A generalization of Stirling, Lah and harmonic numbers with some computational applications

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Abstract

In this article we give a generalization of the multiparameter non-central Stirling numbers of the first and second kinds, Lah numbers and harmonic uumbers. Some new combinatorial identities, new explicit formulas and many relations between different types of Stirling numbers and generalized harmonic numbers are found. Moreover, some interesting special cases of the generalized multiparameter non-central Stirling numbers are deduced. Furthermore, a matrix representation of the results obtained is given and a computer program is written using Maple and executed for calculating GMPNSN-1 and their inverse (GMPNSN-2) along with some of their interesting special cases.

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1. Introduction

Stirling numbers of the first and second kinds were introduced by James Stirling in 1730 (see [18]). Many generalizations and extensions of these numbers have been studied (see [1], [6],[11] [13], and [19]). Moreover, they have many combinatorial, probabilistic and statistical applications (see [4], [7], and [15]). Through this article we use the following notations. The falling and rising factorials are defined by $(t)_n = t(t-1)\cdots(t-n+1), (t)_0 = 1$ and $\langle t\rangle_n = t(t+1)\cdots(t+n-1), \langle t\rangle_0 = 1$, respectively. The generalized factorial of t of order n and scale parameter ν (see [7] and [9]) is $(\nu t)_n = \nu t(\nu t-1)\cdots(\nu t-n+1)(n \in \mathbb{N} := \{1,2,\cdots\}), (\nu t)_0 = 1$ and generally

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$(t;\bar{\alpha})_n = \prod_{j=0}^{n-1}(t-\alpha_j), (t;\bar{\alpha})_0 = 1, (\alpha_k)_l = \prod_{j=0, j\neq k}^{l}(\alpha_k - \alpha_j), k \leq l$, where $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ are real numbers or briefly denoted by $\bar{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$. Comtet [6, 7], defined $s_{\bar{\alpha}}(n,k)$ and $S_{\bar{\alpha}}(n,k)$, the generalized Stirling numbers of the first and the second kinds associated with $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$, by

$$(t,\bar{\alpha})_n = \sum_{k=0}^{n} s_{\bar{\alpha}}(n,k)t^k, \tag{1.1}$$

$$t^n = \sum_{k=0}^{n} S_{\bar{\alpha}}(n,k)(t;\bar{\alpha})_k. \tag{1.2}$$

Koutras [13] defined $s(n,k;a)$ and $S(n,k;a)$, the non-central Stirling numbers of the first and the second kinds by

$$(t)_n = \sum_{k=0}^{n} s(n,k;a)(t-a)^k, \tag{1.3}$$

$$(t-a)^n = \sum_{k=0}^{n} S(n,k;a)(t)_k. \tag{1.4}$$

Charalambides (see [4]) defined the generalized factorial coefficients (or C-numbers) by

$$(\nu t)_n = \sum_{k=0}^{n} C(n,k;\nu)(t)_k, \ (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.5}$$

For $\nu = -1$, we have $L(n,k) = C(n,k;-1)$, hence

$$(-t)_n = \sum_{k=0}^{n} L(n,k)(t)_k, \ (n \in \mathbb{N}_0), \tag{1.6}$$

where $L(n,k)$ are Lah numbers. Since $(-t)_k = (-1)^n \langle t \rangle_n$, we get

$$\langle t \rangle_n = (t+n-1)_n = \sum_{k=0}^{n} |L(n,k)|(t)_k, \ (n \in \mathbb{N}_0), \tag{1.7}$$

where $|L(n,k)| = (-1)^n L(n,k)$ are the signless Lah numbers.
The author (see [9]) derived a generalization of the non-central Stirling numbers of the first and second kinds by

$$(t;\bar{\alpha})_n = \sum_{k=0}^{n} s_a(n,k;\bar{\alpha})(t-a)^k, \tag{1.8}$$

$$(t-a)^n = \sum_{k=0}^{n} S_a(n,k;\bar{\alpha})(t;\bar{\alpha})_k, \tag{1.9}$$

where a, t and α_i $(i = 0, 1, \cdots, n-1)$ are real numbers, and $\bar{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$. Moreover, the author (see [10]) defined the multiparameter non-central Stirling numbers of the first and second kinds by

$$(t)_n = \sum_{k=0}^{n} s(n,k;\bar{\alpha})(t;\bar{\alpha})_k, \tag{1.10}$$

$$(t;\bar{\alpha})_n = \sum_{k=0}^{n} S(n,k;\bar{\alpha})(t)_k. \tag{1.11}$$

Hsu and Shiu [12] defined the generalized Stirling-type pair $\{S^1(n,k), S^2(n,k)\} \equiv \{S(n,k;\alpha,\beta,r), S(n,k;\alpha,\beta,-r)\}$ by

$$(t|\alpha)_n = \sum_{k=0}^{n} S^1(n,k)(t-r|\beta)_k, \tag{1.12}$$

$$(t|\beta)_n = \sum_{k=0}^{n} S^2(n,k)(t+r|\alpha)_k, \tag{1.13}$$

where $n \in \mathbb{N}_0$ and the parameters α, β and r are real or complex numbers, with $(\alpha,\beta,r) \neq (0,0,0)$. These numbers satisfy the recurrence relation

$$S^1(n+1,k) = S^1(n,k-1) + (k\beta - n\alpha + r)S^1(n,k). \tag{1.14}$$

Our goal in this article, Sections 2 and 3, is to derive a generalization of the multiparameter non-central Stirling numbers of the first and second kinds, Lah and harmonic numbers. Recurrence relations, generating functions, explicit formulas and many relations between different types of Stirling numbers and generalized harmonic numbers are given. Moreover, some interesting special cases and new combinatorial identities are deduced. Furthermore, in Section 4, an algorithm is given and a computer program is written using Maple and executed for calculating GMPNSN-1 and their inverse, GMPNSN-2, along with some of their interesting special cases. Finally, a matrix representation of some results obtained is derived.

2. The generalized multiparameter non-central Stirling numbers of the first kind (GMPNSN-1)

Definition 2.1. *We define the generalized multiparameter non-central Stirling numbers of the first kind (briefly denoted by GMPNSN-1) by*

$$(\nu t)_n = \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})(t;\bar{\alpha})_k, s(0,0;\nu,\bar{\alpha}) = 1 \textit{ and } s(n,k;\nu,\bar{\alpha}) = 0 \textit{ if } k < n. \tag{2.1}$$

Theorem 2.1. *The GMPNSN-1 satisfy the recurrence relation*

$$s(n+1,k;\nu,\bar{\alpha}) = \nu s(n,k-1;\nu,\bar{\alpha}) + (\nu\alpha_k - n)s(n,k;\nu,\bar{\alpha}). \tag{2.2}$$

Proof. Since $(\nu t)_{n+1} = (\nu t)_n(\nu t - n) = (\nu t)_n[(\nu t - \nu\alpha_k) + (\nu\alpha_k - n)]$ $= (\nu t)_n \nu(t - \alpha_k) + (\nu t)_n(\nu\alpha_k - n)$, using (2.1) we have

$$\sum_{k=0}^{n+1} s(n+1,k;\nu,\bar{\alpha})(t;\bar{\alpha})_k = \nu \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})(t-\alpha_k)(t;\bar{\alpha})_k$$
$$+ \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})(\nu\alpha_k - n)(t;\bar{\alpha})_k = \nu \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})(t;\bar{\alpha})_{k+1}$$
$$+ \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})(\nu\alpha_k - n)(t;\bar{\alpha})_k = \nu \sum_{k=0}^{n} s(n,k-1;\nu,\bar{\alpha})(t;\bar{\alpha})_k$$
$$+ \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})(\nu\alpha_k - n)(t;\bar{\alpha})_k.$$

Equating the coefficients of $(t;\bar{\alpha})_k$ on both sides yields (2.2). □

Theorem 2.2. *The numbers $s(n,k;\nu,\bar{\alpha})$ have the exponential generating function*

$$\varphi_k(t;\nu,\bar{\alpha}) = \sum_{j=0}^{k} \frac{(1+t)^{\nu\alpha_j}}{(\alpha_j)_k}. \tag{2.3}$$

Proof. Let

$$\varphi_k(t;\nu,\bar{\alpha}) = \sum_{n=k}^{\infty} s(n,k;\nu,\bar{\alpha})\frac{t^n}{n!}. \tag{2.4}$$

Using (2.2), it is easy to show that $s(n,0;\nu,\bar{\alpha}) = (\nu\alpha_0)_n$, hence

$$\varphi_0(t;\nu,\bar{\alpha}) = \sum_{n=0}^{\infty} s(n,0;\nu,\bar{\alpha})\frac{t^n}{n!} = \sum_{n=0}^{\infty} (\nu\alpha_0)_n \frac{t^n}{n!} = (1+t)^{\nu\alpha_0}.$$

