On (s, t)-relaxed $L(2, 1)$-labelings of the hexagonal lattice *

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Abstract

Suppose G is a graph. Let u be a vertex of G. A vertex v is called an i-neighbor of u if $d_G(u, v) = i$. A 1-neighbor of u is simply called a neighbor of u. Let s and t be two nonnegative integers. Suppose f is an assignment of nonnegative integers to the vertices of G. If the following three conditions are satisfied, then f is called an (s, t)-relaxed $L(2, 1)$-labeling of G: (1) for any two adjacent vertices u and v of G, $f(u) \neq f(v)$; (2) for any vertex u of G, there are at most s neighbors of u receiving labels from $\{f(u)-1, f(u)+1\}$; (3) for any vertex u of G, the number of 2-neighbors of u assigned the label $f(u)$ is at most t. The minimum span of (s, t)-relaxed $L(2, 1)$-labelings of G is called the (s, t)-relaxed $L(2, 1)$-labeling number of G, denoted by $\lambda_{2,1}^{s,t}(G)$. It is clear that $\lambda_{2,1}^{0,0}(G)$ is the so called $L(2, 1)$-labeling number of G. In this paper, the (s, t)-relaxed $L(2, 1)$-labeling number of the hexagonal lattice is determined for each pair of two nonnegative integers s and t. And this provides a series of channel assignment schemes for the corresponding channel assignment problem on the hexagonal lattice.

Keywords: channel assignment, $L(2, 1)$-labeling, (s, t)-relaxed $L(2, 1)$-labeling, hexagonal lattice

1 Introduction

An $L(2, 1)$*-labeling* f of a graph G is an assignment f of nonnegative integers to the vertices of G such that $|f(u) - f(v)| \geq 2$ if $uv \in E(G)$, and $|f(u) - f(v)| \geq 1$ if $d_G(u, v) = 2$, where $d_G(u, v)$ is the length (number of edges) of a shortest path between u and v in G. The conditions in the definition involving distances are called the *distance one condition* and the *distance two condition*, respectively. Given a graph G, for an $L(2, 1)$-labeling

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f of G, elements of the image of f are called labels, and we define the *span* of f, denoted by $span(f)$, to be the difference between the maximum and minimum vertex labels of f. The $L(2,1)$-*labeling number*, denoted by $\lambda(G)$, is the minimum span over all $L(2,1)$-labelings of G. Distance two labelings of graphs were introduced by Griggs and Yeh [7] and were studied extensively since then, see surveys [4, 8, 16].

$L(2,1)$-labeling was motivated by a kind of channel assignment problem. The graph (referred as the interference graph) describes the network of transmitters. Labels of vertices correspond to channels assigned to transmitters. To avoid interference, "close" transmitters (corresponding to vertices that are at distance two) are required to receive different channels and "very close" transmitters (corresponding to adjacent vertices) are required to receive channels that are at least two channels apart. The main aim is to minimize the span of channels assigned to transmitters.

However, problems may be arisen if the channel resource is limited (or equivalently the channel span is restricted). Suppose we are given the channel span λ. Let G be the interference graph for a channel assignment instance. If $\lambda(G) > \lambda$, then it is impossible to produce an $L(2,1)$-labeling of G with span λ. One solution to this problem is to construct a function f from the vertices of G to integers $0, 1, \ldots, \lambda$ such that f is as "close" to an $L(2,1)$-labeling of G as possible. We next present a method to measure the "distance" between a function and an $L(2,1)$-labeling. We need a concept called (s,t)-relaxed $L(2,1)$-labeling of a graph which was introduced in [11].

Let G be a graph. Let u be a vertex of G. A vertex v is called an *i-neighbor* (resp. *i^--neighbor*) of u if $d_G(u,v) = i$ (resp. $d_G(u,v) \leq i$). A 1-neighbor of u is simply called a neighbor of u. Let s and t be two nonnegative integers. Suppose f is an assignment of nonnegative integers to the vertices of G. If the following three conditions are satisfied, then f is called an *(s,t)-relaxed $L(2,1)$-labeling* of G:

(1) for any two adjacent vertices u and v of G, $f(u) \neq f(v)$;

(2) for any vertex u of G, there are at most s neighbors of u receiving labels from $\{f(u)-1, f(u)+1\}$;

(3) for any vertex u of G, the number of 2-neighbors of u assigned the label $f(u)$ is at most t.

