lemma 2.2, we have $\lambda(G) \leq \alpha(G) + 2$.

By Lemma 2.1, we have $\alpha(G) + 2 \leq \lambda(b_v^{\frac{n}{2}-3}(3,1,3))$. So far, this completes the proof. □

Recall that $Q(G) = D(G) + A(G)$ is the *signless Laplacian matrix* of graph G. We denote by $\mu(G)$ the largest eigenvalue of $Q(G)$.

Lemma 3.11 [2]. *If G is a connected graph, then $\lambda(G) \leq \mu(G)$, with equality if and only if G is bipartite.*

Theorem 3.12. *If $d_{b(3,0,3)}(v) = 4$, then $\lambda(b_v^{\alpha-2}(3,0,3))$ is the largest one of three roots of the equation $x^3 - (\alpha+6)x^2 + (3\alpha+10)x - n = 0$.*

Proof. Let $b_v^{\alpha-2}(3,0,3)$ be as in Fig. 4. By Lemma 3.9, we know that $\lambda(b_v^{\alpha-2}(3,0,3)) = \lambda(b_v^{\alpha-2}(3,0,3) - u_1u_2 - u_3u_4)$. Let $H = b_v^{\alpha-2}(3,0,3) - u_1u_2 - u_3u_4$. Then H is a bipartite graph, and so by Lemma 3.11, $\lambda(H) = \mu(H)$.

Suppose that y is the principal eigenvector of $Q(H)$ corresponding to $\mu(H)$. Then $Q(H)y = \mu(H)y$. Let $\mu = \mu(H)$ and y_v be the eigencomponent of y corresponding to the vertex v. By symmetry we have

$$\mu y_{u_1} = y_{u_1} + y_v, \mu y_w = y_w + y_u, \mu y_u = 2y_u + y_v + y_w,$$

$$\mu y_v = (2\alpha - n + 5)y_{u_1} + (\alpha+2)y_v + (n - \alpha - 3)y_u.$$

Simplifying the above system of equation, we finally get

$$\mu^3 - (\alpha+6)\mu^2 + (3\alpha+10)\mu - n = 0.$$

Suppose that $f(x) = x^3 - (\alpha+6)x^2 + (3\alpha+10)x - n$. Then $f'(x) = 3x^2 - 2(\alpha+6)x + 3\alpha + 10$, and so $f(x) = x^3 - (\alpha+6)x^2 + (3\alpha+10)x - n$ is strictly increasing if $x \in (\frac{\alpha+6+\sqrt{\alpha^2+3\alpha+6}}{3}, +\infty)$. By Lemma 2.1, we know that $\mu(H) \geq \alpha + 3$. Note that $\alpha + 3 > \frac{\alpha+6+\sqrt{\alpha^2+3\alpha+6}}{3}$. We have $\mu(H) > \frac{\alpha+6+\sqrt{\alpha^2+3\alpha+6}}{3}$, which shows that $\mu(H)$ is largest among three roots of $x^3 - (\alpha+6)x^2 + (3\alpha+10)x - n = 0$. □

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