Differentiating (2.4) with respect to t we have

$$\varphi_k'(t;\nu,\bar{\alpha}) = \sum_{n=k}^{\infty} s(n,k;\nu,\bar{\alpha})\frac{t^{n-1}}{(n-1)!}.$$

Using (2.2), we get

$$\varphi_k'(t;\nu,\bar{\alpha}) = \sum_{n=k}^{\infty}[\nu s(n-1,k-1;\nu,\bar{\alpha}) + (\nu\alpha_k - (n-1))s(n-1,k;\nu,\bar{\alpha})]\frac{t^{n-1}}{(n-1)!}$$
$$= \nu\sum_{n=k-1}^{\infty} s(n,k-1;\nu,\bar{\alpha})\frac{t^n}{n!} + \nu\alpha_k\sum_{n=k}^{\infty} s(n,k;\nu,\bar{\alpha})\frac{t^n}{n!} - t\sum_{n=k}^{\infty} s(n,k;\nu,\bar{\alpha})\frac{t^{n-1}}{(n-1)!},$$

hence

$$\varphi_k'(t;\nu,\bar{\alpha}) = \nu\varphi_{k-1}(t;\nu,\bar{\alpha}) + \nu\alpha_k\varphi_k(t;\nu,\bar{\alpha}) - t\varphi_k'(t;\nu,\bar{\alpha}),$$
$$\varphi_k'(t;\nu,\bar{\alpha}) - \frac{\nu\alpha_k}{1+t}\varphi_k(t;\nu,\bar{\alpha}) = \frac{\nu}{1+t}\varphi_{k-1}(t;\nu,\bar{\alpha}).$$

Solving this difference-differential equation we get (2.3). □

Theorem 2.3. *The numbers $s(n,k;\nu,\bar{\alpha})$ have the explicit formula*

$$s(n,k;\nu,\bar{\alpha}) = n!\sum_{r=k}^{n}\frac{(-1)^{n-r}\nu^r}{r!}\sum_{l_1+\cdots+l_r=n, l_m\geq 1}\frac{1}{l_1\cdots l_r}\sum_{j=0}^{k}\frac{\alpha_j^r}{(\alpha_j)_k}, m = 1,2,\cdots,r. \tag{2.5}$$

Proof. Using (2.3) we get

$$\varphi_k(t;\nu,\bar{\alpha}) = \sum_{j=0}^{k}\frac{(1+t)^{\nu\alpha_j}}{(\alpha_j)_k} = \sum_{j=0}^{k}\frac{e^{\nu\alpha_j log(1+t)}}{(\alpha_j)_k}$$
$$= \sum_{j=0}^{k}\frac{1}{(\alpha_j)_k}\sum_{r=0}^{\infty}\frac{(\nu\alpha_j)^r}{r!}(\sum_{l=1}^{\infty}(-1)^{l-1}\frac{t^l}{l})^r\sum_{j=0}^{k}\frac{1}{(\alpha_j)_k}\sum_{r=0}^{\infty}\frac{(\nu\alpha_j)^r}{r!}\prod_{i=0}^{r}\sum_{l_i=1}^{\infty}(-1)^{l_i-1}\frac{t^{l_i}}{l_i}$$
$$= \sum_{j=0}^{k}\frac{1}{(\alpha_j)_k}\sum_{r=0}^{\infty}\frac{(\nu\alpha_j)^r}{r!}\sum_{n=r}^{\infty}\sum_{l_1+\cdots+l_r=n}(-1)^{n-r}\frac{t^n}{l_1\cdots l_n}$$
$$= \sum_{n=0}^{\infty}(\sum_{j=0}^{k}\frac{1}{(\alpha_j)_k}\sum_{r=k}^{n}\frac{(\nu\alpha_j)^r}{r!}\sum_{l_1+\cdots+l_r=n}\frac{(-1)^{n-r}}{l_1\cdots l_r})t^n. \tag{2.6}$$

Thus, by virtue of (2.4), we obtain (2.5). □

Let

$$\bar{s}(n,k;\nu,\bar{\alpha}) := \nu^{-k}s(n,k;\nu,\bar{\alpha}). \tag{2.7}$$

Then using (2.3), it is easy to prove the following lemma.

Lemma 2.1. *The numbers $\bar{s}(n,k;\nu,\bar{\alpha})$ satisfy the recurrence relation*

$$\bar{s}(n,k;\nu,\bar{\alpha}) = \bar{s}(n-1,k-1;\nu,\bar{\alpha}) + (\nu\alpha_k - n + 1)\bar{s}(n-1,k;\nu,\bar{\alpha}). \tag{2.8}$$

Since $(\nu t)_n = \sum_{k=0}^{n} s(n,k)(\nu t)^k = \sum_{k=0}^{n} s(n,k)\nu^k t^k$, using (1.2) and (2.1), we get

$$\begin{aligned}\sum_{k=0}^{n} s(n,l;\nu,\bar{\alpha})(t;\bar{\alpha})_l &= \sum_{k=0}^{n} s(n,k)\nu^k \sum_{l=0}^{k} S_{\bar{\alpha}}(k,l)(t;\bar{\alpha})_l \\ &= \sum_{l=0}^{n}\left(\sum_{k=l}^{n} s(n,k)\nu^k S_{\bar{\alpha}}(k,l)\right)(t;\bar{\alpha})_l.\end{aligned} \tag{2.9}$$

Equating the coefficients of $(t;\bar{\alpha})_l$ on both sides gives the identity

$$s(n,l;\nu,\bar{\alpha}) = \sum_{k=l}^{n} s(n,k)\nu^k S_{\bar{\alpha}}(k,l). \tag{2.10}$$

Similarly, from (2.1) by using (1.2), we have

$$\sum_{i=0}^{n} s(n,i(\nu t)^i = \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})\sum_{i=0}^{k} s_{\bar{\alpha}}(k,i)t^i = \sum_{i=0}^{n}\sum_{k=i}^{n} s(n,k;\nu,\bar{\alpha})s_{\bar{\alpha}}(k,i)t^i. \tag{2.11}$$

Equating the coefficients of t^i on both sides we have the identity

$$s(n,i)\nu^i = \sum_{k=i}^{n} s(n,k;\nu,\bar{\alpha})s_{\bar{\alpha}}(k,i). \tag{2.12}$$

Since $(\nu t)_n = \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})(t;\bar{\alpha})$, from (1.5) and (1.11), we have

$$\begin{aligned}\sum_{l=0}^{n} C(n,l;\nu)(t)_l &= \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})\sum_{l=0}^{k} S(k,l;\bar{\alpha})(t)_l \\ &= \sum_{l=0}^{n}\left(\sum_{k=l}^{n} s(n,k;\nu,\bar{\alpha})S(k,l;\bar{\alpha})\right)(t)_l, \textit{ hence}\end{aligned}$$

$$C(n,l;\nu) = \sum_{k=l}^{n} s(n,k;\nu,\bar{\alpha})S(k,l;\bar{\alpha}). \tag{2.13}$$

On the other hand (see [4, p. 304 and p. 306]) $C(n,l;\nu) = \frac{1}{l!}\sum_{r=0}^{l}(-1)^{l-r}\binom{l}{r}(\nu r)_n$, whence we have the following combinatorial identities

$$\sum_{k=l}^{n} s(n,k;\nu,\bar{\alpha})S(k,l;\bar{\alpha}) = \frac{1}{l!}\sum_{r=0}^{l}(-1)^{l-r}\binom{l}{r}(\nu r)_n \quad , \tag{2.14}$$

$$\sum_{k=l}^{n} s(n,k;\nu,\bar{\alpha})S(k,l;\bar{\alpha}) = \sum_{r=l}^{n} s(n,r)S_{\bar{\alpha}}(r,l)\nu^r = \sum_{r=l}^{n} s(n,r;\nu,\bar{0})S_{\bar{\alpha}}(r,l). \quad (2.15)$$

Moreover we will handle the following interesting special cases of (2.1):
i) If $\alpha_i = 0$ $(n = 0,1,\cdots)$ we have $(\nu t)_n = \sum_{k=0}^{n} s(n,k;\nu,\bar{0})t^k$,

$$\sum_{k=0}^{n} s(n,k)(\nu t)^k = \sum_{k=0}^{n} s(n,k;\nu,\bar{0})t^k.$$

Equating the coefficients of t^k on both sides we get

$$s(n,k;\nu,\bar{0}) = s(n,k)\nu^k, \quad (2.16)$$

where $s(n,k)$ are the usual Stirling numbers of the first kind defined by $t^n = \sum_{k=0}^{n} s(n,k)t^k$, where $s(0,0) = s(n,n) = 1$ and $s(n,k) = 0$ for $k > n$.
ii) If $\alpha_i = i$ $(n = 0,1,\cdots,n-1)$, then

$$(\nu t)_n = \sum_{k=0}^{n} s(n,k;\nu,\bar{i})(t)_k = \sum_{k=0}^{n} C(n,k;\nu)(t)_k.$$