The *span* of f, denoted by $span(f)$, is the difference between the maximum and minimum labels used under f. Without loss of generality, we assume that the minimum label of an (s,t)-relaxed $L(2,1)$-labeling is always 0. Then the span of f is the maximum vertex label. The minimum span of (s,t)-relaxed $L(2,1)$-labelings of G is called the *(s,t)-relaxed $L(2,1)$-labeling number* of G, denoted by $\lambda_{2,1}^{s,t}(G)$. An (s,t)-relaxed $L(2,1)$-

labeling of a graph using labels in $\{0, 1, \ldots, k\}$ is called an *(s,t)-relaxed k-$L(2,1)$-labeling.*

Let (s,t) and (s',t') be two pairs of nonnegative integers. If $s \le s'$ and $t \le t'$, then we say (s,t) is less than or equal to (s',t'), and is written as $(s,t) \preceq (s',t')$. This defines a partial order on the set of all pairs of nonnegative integers. We clearly have the following lemma.

Lemma 1.1 *Let (s,t) and (s',t') be two pairs of nonnegative integers. If $(s,t) \preceq (s',t')$, then $\lambda_{2,1}^{s,t}(G) \ge \lambda_{2,1}^{s',t'}(G)$.*

Denote by $\Delta(G)$ the maximum degree of a graph G and $\Delta_2(G)$ the maximum number of 2-neighbors of a vertex of G. The following lemma is easy to see.

Lemma 1.2 *If $s \ge \Delta(G)$ and $t \ge \Delta_2(G)$, then $\lambda_{2,1}^{s,t}(G) = \chi(G) - 1$.*

Since $\lambda_{2,1}^{0,0}(G) = \lambda_{2,1}(G)$, for any pair (s,t) of nonnegative integers, the following inequality holds.

$$\chi(G) - 1 \le \lambda_{2,1}^{s,t}(G) \le \lambda_{2,1}(G).$$

Thus if $\lambda \ge \chi(G) - 1$ then there are some pairs (s,t) with $\lambda_{2,1}^{s,t}(G) \le \lambda$. Let f be a function from the vertices of G to integers $0, 1, \ldots, \lambda$, the distance from f to an $L(2,1)$-labeling can be measured by the maximal pair (s,t) such that f is an (s,t)-relaxed $L(2,1)$-labeling of G. For the given span λ, the candidate relaxing schemes are those maximal pairs (s,t) such that G has an (s,t)-relaxed $L(2,1)$-labeling. By comparing the levels of interference of the corresponding (s,t)-relaxed $L(2,1)$-labelings for these maximal pairs, one may choose the best one for the practical use. Thus in some sense, one makes the full use of the given channels in this way.

We claim that $\lambda_{2,1}^{s,t}(G) = \lambda_{2,1}^{2t+2,t}(G)$ for $s \ge 2t+2$. Suppose f is an (s,t)-relaxed $L(2,1)$-labeling of G. It is clear that there are at most $t+1$ neighbors of u receiving the label $f(u)+1$ (resp. $f(u)-1$). Therefore, for each vertex u, there are at most $2t+2$ neighbors of u that could have labels adjacent to $f(u)$. Note that a $(2,0)$-relaxed k-$L(2,1)$-labeling of a graph G is actually a $(k+1)$-coloring of G^2 (the square of G) using colors $0, 1, \ldots, k$. It follows that $\lambda_{2,1}^{s,0}(G) = \lambda_{2,1}^{2,0}(G) = \chi(G^2) - 1$ for $s \ge 2$.

Lattices are frequently used models for channel assignment problems. Distance two labelings of lattices (including hexagonal, square and triangular lattices) have been investigated extensively, see [2,3,6,9,10,13–15,17].

For any two nonnegative integers s and t, the (s,t)-relaxed $L(2,1)$-labeling numbers of the square lattice and the triangular lattice are determined in [5] and [12], respectively. For all nonnegative integers s and t, lower and upper bounds of the (s,t)-relaxed $L(2,1)$-labeling number for trees are obtained in [11]. These bounds are proved to be sharp.

In this paper, we determine the (s,t)-relaxed $L(2,1)$-labeling number of the hexagonal lattice for any two nonnegative integers s and t. And this provides a series of channel assignment schemes for the corresponding channel assignment problem on the hexagonal lattice.

2 Main theorem

Let $\mathbf{e_1} = (1,0)^T$, $\mathbf{e_2} = (0,1)^T$ and $\mathbf{f} = (1/2, \sqrt{3}/2)^T$ be three vectors in the Euclidean plane. The *triangular lattice* Γ_3 is an infinite graph with vertex set $\{x\mathbf{e_1} + y\mathbf{f} : x, y \in \mathbb{Z}\}$, and with two different vertices (x_1, y_1), (x_2, y_2) adjacent if the Euclidean distance between them is 1. The *square lattice* Γ_4 is an infinite graph with vertex set $\{x\mathbf{e_1} + y\mathbf{e_2} : x, y \in \mathbb{Z}\}$, and with two different vertices (x_1, y_1), (x_2, y_2) adjacent if the Euclidean distance between them is 1. If two vertices (x_1, y_1) and (x_2, y_2) in Γ_i $(i = 3, 4, 6)$ are adjacent, then we write the edge joining them by $(x_1, y_1)(x_2, y_2)$.

The hexagonal lattice Γ_6 is the subgraph of Γ_3 induced by the vertex set $V(\Gamma_3) \setminus \{(x, x+3y+1) : x, y \in \mathbb{Z}\}$. One can also view the hexagonal lattice Γ_6 as a spanning subgraph of Γ_4 with edge set $E(\Gamma_4) \setminus E^*$, where $E^* = \{(x,y)(x+1,y) : x, y \in \mathbb{Z} \text{ and } x+y \text{ is odd}\}$. Please see Figure 1 for illustrations. We shall use the coordinates to express the vertices of Γ_6 in the proof of the following theorem.

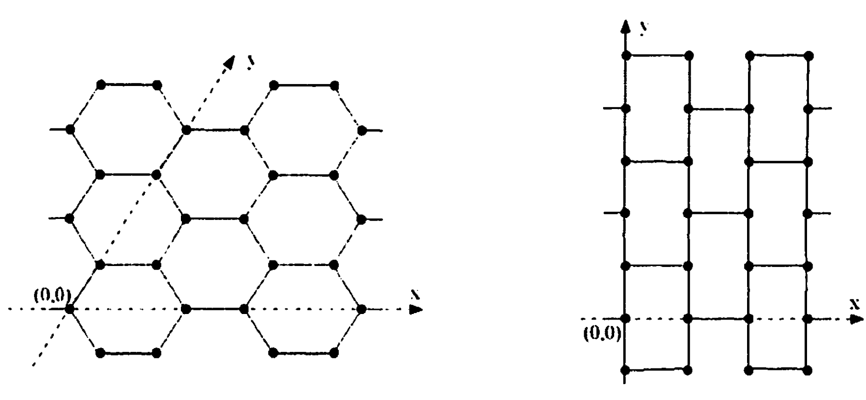

Figure 1: Two drawings of the hexagonal lattice Γ_6.

Note that any vertex of Γ_6 has three neighbors and six 2-neighbors, we only need to deal with (s,t)-relaxed $L(2,1)$-labelings for $0 \le s \le 3$ and $0 \le t \le 6$. For two integers a and b with $a \le b$, let $[a,b]$ denote the set of integers $a, a+1, \ldots, b$.

Observe that in an $L(2,1)$-labeling f of a graph G with span $\Delta(G)+1$ vertices of maximum degree must receive labels in $\{0, \Delta(G)+1\}$. It follows that $\lambda_{2,1}(\Gamma_6) \ge 5$. On the other hand, $f(i,j) = 3i + 2j \pmod 6$ is an $L(2,1)$-labeling of Γ_6 with span 5. Thus $\lambda_{2,1}(\Gamma_6) = 5$. See also [1] for this result. Therefore $\lambda_{2,1}^{0,0}(\Gamma_6) = \lambda_{2,1}(\Gamma) = 5$.