So we have

$$s(n,k;\nu,\bar{i}) = C(n,k;\nu). \quad (2.17)$$

iii) If $\alpha_i = a$ $(n = 0,1,\cdots,n-1)$, then we have

$$(\nu t)_n = \sum_{k=0}^{n} s(n,k;\nu,\bar{a})(t-a)^k, \quad (2.18)$$

where $\bar{a}, \bar{i}$ and $\bar{0}$ are defined by $\bar{a} = (a,a,\cdots,a), \bar{i} = (0,1,\cdots,n-1)$ and $\bar{0} = (0,0,\cdots,0)$, respectively.
We call $s(n,k;\nu,\bar{a})$ the generalized non-central Stirling numbers of the first kind.
Notice that setting $\nu = 1$ in (2.18), we find $s(n,k;1,\bar{a}) = s(n,k,a)$.
From (1.3) we have $(\nu t)_n = \sum_{k=0}^{n} s(n,k;a)(\nu t - a)^k = \sum_{k=0}^{n} s(n,k;\nu a)(\nu t - \nu a)^k$.
Using (2.18) gives $\sum_{k=0}^{n} s(n,k;\nu,\bar{a})(t-a)^k = \sum_{k=0}^{n} s(n,k;\nu a)\nu^k(t-a)^k$.
We, therefore, have

$$s(n,k;\nu,\bar{a}) = s(n,k;\nu a)\nu^k. \quad (2.19)$$

From (1.3) and (2.18), $(\nu t)_n = \sum_{k=0}^{n} s(n,k;a)(\nu t - a)^k$ and so $\sum_{k=0}^{n} s(n,k;\nu,\bar{a})(t-a)^k = \sum_{k=0}^{n} s(n,k;a)\sum_{i=0}^{k}\binom{k}{i}(-a)^{k-i}(\nu t)^i$. Then we get

$$\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i}(-a)^{k-i}s(n,k;\nu,\bar{a})\right)t^i = \sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i}(-a)^{k-i}s(n,k;a)\right)(\nu t)^i.$$

Hence we have the identity

$$\nu^i \sum_{k=i}^{n} \binom{k}{i} (-a)^{k-i} s(n,k,a) = \sum_{k=i}^{n} \binom{k}{i} (-a)^{k-i} s(n,k;\nu,\bar{a}). \tag{2.20}$$

Moreover, $(\nu t)_n = \sum_{k=0}^{n} s(n,k;\nu,\bar{a})(t-a)^k = \sum_{k=0}^{n} s(n,k;\nu,\bar{a}) \sum_{i=0}^{k} \binom{k}{i} (-a)^{k-i} t^i$, whence

$$\sum_{i=0}^{n} s(n,i)(\nu t)^i = \sum_{i=0}^{n} \left(\sum_{k=i}^{n} \binom{k}{i} (-a)^{k-i} s(n,k;\nu,\bar{a}) \right) t^i.$$

Equating the coefficients of t^i yields

$$\nu^i s(n,i) = \sum_{k=i}^{n} \binom{k}{i} (-a)^{k-i} s(n,k;\nu,\bar{a}). \tag{2.21}$$

Therefore, from (2.20) and (2.21) we get

$$s(n,i) = \sum_{k=i}^{n} \binom{k}{i} (-a)^{k-i} s(n,k;a). \tag{2.22}$$

Furthermore, we consider the following two special cases
a) If $\nu = 1$, the GMPNSN-1 reduces to $s(n,k;\bar{a})$ the multiparameter non-central Stirling numbers of the first kind (see [10]). Hence we have

$$s(n,k;1,\bar{\alpha}) = s(n,k;\bar{\alpha}). \tag{2.23}$$

Thus, we have the following new explicit formula for the multiparameter non-central Stirling numbers of the first kind, see ([2, Theorem 3.2]).

Theorem 2.4. *The following identity holds true*

$$s(n,k;\bar{\alpha}) = \sum_{\substack{\sigma_n = k, \\ i_j \in \{0,1\}}} \binom{i_1 + \alpha_{i_1}}{1 - i_1} \binom{i_2 + \alpha_{i_1+i_2} - 1}{1 - i_2} \cdots \binom{i_n + \alpha_{i_1+\cdots+i_n} - n + 1}{1 - i_n}, \tag{2.24}$$

where $\sigma_n := i_0 + i_1 + \cdots + i_n$ *with* $i_0 = 0$.

b) If $\nu = -1$, then from (2.1) we get

$$(-t)_n = \sum_{k=0}^{n} s(n,k;-1,\bar{\alpha})(t;\bar{\alpha})_k = \sum_{k=0}^{n} L_1(n,k;\bar{\alpha})(t;\bar{\alpha})_k. \tag{2.25}$$

We call $L_1(n,k;\bar{\alpha}) = s(n,k;-1,\bar{\alpha})$ the generalized Lah numbers of the first kind associated with the sequence $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$.

Corollary 2.1.

$$L_1(n,k;\bar{\alpha}) = (-1)^k \sum_{\substack{\sigma_n=k,\\ i_j\in\{0,1\}}} \binom{i_1-\alpha_{i_1}}{1-i_1}\binom{i_2-\alpha_{i_1+i_2}-1}{1-i_2}\cdots\binom{i_n-\alpha_{i_1+\cdots+i_n}-n+1}{1-i_n}. \tag{2.26}$$

Proof. The proof follows from (2.24), using (2.7) and (2.25), by replacing α_i by $-\alpha_i$ $(i=0,1,\cdots,n-1)$. □

Since $(-t)_n = (-1)^n(t+n-1)_n = (-1)^n\langle t\rangle_n$, we get

$$(t+n-1) = \langle t\rangle_n = \sum_{k=0}^{n} |L_1(n,k;\bar{\alpha})|(t;\bar{\alpha}), \tag{2.27}$$

where $|L_1(n,k;\bar{\alpha})| = (-1)^n L_1(n,k;\bar{\alpha})$ are the signless generalized Lah numbers of the first kind associated with the sequence $\alpha_0,\alpha_1,\cdots,\alpha_{n-1}$.
Moreover, this gives us the following special cases of (2.25).
i) If $\alpha_i = 0$ $(i=0,1,\cdots,n-1)$, then we get

$$(-t)_n = \sum_{k=0}^{n} s(n,k;-1,\bar{0})t^k = \sum_{k=0}^{n} L_1(n,k;\bar{0}) = \sum_{k=0}^{n} s(n,k)(-t)^k.$$

Hence we have

$$s(n,k;-1,\bar{0}) = L_1(n,k;\bar{0}) = (-1)^k s(n,k). \tag{2.28}$$

ii) If $\alpha_i = i$ $(i=0,1,\cdots,n-1)$,

$$(-t)_n = \sum_{k=0}^{n} s(n,k;-1,\bar{i})(t)_k \sum_{k=0}^{n} L_1(n,k;\bar{i})(t)_k = \sum_{k=0}^{n} L(n,k)(t)_k.$$

Hence we have

$$s(n,k;-1,\bar{i}) = L_1(n,k;\bar{i}) = C(n,k;-1) = L(n,k). \tag{2.29}$$

Setting $\alpha_i = i$ $(i=0,1,\cdots,n-1)$ in (2.25) gives the following new explicit formula for $L(n,k)$ (see [3]):

$$L(n,k) = (-1)^k \sum_{\substack{\sigma_n=k,\\ i_j\in\{0,1\}}} \binom{-1-i_1}{1-i_1}\binom{-2-(i_1+i_2)}{1-i_2}\cdots\binom{-n+1-(i_1+i_2\cdots+i_n)}{1-i_n}, \tag{2.30}$$

iii) If $\alpha_i = a$ $(i = 0, 1, \cdots, n-1)$,

$$(-t)_n = \sum_{k=0}^{n} s(n,k;-1,\bar{a})(t-a)^k = \sum_{k=0}^{n} L_1(n,k;\bar{a})(t-a)^k$$

$$= \sum_{k=0}^{n} s(n,k,a)(-t-a)^k = \sum_{k=0}^{n} s(n,k;a)(-1)^k(t+a)^k = \sum_{k=0}^{n} s(n,k;-a)(-1)^k(t-a)^k.$$