Theorem 2.1 $\lambda_{2,1}^{s,t}(\Gamma_6) = \begin{cases} 1 & \textit{if } s = 3, t = 6, \\ 2 & \textit{if } s = 3, t \in [2,5], \ \textit{or } s \in [0,2], t = 6, \\ 3 & \textit{if } s = 0, t \in [4,5], \ \textit{or } s = 1, t \in [2,5], \ \textit{or} \\ & \quad s = 2, t \in [0,5], \ \textit{or } s = 3, t \in [0,1], \\ 4 & \textit{if } s = 0, t \in [2,3], \ \textit{or } s = 1, t \in [0,1], \\ 5 & \textit{if } s = 0, t \in [0,1]. \end{cases}$

Proof. (1) Since Γ_6 is bipartite, $\chi(\Gamma_6) = 2$ and so $\lambda_{2,1}^{3,6}(\Gamma_6) = \chi(\Gamma_6) - 1 = 1$. Note that the 2-coloring of Γ_6 is unique up to the permutation of colors. It follows that $\lambda_{2,1}^{s,t}(\Gamma_6) \ge 2$ for all pairs $(s,t) \ne (3,6)$.

(2) For this case, by Lemma 1.1, it suffices to show that $\lambda_{2,1}^{0,6}(\Gamma_6) \le 2$ and $\lambda_{2,1}^{3,2}(\Gamma_6) \le 2$. Suppose c is a 2-coloring of Γ_6 using colors 1 and 2. For any vertex v of Γ_6, let $f(v) = 0$ if $c(v) = 1$ and $f(v) = 2$ if $c(v) = 2$. Then it is clear that f is a $(0,6)$-relaxed 2-$L(2,1)$-labeling of Γ_6. Thus $\lambda_{2,1}^{0,6}(\Gamma_6) \le 2$. Let $f(i,j) = 2 \cdot \lfloor (i+3j)/2 \rfloor \pmod 3$. It is straightforward to check that f is a $(3,2)$-relaxed 2-$L(2,1)$-labeling of Γ_6 (see Figure 2 for an illustration). Therefore $\lambda_{2,1}^{3,2}(\Gamma_6) \le 2$.

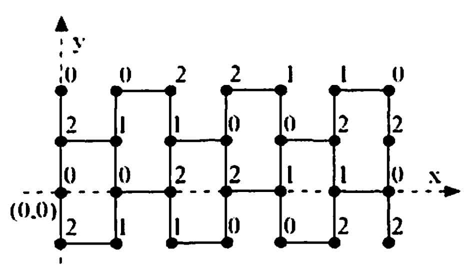

Figure 2: A $(3,2)$-relaxed 2-$L(2,1)$-labeling of Γ_6.

(3) By Lemma 1.1, it suffices to show that $\lambda_{2,1}^{s,t}(\Gamma_6) \ge 3$ for $(s,t) \in$

$\{(2,5),(3,1)\}$ and $\lambda_{2,1}^{s,t}(\Gamma_6) \le 3$ for $(s,t) \in \{(0,4),(1,2),(2,0)\}$.

We claim that $\lambda_{2,1}^{2,5}(\Gamma_6) \ge 3$. Suppose $\lambda_{2,1}^{2,5}(\Gamma_6) \le 2$. Let f be a $(2,5)$-relaxed $L(2,1)$-labeling of Γ_6 with span 2. If some vertex u is assigned the label 1, then each neighbor of u must receive label 0 or 2. It follows that u has three neighbors having labels adjacent to $f(u)$, which is a contradiction. Thus no vertex is assigned label 1. Let u be a vertex with label 0. Then the three neighbors of u must have the same label 2, implying that the six 2-neighbors of u must have the same label as u. This is a contradiction. Thus $\lambda_{2,1}^{2,5}(\Gamma_6) \ge 3$.

Now we verify that $\lambda_{2,1}^{3,1}(\Gamma_6) \ge 3$. Suppose that $\lambda_{2,1}^{3,1}(\Gamma_6) \le 2$ and let f be a $(3,1)$-relaxed $L(2,1)$-labeling of Γ_6 with span 2. If the label 1 is not used by f, then as in the previous paragraph, the six 2-neighbors of a vertex u with label 0 must have the same label 0 as u, implying a contradiction. Thus there is a vertex, say u, with label 1. Please refer to Figure 3 for names and labels of vertices around u. Since f is a $(3,1)$-relaxed $L(2,1)$-labeling of Γ_6, two of the three neighbors must have the same label 0 (or 2) and the remaining one must have the label 2 (or 0). By symmetry, we may assume that u_1, u_2 are labeled 0 and u_3 is labeled 2. Then $f(v), f(w), f(x), f(y) \neq 0$. Since u has at most one 2-neighbor labeled 1 as u, $f(v) = f(w) = 1$ will not happen. If $\{f(v), f(w)\} = \{1,2\}$, then $f(z) = 0$ and z has two 2-neighbors u_1 and u_2 with label 0, a contradiction. Therefore $f(v) = f(w) = 2$. Since $f(x) \neq 0$ and $f(w) = 2$, we must have $f(x) = 1$, otherwise v has two 2-neighbors having the same label as v. Similarly, we have $f(y) = 1$. But this is a contradiction since then x and y are two 2-neighbors of u with the same label as u.

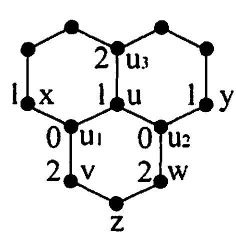

Figure 3: Lower bound about $\lambda_{2,1}^{3,1}(\Gamma_6)$.

Next we give a $(0,4)$-relaxed 3-$L(2,1)$-labeling f of Γ_6 in Figure 4 and a $(1,2)$-relaxed 3-$L(2,1)$-labeling g of Γ_6 in Figure 5. The two labelings just satisfy: $f(0,0) = 0$, $f(1,0) = 3$, $f(2,0) = 0$, $f(3,0) = 3$, $f(4,0) = 1$, $f(5,0) = 2$ and $f(i+6,j) = f(i,j)$, $f(i+3,j+1) = f(i,j)$ for any two integers i and j. $g(0,0) = 0$, $g(1,0) = 0$, $g(2,0) = 3$, $g(3,0) = 3$, $g(4,0) = 2$,

$g(5,0) = 1$ and $g(i+6,j) = g(i,j)$, $g(i+3,j+1) = g(i,j)$ for any two integers i and j.

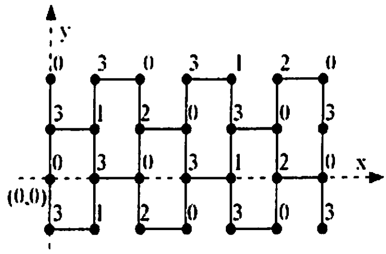

Figure 4: A $(0,4)$-relaxed 3-$L(2,1)$-labeling of Γ_6.

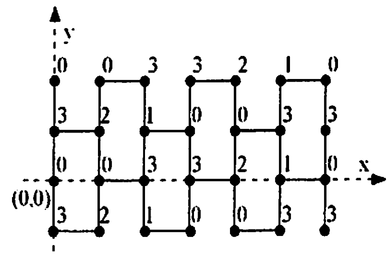

Figure 5: A $(1,2)$-relaxed 3-$L(2,1)$-labeling of Γ_6.

$\lambda_{2,1}^{2,0}(\Gamma_6) \leq 3$ is trivial since $\lambda_{2,1}^{2,0}(\Gamma_6) = \chi(\Gamma_6^2) - 1$ and it is easy to see that $\chi(\Gamma_6^2) = 4$.

(4) By Lemma 1.1, it suffices to show that $\lambda_{2,1}^{s,t}(\Gamma_6) \geq 4$ for $(s,t) \in \{(0,3),(1,1)\}$ and $\lambda_{2,1}^{s,t}(\Gamma_6) \leq 4$ for $(s,t) \in \{(0,2),(1,0)\}$.

Firstly, we verify that $\lambda_{2,1}^{0,3}(\Gamma_6) \geq 4$. Suppose $\lambda_{2,1}^{0,3}(\Gamma_6) \leq 3$. Let f be a $(0,3)$-relaxed $L(2,1)$-labeling of Γ_6 with span 3. We observe that if some vertex is assigned the label 1 (or 2), then its three neighbors must receive the same label 3 (or 0). We claim that if some vertex is assigned the label 1 (or 2), then none of its 2-neighbors can receive the same label 1 (or 2). Otherwise, if a vertex u and one of its 2-neighbor v are assigned the same label 1 (or 2), then the common neighbor of u and v, say w, must receive the label 3 (or 0). Since all neighbors of u and v have label 3 (or 0), at least four 2-neighbors of w have the same label 3 (or 0). This is a contradiction. Hence, if some vertex is assigned the label 0 (or 3), then at least two of its three neighbors must receive the label 3 (or 0). Note that using only the two labels 0 and 3 one cannot produce a $(0,3)$-relaxed $L(2,1)$-labeling of Γ_6. Thus f assigns 1 or 2 to some vertex of Γ_6. Without loss of generality, assume there is some vertex u with $f(u) = 2$. Then the three neighbors of u (u_1, u_2 and u_3) are assigned the same label 0. Please see Figure 6 for the locations of these vertices. Let u_4 and u_5 be the other two neighbors of u_1. Then $f(u_4) = f(u_5) = 3$. For the same reason, one of the other two neighbors of u_4 (u_6 and u_7) and one of the other two neighbors of u_5 (u_8 and u_9) must receive the label 0. It follows that at least four 2-neighbors of