Hence we have

$$s(n,k;-1,\bar{a}) = L_1(n,k;\bar{a}) = (-1)^k s(n,k;-a). \tag{2.31}$$

Since $(-t)_n = \sum_{k=0}^{n} s(n,k;-1,\bar{a})(t-a)^k = \sum_{k=0}^{n} s(n,k;-1,\bar{a}) \sum_{l=0}^{k} (-a)^{k-l} \binom{k}{l} t^l$, we find

$$\sum_{l=0}^{n} s(n,l)(-t)^l = \sum_{l=0}^{n} (\sum_{k=l}^{n} \binom{k}{l} (-a)^{k-l} s(n,k;-1,\bar{a}) t^l, \tag{2.32}$$

which yields

$$(-1)^l s(n,l) = L_1(n,l;\bar{0}) = \sum_{k=l}^{n} \binom{k}{l} (-a)^{k-l} s(n,k;-1,\bar{a}). \tag{2.33}$$

Also, since

$$(-t)_n = \sum_{k=0}^{n} L(n,k)(t)_k = \sum_{k=0}^{n} L(n,k) \sum_{l=0}^{k} s(k,l) t^l = \sum_{l=0}^{n} \left(\sum_{k=l}^{n} L(n,k) s(k,l) \right) t^l,$$

from (2.32), we obtain

$$\sum_{l=0}^{n} (\sum_{k=l}^{n} L(n,k) s(k,l)) t^l = \sum_{l=0}^{n} (\sum_{k=l}^{n} \binom{k}{l} (-a)^{k-l} s(n,k;-1,\bar{a})) t^l.$$

Hence we have the identity

$$\sum_{k=l}^{n} L(n,k) s(k,l) = \sum_{k=l}^{n} \binom{k}{l} (-a)^{k-l} s(n,k;-1,\bar{a}). \tag{2.34}$$

Moreover, from (2.28), (2.33) and (2.34) we have

$$\sum_{k=l}^{n} L(n,k) s(k,l) = (-1)^l s(n,l) = L_1(n,l;\bar{0}) = s(n,l;-1,\bar{0}). \tag{2.35}$$

3. The generalized multiparameter non-central Stirling numbers of the second kind (GMPNSN-2)

Definition 3.1. *We define the generalized multiparameter non-central Stirling unmbers of the second kind (briefly denoted by GMPNSN-2) by*

$$(t;\bar{\alpha})_n = \sum_{k=0}^{n} S(n,k;\nu,\bar{\alpha})(\nu t)_k, S(0,0,\nu,\bar{\alpha}) = 1, S(n,k,\nu,\bar{\alpha}) = 0 if k > n. \tag{3.1}$$

Theorem 3.1. *The GMPNSN-2 satisfies the recurrence relation*

$$S(n,k;\nu,\bar{\alpha}) = \frac{1}{\nu}S(n-1,k-1;\nu,\bar{\alpha}) + (\frac{k}{\nu} - \alpha_{n-1})S(n-1,k;\nu,\bar{\alpha}). \qquad (3.2)$$

Proof. Since $(t;\bar{\alpha})_n = (t;\bar{\alpha})_{n-1}(t-\alpha_{n-1}) = (t;\bar{\alpha})_{n-1}\left[\frac{1}{\nu}(\nu t - k) + (\frac{k}{\nu} - \alpha_{n-1})\right]$.
Using (3.1) we get

$$\sum_{k=0}^{n} S(n,k;\nu,\bar{\alpha})(\nu t)_k = \frac{1}{\nu}\sum_{k=0}^{n-1} S(n-1,k;\nu,\bar{\alpha})(\nu t)_{k+1}$$
$$+\sum_{k=0}^{n-1} S(n-1,k;\nu,\bar{\alpha})(\frac{k}{\nu} - \alpha_{n-1})(\nu t)_k = \frac{1}{\nu}\sum_{k=1}^{n} S(n-1,k-1;\nu,\bar{\alpha})(\nu t)_k$$
$$+\sum_{k=0}^{n-1} S(n-1,k;\nu,\bar{\alpha})(\frac{k}{\nu} - \alpha_{n-1})(\nu t)_k.$$

Comparing the coefficients of $(\nu t)_k$ we get (3.2) □

Set

$$\tilde{S}(n,k;\nu,\bar{\alpha}) := \nu^n S(n,k;\nu,\bar{\alpha}). \qquad (3.3)$$

It is easy to prove the following lemma.

Lemma 3.1. *The numbers $\tilde{S}(n,k;\nu,\bar{\alpha})$ satisfy the recurrence relation:*

$$\tilde{S}(n,k;\nu,\bar{\alpha}) = \tilde{S}(n-1,k-1;\nu,\bar{\alpha}) + (k - \nu\alpha_{n-1})\tilde{S}(n-1,k;\nu,\bar{\alpha}). \qquad (3.4)$$

The following special cases give us many types of Stirling numbers as a consequence of (3.1).
i) If $\alpha_i = 0$ $(i = 0,1,\cdots,n-1)$,

$$t^n = \sum_{k=0}^{n} S(n,k;\nu,\bar{0})(\nu t)_k. \qquad (3.5)$$

We thus have $(\nu t)^n = \nu^n \sum_{k=0}^{n} S(n,k;\nu,\bar{0})(\nu t)_k = \sum_{k=0}^{n} S(n,k)(\nu t)_k$.
Hence we get

$$\nu^n S(n,k;\nu,\bar{0}) = S(n,k), \qquad (3.6)$$

where $S(n,k)$ are the usual Stirling numbers of the second kind defined by $t^n = \sum_{k=0}^{n} S(n,k)(t)_k$, where $S(0,0) = S(n,n) = 1$ and $S(n,k) = 0$ for $k > n$.
From (3.5) we have

$$t^n = \sum_{k=0}^{n} S(n,k;\nu,\bar{0}) \sum_{l=0}^{k} s(k,l)\nu^l t^l = \sum_{l=0}^{n}\left(\sum_{k=l}^{n} \nu^l S(n,k;\nu,\bar{0})s(k,l)\right)t^l.$$

Hence we get

$$\sum_{k=l}^{n} v^l S(n,k;v,\bar{0})s(k,l) = \sum_{k=l}^{n} S(n,k;v,\bar{0})s(k,l;v,\bar{0}) = \delta_{n,l}, \tag{3.7}$$

where $\delta_{n,l}$ is the Kronecker delta.

ii) If $\alpha_i = i$ $(i = 0, 1, \cdots, n-1)$, then $(t)_n = \sum_{k=0}^{n} S(n,k;v,\bar{i})(vt)_k$. Replacing t by $v^{-1}t$, we have

$(v^{-1}t)_n = \sum_{k=0}^{n} S(n,k;v,\bar{i})(t)_k$. Using (1.5) we get $(v^{-1}t)_n = \sum_{k=0}^{n} C(n,k;v^{-1})(t)_k = \sum_{k=0}^{n} S(n,k;v,\bar{i})(t)_k$. Hence we get

$$S(n,k,v,\bar{i}) = C(n,k;v^{-1}). \tag{3.8}$$

Also, using (1.5), we find

$$(t)_n = \sum_{k=0}^{n} S(n,k;v,\bar{i}) \sum_{l=0}^{k} C(k,l;v)(t)_l = \sum_{l=0}^{n} \left(\sum_{k=l}^{n} S(n,k;v,\bar{i})C(k,l;v) \right) (t)_l.$$

Hence we obtain

$$\sum_{k=l}^{n} S(n,k;v,\bar{i})C(k,l;v) = \delta_{n,l}. \tag{3.9}$$

iii) If $\alpha_i = a$ $(i = 0, 1, \cdots, n-1)$, then we have

$$(t-a)^n = \sum_{k=0}^{n} S(n,k,v,\bar{a})(vt)_k. \tag{3.10}$$

We call $S(n,k;v,\bar{a})$ the generalized non-central Stirling numbers of the second kind.

It is worth noting that if $v = 1$ in (3.10), then $S(n,k;1,\bar{a}) = S(n,k;a)$.

From (1.4) we have $v^n(t-a)^n = \sum_{k=0}^{n} S(n,k;va)(vt)_k$. Making use of (3.10) we get $v^n \sum_{k=0}^{n} S(n,k;v,\bar{a})(vt)_k = \sum_{k=0}^{n} S(n,k;va)(vt)_k$.