u_1 receive the same label 0 as u_1, which is a contradiction. So $\lambda_{2,1}^{0,3}(\Gamma_6) \geq 4$.

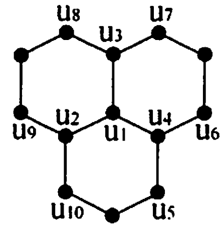

Figure 6: Lower bound about $\lambda_{2,1}^{0,3}(\Gamma_6)$.

Secondly, we prove that $\lambda_{2,1}^{1,1}(\Gamma_6) \geq 4$. Suppose that $\lambda_{2,1}^{1,1}(\Gamma_6) \leq 3$. By symmetry, if all $(1,1)$-relaxed 3-$L(2,1)$-labeling of Γ_6 does not use the label 1, then all $(1,1)$-relaxed 3-$L(2,1)$-labeling of Γ_6 does not use the label 2. However, only using the labels 0 and 3 one can not get a $(1,1)$-relaxed $L(2,1)$-labeling of Γ_6. So there exist a $(1,1)$-relaxed 3-$L(2,1)$-labeling f of Γ_6 which uses the label 1. Suppose $f(u) = 1$ for some vertex u of Γ_6 (See Figure 7). Since at most one neighbor of u can receive the label 0 or 2 and the three neighbors of u can not have the same label, by symmetry, we only need to deal with the following two cases.

Case 1. $f(u_1) = f(u_2) = 3$ and $f(u_3) = 2$.

Since $f(u_3) = 2$ and $f(u) = 1$, we have $f(v_1) = f(v_2) = 0$. This together with $f(u_1) = f(u_2) = 3$ imply that $f(x_1), f(x_2), f(y_1), f(y_2) \in \{1,2\}$. If $f(x_1) = 1$ and $f(x_2) = 2$, then x_1 would have two neighbors receiving labels adjacent to $f(x_1)$, a contradiction. Therefore $f(x_1) = 2$ and $f(x_2) = 1$. Similarly, we must have $f(y_1) = 2$ and $f(y_2) = 1$. But this is a contradiction since then u_3 has two 2-neighbors having the same label as itself.

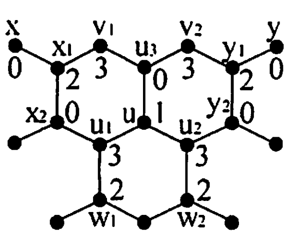

Figure 7: Case 1.

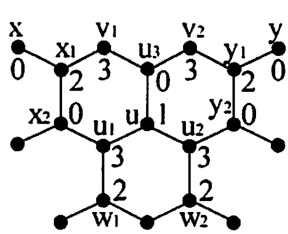

Figure 8: Case 2.

Case 2. $f(u_1) = f(u_2) = 3$ and $f(u_3) = 0$.