Hence we get

$$v^n S(n,k;v,\bar{a}) = S(n,k;va). \tag{3.11}$$

From (3.10) we have

$$\sum_{i=0}^{n} \binom{n}{i} (-a)^{n-i} t^i = \sum_{k=0}^{n} S(n,k;v,\bar{a}) \sum_{i=0}^{k} s(k,i)(vt)^i = \sum_{i=0}^{n} \left(\sum_{k=i}^{n} S(n,k;v,\bar{a})s(k,i) \right) v^i t^i.$$

Hence we get the combinatorial identity

$$\sum_{k=i}^{n} S(n,k;v,\bar{a})s(k,i) = v^{-i} \binom{n}{i} (-a)^{n-i}. \tag{3.12}$$

Using (3.1) we have $(t;\bar{\alpha})_n = \sum_{k=0}^{n} S(n,k;v,\bar{\alpha}) \sum_{l=0}^{k} s(k,l)(vt)^l$, then from (1.1) $\sum_{l=0}^{n} s_{\bar{\alpha}}(n,l)t^l = \sum_{l=0}^{n} \left(\sum_{k=l}^{n} S(n,k;v,\bar{\alpha})s(k,l) \right) v^l t^l$. Hence we get an identity

$$s_{\bar{\alpha}}(n,l) = \sum_{k=l}^{n} S(n,k;v,\bar{\alpha})s(k,l)v^l. \tag{3.13}$$

This gives a relation between Comtet numbers, Stirling umbers and GMPNSN-2
Moreover, since $s_{\bar{\alpha}}(n,l) = \sum_{k=l}^{n} S(n,k;\bar{\alpha})s(k,l)$, see ([10, Eq. (3.3)]), we have the combinatorial identity

$$\sum_{k=l}^{n} S(n,k;\bar{\alpha})s(k,l) = \sum_{k=l}^{n} S(n,k;\nu,\bar{\alpha})s(k,l)\nu^{l}. \tag{3.14}$$

Since $(t;\bar{\alpha})_n = \sum_{k=0}^{n} S(n,k;\nu,\bar{\alpha})(\nu t)_k$, using (1.5) and (1.11), we have

$$\sum_{l=0}^{n} S(n.l;\bar{\alpha})(t)_l = \sum_{k=0}^{n} S(n,k;\nu,\bar{\alpha}) \sum_{l=0}^{k} C(k,l;\nu)(t)_l = \sum_{l=0}^{n} \left(\sum_{k=l}^{n} S(n,k;\nu,\bar{\alpha})C(k,l;\nu) \right) (t)_l. \tag{3.15}$$

Hence we obtain a new identity

$$S(n,l;\bar{\alpha}) = \sum_{k=l}^{n} S(n,k;\nu,\bar{\alpha})C(k,l;\nu). \tag{3.16}$$

This gives a connection between the multiparameter non-central Stirling numbers, the generalized multiparameter non-central Stirling numbers and C-numbers.
From (1.9) we have $(t-a)^n = \sum_{k=0}^{n} S_a(n,k;\bar{\alpha})(t;\bar{\alpha})_k$. Then using (1.1) and (3.10),

$$\sum_{k=0}^{n} S(n,k;\nu,\bar{a}) \sum_{l=0}^{k} s(k,l)(\nu t)^l = \sum_{k=l}^{n} S_a(n,k;\bar{a}) \sum_{l=0}^{k} s_{\bar{\alpha}}(k,l;\nu)t^l,$$

$$\sum_{l=0}^{n} (\sum_{k=l}^{n} S(n,k;\nu,\bar{a})s(k,l))(\nu t)^l = \sum_{l=0}^{n} (\sum_{k=l}^{n} S_a(n,k;\bar{a})s_{\bar{\alpha}}(k,l;\nu))t^l.$$

Hence we have the identity

$$\sum_{k=l}^{n} S(n,k;\nu,\bar{a})s(k,l)\nu^l = \sum_{k=l}^{n} S_a(n,k;\bar{a})s_{\bar{\alpha}}(k,l). \tag{3.17}$$

We derive the following two special cases of (3.1):
a) If $\nu = 1$, then $(t;\bar{\alpha})_n = \sum_{k=0}^{n} S(n,k;1,\bar{\alpha})(t)_k = \sum_{k=0}^{n} S(n,k;\bar{\alpha})(t)_k$. Hence we get

$$S(n,k;1,\bar{\alpha}) = S(n,k;\bar{\alpha}), \tag{3.18}$$

and so, GMPNSN-2 reduces to $S(n,k;\bar{\alpha})$, the multiparamater non-central Stirling numbers of the second kind (see [10]).
Thus we have the following theorem (see [2, Theorem 3.1]).

Theorem 3.2. *The numbers $S(n,k;\bar{\alpha})$ have the explicit formula*

$$S(n,k;\bar{\alpha}) = \sum_{\substack{\sigma_{n-1}=n-k,\\ i_j\in\{0,1\}}} \binom{-\alpha_0}{i_0}\binom{-\alpha_1+1-i_0}{i_1}\cdots\binom{-\alpha_{n-1}+n-1-(i_0+\cdots+i_{n-2})}{i_{n-1}}. \tag{3.19}$$

where $\sigma_n := i_0 + i_1 + \cdots + i_n$.

b) If $v = -1$, then we have

$$(t;\bar{\alpha})_n = \sum_{k=0}^{n} S(n,k;-1,\bar{\alpha})(-t)_k = \sum_{k=0}^{n} L_2(n,k;\bar{\alpha})(-t)_k. \tag{3.20}$$

We call $L_2(n,k;\bar{\alpha}) = S(n,k;-1,\bar{\alpha})$ the generalized Lah numbers of the second kind associated with the sequence $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. Since $(-t)_n = (-1)^n(t+n-1)_n = (-1)^n\langle t\rangle_n$, we get

$$(t;\bar{\alpha})_n = \sum_{k=0}^{n} L_2(n,k;\bar{\alpha})(-1)^k\langle t\rangle_k = \sum_{k=0}^{n} |L_2(n,k;\bar{\alpha})|\,\langle t\rangle_k, \tag{3.21}$$

where $|L_2(n,k;\bar{\alpha})| = (-1)^k L_2(n,k;\bar{\alpha})$ are signless generalized Lah numbers of the second kind.

Corollary 3.1. *The generalized Lah numbers of the second kind $L_2(n,k;\bar{\alpha})$ have the following explicit formula*

$$L_2(n,k;\bar{\alpha}) = (-1)^n \sum_{\substack{\sigma_{n-1}=n-k,\\ i_j\in\{0,1\}}} \binom{\alpha_0}{i_0}\binom{\alpha_1+1-i_0}{i_1}\cdots\binom{\alpha_{n-1}+n-1-(i_0+\cdots+i_{n-2})}{i_{n-1}}. \tag{3.22}$$

Proof. The proof follows from (3.19), using (3.3) and (3.20), by replacing α_i by $-\alpha_i$ $(i = 0,1,\cdots,n-1)$. □

This gives us the following special cases of (3.20):
i) If $\alpha_i = 0$ $(i = 0,1,\cdots,n-1)$, then $t^n = \sum_{k=0}^{n} S(n,k;-1,\bar{0})(-t)_k$, replacing t by $-t$ we get $(-1)^n t^n = \sum_{k=0}^{n} S(n,k;-1,\bar{0})(t)_k = (-1)^n \sum_{k=0}^{n} S(n,k)(t)_k$.
Hence we have

$$S(n,k;-1,\bar{0}) = L_2(n,k;\bar{0}) = (-1)^n S(n,k). \tag{3.23}$$

ii) If $\alpha_i = i$ $(i = 0,1,\cdots,n-1)$, then $(t)_n = \sum_{k=0}^{n} S(n,k;-1,\bar{i})(-t)_k$, replacing t by $-t$ we get
$(-t)_n = \sum_{k=0}^{n} S(n,k;-1,\bar{i})(t)_k = \sum_{k=0}^{n} L(n,k)(t)_k$.
Hence we have

$$S(n,k;-1,\bar{i}) = L_2(n,k;\bar{i}) = L(n,k). \tag{3.24}$$

Thus, setting $\alpha_i = i$ $(i = 0,1,\cdots,n-1)$ in (3.22) gives (see [3])

$$L(n,k) = (-1)^n \sum_{\substack{\sigma_{n-1}=n-k,\\ i_j\in\{0,1\}}} \binom{2-i_0}{i_1}\binom{4-(i_0+i_1)}{i_2}\cdots\binom{2(n-1)-(i_0+\cdots+i_{n-2})}{i_{n-1}}. \tag{3.25}$$

iii) If $\alpha_i = a$ $(i = 0,1,\cdots,n-1)$, then $(t-a)^n = \sum_{k=0}^{n} S(n,k;-1,\bar{a})(-t)_k$, replacing t by $-t$, we get

$(-1)^n(t+a)^n = \sum_{k=0}^{n} S(n,k;-1,\bar{a})(t)_k = (-1)^n \sum_{k=0}^{n} S(n,k;-a)(t)_k$.
Hence we get

$$S(n,k;-1,\bar{a}) = L_2(n,k;\bar{a}) = (-1)^n S(n,k;-a). \tag{3.26}$$

Since

$$(\nu t)_n = \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha})(t;\bar{\alpha})_k = \sum_{k=0}^{n} s(n,k;\nu,\bar{\alpha}) \sum_{l=0}^{k} S(k,l;\nu,\bar{\alpha})(\nu t)_l$$

$= \sum_{l=0}^{n} \left(\sum_{k=l}^{n} s(n,k;\nu,\bar{\alpha})S(k,l;\nu,\bar{\alpha})\right)(\nu t)_l,$

we have

$$\sum_{k=l}^{n} s(n,k;\nu,\bar{\alpha})S(k,l;\nu,\bar{\alpha}) = \delta_{n,l}. \tag{3.27}$$

This gives the orthogonality relation of GMPNSN-1 and GMPNSN-2.
Since

$$(\nu_1\nu_2^{-1}t)_n = \sum_{k=0}^{n} s(n,k;\nu_1,\bar{\alpha})(\nu_2^{-1}t;\bar{\alpha})_k = \sum_{k=0}^{n} s(n,k;\nu_1,\bar{\alpha}) \sum_{l=0}^{k} S(k,l;\nu_2,\bar{\alpha})(t)_l$$
$$= \sum_{l=0}^{n} \left(\sum_{k=l}^{n} s(n,k;\nu_1,\bar{\alpha})S(k,l;\nu_2,\bar{\alpha})\right)(t)_l.$$

From (1.5) we have $(\nu_1\nu_2^{-1}t)_n = \sum_{l=0}^{n}\left(\sum_{k=l}^{n} C(n,k;\nu_1)C(k,l;\nu_2^{-1})\right)(t)_l$. Hence we get a new identity

$$\sum_{k=l}^{n} s(n,k;\nu_1,\bar{\alpha})S(k,l;\nu_2,\bar{\alpha}) = \sum_{k=l}^{n} C(n,k;\nu_1)C(k,l;\nu_2^{-1}) = C(n,l;\nu_1\nu_2^{-1}). \tag{3.28}$$

It is worth noting that if $\nu_1 = \nu_2 = \nu$, we get

$$\sum_{k=l}^{n} s(n,k;\nu,\bar{\alpha})S(k,l;\nu,\bar{\alpha}) = C(n,l;1) = \delta_{n,l}. \tag{3.29}$$

Finally we find some relations between the generalized multiparameter non-central Stirling numbers and usual Stirling numbers, q-Stirling numbers p,q-Stirling numbers and a,d-progressive p,q-Stirling numbers (see [14]).
Putting $\alpha_i = q^i - 1 (i = 0,1,\cdots,n-1)$ in (2.10) and (3.13), respectively (see [14, Tables 1 and 9]) we get

$$s(n,l;\nu,\bar{q^i}-1) = \sum_{k=l}^{n} (q-1)^{k-l} s(n,k)\nu^k S_q[k,l] = \sum_{k=l}^{n} (q-1)^{k-l} S_q[k,l] s(n,k;\nu,\bar{0}), \tag{3.30}$$

$$(q^i-1)^{n-l} c_q[n,l] = \sum_{k=l}^{n} S(n,k;\nu,\bar{q^i}-1)s(k,l)\nu^l = \sum_{k=l}^{n} S(n,k;\nu,\bar{q^i}-1)s(n,k;\nu,\bar{0}), \tag{3.31}$$

where $S_q[k,l]$ and $c_q[k,l]$ are the usual q-Striling numbers and signless q-Stirling numbers of the second and first kind, respectively.
Similarly, putting $\alpha_i = [i]_{p,q}$ $(i = 0,1,\cdots,n-1)$ in (2.10) and (3.13), respectively (see [14,

Tables 1 and 8]), we get

$$s(n,k;\nu,[i]_{p,q}) = \sum_{k=l}^{n} s(k,l)\nu^l S_{p,q}[k,l] = \sum_{k=l}^{n} s(n,k;\nu,\bar{0}) S_{p,q}[k,l], \tag{3.32}$$

$$c_{p,q}[n,l] = \sum_{k=l}^{n} S(n,k;\nu,[i]_{p,q}) s(k,l)\nu^l = \sum_{k=l}^{n} S(n,k;\nu,[i]_{p,q}) s(n,k;\nu,\bar{0}), \tag{3.33}$$

where $S_{p,q}[k,l]$ and $c_{p,q}[n,l]$ are p,q-Stirling numbers of the second and first kind, respectively.
Moreover, putting $\alpha_i = [a+id]_{p,q}$ $(i = 0,1,\cdots,n-1)$ in (2.10) and (3.13), respectively (see [14, Tables 1 and 13]), we obtain

$$S(n,k;\nu,[a+id]_{p,q}) = \sum_{k=l}^{n} s(k,l)\nu^l S_{a,d}^{p,q}[k,l] = \sum_{k=l}^{n} s(n,k;\nu,\bar{0}) S_{a,d}^{p,q}[k,l], \tag{3.34}$$

$$c_{a,d}^{p,q}[n,l] = \sum_{k=l}^{n} S(n,k;\nu,[a+id]_{p,q}) s(k,l)\nu^l = \sum_{k=l}^{n} S(n,k;\nu,[a+id]_{p,q}) s(n,k;\nu,\bar{0}), \tag{3.35}$$

where $S_{a,d}^{p,q}[k,l]$ and $c_{a,d}^{p,q}[n,l]$ are the a,d-progressive p,q-Stirling numbers of the second and first kind, respectively.

Remark 1. *From* (1.10), $(t)_n = \sum_{k=0}^{n} s(n,k;\nu\bar{\alpha})(t;\nu\bar{\alpha})_k = \sum_{k=0}^{n} s(n,k;\nu\bar{\alpha})\nu^k(\frac{t}{\nu};\bar{\alpha})_k$, *setting* $\frac{t}{\nu} = x$ *gives* $(\nu x)_n = \sum_{k=0}^{n} s(n,k;\nu\bar{\alpha})(x;\bar{\alpha})_k$. *Then by virtue of* (2.1) *we get*

$$s(n,k;\nu,\bar{\alpha}) = s(n,k;\nu\bar{\alpha})\nu^k, \tag{3.36}$$

where $\nu\bar{\alpha} := (\nu\alpha_0, \nu\alpha_1, \cdots, \nu\alpha_{n-1})$.

Corollary 3.2. *The numbers* $s(n,k;\nu,\bar{\alpha})$ *have the explicit formula*

$$s(n,k;\nu,\bar{\alpha}) = \nu^k \sum_{\substack{\sigma_n=k,\\ i_j\in\{0,1\}}} \binom{i_1+\nu\alpha_{i_1}}{1-i_1}\binom{i_2+\nu\alpha_{i_1+i_2}-1}{1-i_2}\cdots\binom{i_n+\nu\alpha_{i_1+\cdots+i_n}-n+1}{1-i_n}. \tag{3.37}$$

Proof. The proof follows using (3.36) and replacing α_j by $\nu\alpha_j$ ($j = 0,1,\cdots,n-1$) in (2.24). □

From (3.37) and (2.5) we obtain a new interesting identity

$$n!\sum_{r=k}^{n}(-1)^{n-r}\frac{\nu^{r-k}}{r!}\sum_{l_1+\cdots+l_r=n,l_m\geq 1}\frac{1}{l_1\cdots l_r}\sum_{j=0}^{k}\frac{\alpha_j^r}{(\alpha_j)_k} = \sum_{\sigma_n=k,i_j\in\{0,1\}}\binom{i_1+\nu\alpha_{i_1}}{1-i_1}$$
$$\binom{i_2+\nu\alpha_{i_1+i_2}-1}{1-i_2}\cdots\binom{i_n+\nu\alpha_{i_1+\cdots+i_n}-n+1}{1-i_n}. \tag{3.38}$$

Remark 2. *Using* (1.11) *and* (3.1) *we get*

$$S(n,k;\nu,\bar{\alpha}) = \nu^{-n} S(n,k;\nu\bar{\alpha}). \tag{3.39}$$

Similarly using (3.39) and (3.19) we can prove the following corollary.

Corollary 3.3. *The numbers $S(n,k;\nu,\bar{\alpha})$ have an explicit formula*

$$S(n,k;\nu,\bar{\alpha}) = \nu^{-n} \sum_{\sigma_{n-1}=n-k, i_j\in\{0,1\}} \binom{-\nu\alpha_{i_0}}{i_0}\binom{-\nu\alpha_{i_1}+1-i_0}{i_1}\cdots \binom{-\nu\alpha_{n-1}+n-1-(i_0+\cdots+i_{n-2})}{i_{n-1}}. \tag{3.40}$$

where $j = 0,1,\cdots,n-1$, and $\sigma_{n-1} := i_0+i_1+\cdots+i_{n-1}$.

Remark 3. *For the special case $\alpha_k = \frac{r+k\beta}{\nu\alpha}$, from* (2.2)*, by using* (1.14)*, see* [12]*,we can prove that*

$$s(n,k;\nu,\bar{\alpha}) = \alpha^{k-n} S^1(n,k)\nu^k. \tag{3.41}$$

Definition 3.2. *The generalized harmonic numbers of order k, associated with the sequence*
$\bar{\alpha} = (\alpha_0,\alpha_1,\cdots,\alpha_{n-1})$, denoted by $H_n(k;\bar{\alpha})$, are defined by $H_n(k;\bar{\alpha}) = \sum_{j=0}^{n-1}\frac{1}{\alpha_j^k}$.