Please see Figure 8 for locations of vertices in the proof of this case. If $f(x_2) = 1$, then x_2 has two neighbors with label 3, implying that u_1 has two 2-neighbors with the same label 3 as itself, a contradiction. So $f(x_2) \neq 1$. If $f(x_2) = 2$, then x_2 has two neighbors receiving label 0, implying that $f(x_1) = 0$ and x_1 has two 2-neighbors with the same label 0 as itself, which is a contradiction. Hence $f(x_2) \neq 2$. It follows that $f(x_2) = 0$ and $f(x_1) \in \{1,2\}$. Similarly, we have $f(y_2) = 0$ and $f(y_1) \in \{1,2\}$. Since $f(u_3) = 0$ and $f(u) = 1$, $f(v_1) \neq 1$ and $f(v_2) \neq 1$. If $f(v_1) = 2$, then $f(x_1) = 1$ and so x_1 has two neighbors receiving the labels adjacent to $f(x_1)$, a contradiction. Thus $f(v_1) \neq 2$ and so $f(v_1) = 3$. Similarly, $f(v_2) = 3$. We claim that $f(x_1) = f(y_1) = 2$ and $f(x) = f(y) = 0$. In fact, if $f(x_1) = 1$, then vertex x_1 has two neighbors receiving label 3, implying that v_1 has two 2-neighbors receiving the same label 3 as itself. Thus $f(x_1) \neq 1$. And so $f(x_1) = 2$, implying $f(x) = 0$. Similarly, $f(y_1) = 2$ and $f(y) = 0$. Now look at the two vertices w_1 and w_2. Since $f(x) = f(x_2) = 0$, $f(w_1) \neq 0$. If $f(w_1) = 1$, then w_1 has two neighbors with label 3, implying that u_1 has two 2-neighbors with the same label 3 as itself. Thus $f(w_1) \neq 1$. It follows that $f(w_1) = 2$. Similarly, $f(w_2) = 2$. Then both w_1 and w_2 have two neighbors receiving the label 0. It follows that the vertex which is the common neighbor of w_1 and w_2 has two neighbors receiving the same label 0, a contradiction. Therefore $\lambda_{2,1}^{1,1}(\Gamma_6) \geq 4$.

Thirdly, let $f(i,j) = 2 \cdot \lfloor (i+3j)/2 \rfloor \pmod 6$. It is straightforward to check that f is a $(0,2)$-relaxed 4-$L(2,1)$-labeling of Γ_6. See Figure 9 for an illustration. Thus $\lambda_{2,1}^{0,2}(\Gamma_6) \leq 4$.

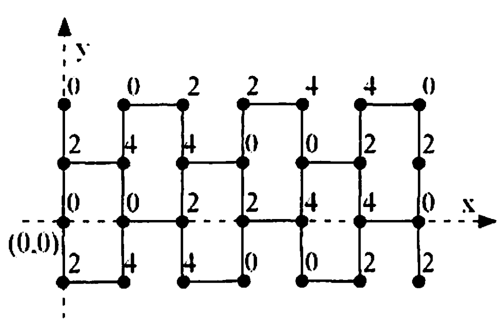

Figure 9: A $(0,2)$-relaxed 4-$L(2,1)$-labeling of Γ_6.

Finally, let $f(i,j) = 2 \cdot \lfloor (1-i+3j)/2 \rfloor \pmod 5$. It is straightforward to check that f is a $(1,0)$-relaxed 4-$L(2,1)$-labeling of Γ_6 (see Figure 10 for an illustration). Therefore $\lambda_{2,1}^{1,0}(\Gamma_6) \leq 4$.

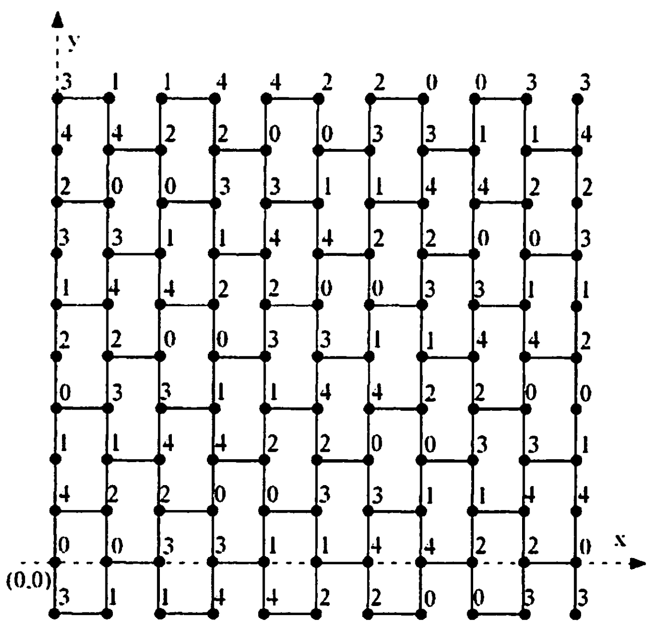

Figure 10: A $(1,0)$-relaxed 4-$L(2,1)$-labeling of Γ_6.

(5) Since $\lambda_{2,1}^{0,0}(\Gamma_6) = \lambda(\Gamma_6) = 5$, we only need to prove that $\lambda_{2,1}^{0,1}(\Gamma_6) \geq 5$. Suppose to the contrary that $\lambda_{2,1}^{0,1}(\Gamma_6) \leq 4$. Let f be a $(0,1)$-relaxed 4-$L(2,1)$-labeling of Γ_6. We claim that the label 2 is not assigned to any vertex of Γ_6 by f. Suppose that $f(u) = 2$ for some vertex u of Γ_6. See Figure 11. Then the three neighbors v, u_1, v_1 must receive the labels 0 or 4. By symmetry, we may assume that $f(v) = 4$, $f(u_1) = f(v_1) = 0$. Then the optional labels for u_2 are $0, 1, 2$. This implies that $f(u_4) \neq 1$. Since $f(u_1) = f(v_1) = 0$, $f(u_4) \neq 0$. So the optional labels for u_4 and u_3 are $2, 3, 4$. It follows that $f(u_3) \neq 3$ and $f(u_4) \neq 3$. If $f(u_3) = 2$, then $f(u_4) = 4$ and $f(z) = 0$ or 4. However, both $f(v_1) = f(u_1) = f(z) = 0$ and $f(v) = f(u_4) = f(z) = 4$ are illegal for f. Therefore $f(u_3) = 4$, implying $f(u_4) = 2$ and $f(u_2) = 0$. Similarly, we have $f(v_3) = 4$, $f(v_4) = 2$ and $f(v_2) = 0$. Since $f(u_2) = f(v_2) = 0$, $f(x_1) \neq 0$ and $f(y_1) \neq 0$. So $f(x_1) = f(y_1) = 4$. Consider labels of vertices around u_4 (resp. v_4), by similar consideration of vertices around u, we get $f(x_2) = 4$, $f(x_3) = 2$ and $f(x_4) = 0$ (resp. $f(y_2) = 4$, $f(y_3) = 2$ and $f(y_4) = 0$). But this

is a contradiction since $f(v) = f(x_2) = f(y_2) = 4$. So the label 2 is not assigned to any vertex of Γ_6 by f.

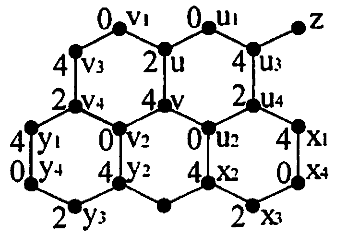

Figure 11: Lower bound about $\lambda_{2,1}^{0,1}(\Gamma_6)$.

By the above discussion, f uses at most four labels 0,1,3 and 4. Since the label 0 must be used by f, there is a vertex, say u, with $f(u) = 0$. Then the three neighbors of u must have labels from $\{3, 4\}$. It follows that the six 2-neighbors of u must have labels from $\{0, 1\}$. Since at most one of its 2-neighbors is labeled 0, at least 5 of them are labeled 1. Let w be a 2-neighbor of u that is at distance greater than 2 from the the 2-neighbor of u with label 0 if there is a 2-neighbor of u with label 0, otherwise let w be any 2-neighbor of u. Then it is clear that w has two 2-neighbors having the same label 1 as itself. Hence $\lambda_{2,1}^{0,1}(\Gamma_6) \geq 5$.

The proof of the theorem is completed. ∎

The results in the main theorem is summarized as the following table.

$\lambda_{2,1}^{s,t}$ t / s	0	1	2	3	4	5	6
0	5	5	4	4	3	3	2
1	4	4	3	3	3	3	2
2	3	3	3	3	3	3	2
3	3	3	2	2	2	2	1

For example, let the interference graph be Γ_6. Suppose the given channel span is 3. Then the maximal pairs (s, t) such that Γ_6 has an (s, t)-relaxed 3-$L(2, 1)$-labeling are $(2, 0)$, $(1, 2)$ and $(0, 4)$. The corresponding (s, t)-relaxed 3-$L(2, 1)$-labelings are presented in the proof of the theorem. Since in practice the total interference level of an assignment of channels can be computed in some way, the best choice may be made after comparing the total interference levels of the (s, t)-relaxed 3-$L(2, 1)$-labelings for

$(s, t) \in \{(2,0), (1,2), (0,4)\}$.

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