Theorem 3.3.

$$\sum_{k=\ell+1}^{n+1} s(n+1,k;\nu,\bar{\alpha})s_{\bar{\alpha}}(k,\ell+1) = n!\sum_{r=0}^{\ell}\frac{(-1)^{n+r}(\nu)^{\ell+1}}{r!}\sum_{k_1+k_2+\cdots+k_r=\ell}\prod_{i=1}^{r}\frac{H_n^{(k_i)}}{k_i}, \tag{3.42}$$

where $H_n^{(r)} = \sum_{k=1}^{n}\frac{1}{k^r}$ are the generalized harmonic numbers of order r, see [5] and [17], *and $H_n = H_n^{(1)} = \sum_{k=1}^{n}\frac{1}{k}$ the usual harmonic numbers, where $H_0 = H_0^{(r)} = 0$ and $H_0^{(0)} = 1$.*

Proof. From (1.1) and (2.1), we have

$$(\nu t)_{n+1} = \nu t(\nu t-1)\cdots(\nu t-n) = \sum_{\ell=0}^{n+1}\sum_{k=\ell}^{n+1} s(n+1,k;\nu,\bar{\alpha})s_{\bar{\alpha}}(k,\ell)t^\ell.$$

Hence we get

$$(-1)^n n!\nu\prod_{j=1}^{n}(1-\frac{\nu t}{j}) = \sum_{\ell=0}^{n+1}\sum_{k=\ell+1}^{n+1} s(n+1,k;\nu,\bar{\alpha})s_{\bar{\alpha}}(k,\ell+1)t^\ell. \tag{3.43}$$

Since

$$\prod_{j=1}^{n}(1-\frac{\nu t}{j}) = \exp(\sum_{j=1}^{n}\log(1-\frac{\nu t}{j})) = \exp(-\sum_{k=1}^{\infty}\sum_{j=1}^{n}\frac{1}{j^k}\frac{\nu^k t^k}{k}) = \exp(-\sum_{k=1}^{\infty}H_n^{(k)}\frac{\nu^k t^k}{k}).$$

Therefore, using Cauchy rule of product of series, then (3.43) takes the form

$$\sum_{\ell=0}^{n+1}\sum_{k=\ell+1}^{n+1} s(n+1,k;\nu,\bar{\alpha})s_{\bar{\alpha}}(k,\ell+1)t^{\ell} = n!\sum_{\ell=0}^{\infty}\sum_{r=0}^{\ell}\frac{(-1)^{n+r}\nu^{\ell+1}}{r!}\sum_{k_1+k_2+\cdots+k_r=\ell}\prod_{i=1}^{r}\frac{H_n^{(k_i)}}{k_i}t^{\ell}.$$

Equating coefficients of t^{ℓ} on both sides yields (3.42). □

For a particular case, setting $n=2$ and $\ell=2$ in (3.42). We get

$$s(3,3;\nu,\bar{\alpha})s_{\bar{\alpha}}(3,3) = 2\nu^3\left(\frac{-H_2^{(2)}}{2}+\frac{(H_2^{(1)})^2}{2}\right) = \nu^3.$$

Hence we have the identity

$$(H_2^{(1)})^2 - H_2^{(2)} = 1.$$

Moreover, setting $n=3$ and $\ell=2$ in (3.42)

$$\begin{aligned} s(4,3;\nu,\overline{\alpha})s_{\overline{\alpha}}(3,3)+s(4,4;\nu,\overline{\alpha})s_{\overline{\alpha}}(4,3) &= \nu^3(\alpha_0\nu-6+\alpha_1\nu+\alpha_2\nu+\alpha_3\nu) \\ &\quad +\nu^4(-\alpha_0-\alpha_1-\alpha_2-\alpha_3) = -3\nu^3(-H_3^{(2)}+(H_3^{(1)})^2) = -6\nu^3. \end{aligned}$$

Thus, we have the identity $(H_3^{(1)})^2 - H_3^{(2)} = 2.$
Furthermore, if putting $n=4, \ell=3$ *and* 4, respectively in (3.42), we get the following identities

$$(H_4^{(1)})^3 - 3H_4^{(1)}H_4^{(2)} + 2H_4^{(3)} = 5/2.$$

$$(H_4^{(1)})^4 - 16(H_4^{(1)})^2H_4^{(2)} + 8H_4^{(1)}H_4^{(3)} + 3(H_4^{(2)})^2 - 6H_4^{(4)} = 1.$$

From (3.42) and (2.12), we have a new interesting identity

$$s(n+1,l+1) = n!\sum_{r=1}^{\ell}\frac{(-1)^{n+r}}{r!}\sum_{k_1+k_2+\cdots+k_r=\ell}\prod_{i=1}^{r}\frac{H_n^{(k_i)}}{k_i}. \tag{3.44}$$

Also, using [5, Eq.(2.9)] and (3.44), we obtain an identity

$$\sum_{r=1}^{\ell} s(n+1,\ell-r+1)H_n^{(r)} = \ell\, n!\sum_{r=1}^{\ell}\frac{(-1)^{n+r-1}}{r!}\sum_{k_1+k_2+\cdots+k_r=\ell}\prod_{i=1}^{r}\frac{H_n^{(k_i)}}{k_i}. \tag{3.45}$$

Remark 4. *Setting* $\ell = 1,2,3$ *and* 4 *in* (3.44), *we get* [5, Eq.(2.10)] *as a special case, and so on for other values of* ℓ. Setting $\ell = n$ and $n-1$, respectively, in (3.42), we obtain the following identities

$$n!\sum_{r=0}^{n}\frac{(-1)^{n+r}}{r!}\sum_{k_1+k_2+\cdots+k_r=n}\frac{H_n^{(k_1)}H_n^{(k_2)}\cdots H_n^{(k_r)}}{k_1k_2\cdots k_r} = 1. \tag{3.46}$$

$$s(n+1,n) = -\binom{n+1}{2} = n!\sum_{r=0}^{n-1}\frac{(-1)^{n+r}}{r!}\sum_{k_1+k_2+\cdots+k_r=n-1}\frac{H_n^{(k_1)}H_n^{(k_2)}\cdots H_n^{(k_r)}}{k_1k_2\cdots k_r}. \tag{3.47}$$

Theorem 3.4.

$$\sum_{\ell=0}^{n+1} s(n+1,\ell;\nu,\bar{\alpha})\prod_{j=0}^{\ell-1}\alpha_j \sum_{\ell_1+2\ell_2+\cdots+k\ell_k=k} \frac{(-1)^{\ell_1+\ell_2+\cdots+\ell_k}}{\ell_1!\ell_2!\cdots\ell_k!}\prod_{i=1}^{k}\left(\frac{H_\ell(i;\bar{\alpha})}{i}\right)^{\ell_i}$$
$$= (-1)^n n!\sum_{r=0}^{k-1}\frac{(-1)^r\nu^k}{r!}\sum_{\ell_1+\ell_2+\cdots+\ell_r=k-1}\prod_{i=1}^{r}\frac{H_n^{(\ell_i)}}{\ell_i}. \quad (3.48)$$

Proof. The left hand side of (2.1) can be written as

$$(\nu t)_{n+1} = (-1)^n n!(\nu t)\prod_{j=1}^{n}\left(1-\frac{\nu t}{j}\right) = (-1)^n n!(\nu t)\exp\left(\sum_{j=1}^{n}\log\left(1-\frac{\nu t}{j}\right)\right)$$
$$= (-1)^n n!(\nu t)\exp\left(-\sum_{k=1}^{\infty}H_n^{(k)}\frac{t^k}{k}\right).$$

Using Cauchy rule of product of series, then we have

$$(\nu t)_{n+1} = (-1)^n n!\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r\nu^k}{r!}\sum_{\ell_1+\ell_2+\cdots+\ell_r=k-1}\prod_{i=1}^{r}\frac{H_n^{(\ell_i)}}{\ell_i}t^k. \quad (3.49)$$

Also, the right hand side of (2.1) can be written as

$$\sum_{\ell=0}^{n+1}s(n+1,\ell;\nu,\bar{\alpha})(t;\bar{\alpha})_\ell = \sum_{\ell=0}^{n+1}s(n+1,\ell;\nu,\bar{\alpha})(-1)^\ell\prod_{i=0}^{\ell-1}\alpha_i\prod_{j=0}^{\ell-1}\left(1-\frac{t}{\alpha_j}\right)$$
$$= \sum_{\ell=0}^{n+1}s(n+1,\ell;\nu,\bar{\alpha})(-1)^\ell\prod_{i=0}^{\ell-1}\alpha_i\exp\left(-\sum_{k=1}^{\infty}H_\ell(k;\bar{\alpha})\frac{t^k}{k}\right)$$
$$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{n+1}s(n+1,\ell;\nu,\bar{\alpha})\prod_{j=0}^{\ell-1}\alpha_j\sum_{\ell_1+2\ell_2+\cdots+k\ell_k=k}\frac{(-1)^{\ell_1+\ell_2+\cdots+\ell_k}}{\ell_1!\ell_2!\cdots\ell_k!}\prod_{i=1}^{k}\left(\frac{H_\ell(i;\bar{\alpha})}{i}\right)^{\ell_i}t^k. \quad (3.50)$$

Equating coefficients of t^k in (3.49) and (3.50) yields (3.48). □

Theorem 3.5.

$$\sum_{\ell=k}^{n}\nu^k S(n,\ell;\nu,\overline{\alpha})s(\ell,k) = \prod_{i=0}^{n-1}\alpha_i\sum_{r=0}^{k}\frac{(-1)^{n+r}}{r!}\sum_{\ell_1+\ell_2+\cdots+\ell_k=k}\prod_{i=1}^{k}\frac{H_n(\ell_i;\bar{\alpha})}{\ell_i}, \quad (3.51)$$

where $s(n,k)$ are the usual Stirling numbers of first kind.

Proof. Since $(t;\overline{\alpha})_{\underline{n}} = \sum_{k=0}^{n}\sum_{\ell=k}^{n}S(n,\ell;\nu,\overline{\alpha})s(\ell,k)(\nu t)^k$, we have

$$(-1)^n\prod_{i=0}^{n-1}\alpha_i\prod_{j=0}^{n-1}\left(1-\frac{t}{\alpha_j}\right) = \sum_{k=0}^{n}\sum_{\ell=k}^{n}S(n,\ell;\nu,\overline{\alpha})s(k,\ell)(\nu t)^k. \quad (3.52)$$

Since

$$\prod_{j=0}^{n-1}(1-\frac{t}{\alpha_j}) = exp(\sum_{j=0}^{n-1} log(1-\frac{t}{\alpha_j}) = exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{n-1}\frac{1}{\alpha_j^k}\frac{t^k}{k}) = exp(-\sum_{k=1}^{\infty} H_n(k;\bar{\alpha})\frac{t^k}{k}),$$

then, using Cauchy rule of products of series (3.52) takes the form

$$\sum_{k=0}^{\infty}(-1)^n\prod_{i=0}^{n-1}\alpha_i\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\ell_1+\ell_2+\cdots+\ell_k=k}\prod_{i=1}^{k}\frac{H_n(\ell_i;\bar{\alpha})}{\ell_i}t^k = \sum_{k=0}^{n}\sum_{\ell=k}^{n} v^k S(n,\ell;v,\overline{\alpha})s(\ell,k)t^k.$$

Equating coefficients of t^k on both sides yields (3.51). □

4. Matrix representation and some computational applications

Let s, S; $s(v,\bar{\alpha}), S(v,\bar{\alpha})$; $s(\bar{\alpha}), S(\bar{\alpha})$; $s_{\bar{\alpha}}$, $S_{\bar{\alpha}}$ and $C(v)$ be $n\times n$ lower triangular matrices, where s and S are the matrices whose entries are the Stirling numbers of the first and second kinds (i.e. $s=[s_{ij}]$ and $S=[S_{ij}]$); $s(v,\bar{\alpha})$ and $S(v,\bar{\alpha})$ are the matrices whose entries are the GMPNSN-1 and GMPNSN-2 (i.e. $s(v,\bar{\alpha})=[s_{ij}(v,\bar{\alpha})]$ and $S(v,\bar{\alpha})=[S_{ij}(v,\bar{\alpha})]$); $s(\bar{\alpha})$ and $S(\bar{\alpha})$ are the matrices whose entries are the multiparameter non-central Stirling numbers of the first and second kinds (i.e. $s(\bar{\alpha})=[s_{ij}(\bar{\alpha})]$ and $S(\bar{\alpha})=[S_{ij}(\bar{\alpha})]$); $s_{\bar{\alpha}}$ and $S_{\bar{\alpha}}$ are the matrices whose entries are Comtet numbers of the first and second kinds (i.e. $s_{\bar{\alpha}}$ $=[(s_{\bar{\alpha}})_{ij}]$ and $S_{\bar{\alpha}}=[(S_{\bar{\alpha}})_{ij}]$) and $C(v)$ is the matrix whose entries are the C-numbers, (i.e. $C(v)=[C_{ij}(v)]$), respectively. Let $A(v)$ be a diagonal matrix whose entries of the main diagonal are v^i, $i=1,2,\cdots,n$, i.e. $A(v)=diag(v,v^2,\cdots,v^n)$. An algorithm is given and a computer program is written using Maple and executed for calculating GMPNSN-1 and their inverse, GMPNSN-2, along with some of their interesting special cases. Therefore, we can derive the matrix representation of our results.

Algorithm: For $n\in\mathbb{N}$, the elements of the $n\times n$ lower triangular matrix $s(v,\bar{\alpha})$ may be calculated as follows:

Set $s_{1,1}(v,\bar{\alpha})=1$

For $i=2$ to n do

Set $s_{i,i}(v,\bar{\alpha})=1$

Calculate
$s_{i,1}(v,\bar{\alpha})=v\prod_{r=0}^{i-2}(-\alpha_r)+(v\alpha_1-i+1)s_{i-1,1}(v,\bar{\alpha})$

Next i

For $i=3$ to n do

For $j=2$ to $i-1$ do

Calculate
$s_{i,j}(v,\bar{\alpha})=vs_{i-1,j-1}(v,\bar{\alpha})+(v\alpha_j-i+1)s_{i-1,j}(v,\bar{\alpha})$

Next j

Next i

Equation (2.13) can be written in the matrix form

$$C(v) = s(v,\bar{\alpha})S(\bar{\alpha}). \tag{4.1}$$

For example, if $n = 3$, then

$$\begin{bmatrix} v & 0 & 0 \\ v(v-1) & v^2 & 0 \\ v(v-1)(v-2) & 3v^2(v-1) & v^3 \end{bmatrix} =$$

$$\begin{bmatrix} v & 0 & 0 \\ v(v\alpha_0 + v\alpha_1 - 1) & v^2 & 0 \\ v^3(\alpha_0^2 + \alpha_1\alpha_0 + \alpha_1^2) - 3v^2(\alpha_0 + \alpha_1) + 2v & v^3(\alpha_0 + \alpha_1 + \alpha_2) - 3v^2 & v^3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\alpha_0 - \alpha_1 + 1 & 1 & 0 \\ \alpha_0\alpha_1 - \alpha_0 + \alpha_0\alpha_2 - \alpha_1 + \alpha_1\alpha_2 + 1 - \alpha_2 & -\alpha_0 - \alpha_1 + 3 - \alpha_2 & 1 \end{bmatrix}.$$

Equation (2.16) is equivalent to

$$s(v,\bar{0}) = sA(v). \tag{4.2}$$

For example, if $n = 3$, then

$$\begin{bmatrix} v & 0 & 0 \\ -v & v^2 & 0 \\ 2v & -3v^2 & v^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} v & 0 & 0 \\ 0 & v^2 & 0 \\ 0 & 0 & v^3 \end{bmatrix}.$$

Let s_a and S_a be $n \times n$ lower triangle matrices whose entries are the non-central Stirling numbers of the first and second kinds, respectively (i.e., $s_a = [(s_a)_{ij}]$ and
$S_a = [(S_a)_{ij}]$).
Equation (2.19) is equivalent to

$$s(v,\bar{a}) = s_{va}A(v). \tag{4.3}$$

For example, if $n = 3$, then

$$\begin{bmatrix} v & 0 & 0 \\ v(2va - 1) & v^2 & 0 \\ v(3v^2a^2 - 6va + 2) & 3v^2(va - 1) & v^3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2va - 1 & 1 & 0 \\ 3v^2a^2 - 6va + 2 & 3va - 3 & 1 \end{bmatrix} \begin{bmatrix} v & 0 & 0 \\ 0 & v^2 & 0 \\ 0 & 0 & v^3 \end{bmatrix}.$$

From (2.17) and (3.8) we get $C^{-1}(v) = C(v^{-1})$ where $C^{-1}(v)$ is the inverse matrix of $C(v)$ and I is the identity matrix, hence

$$C^{-1}(v)C(v^{-1}) = C(v^{-1})C^{-1}(v) = I. \tag{4.4}$$

For example, if $n = 3$, then

$$\begin{bmatrix} \frac{1}{v} & 0 & 0 \\ -\frac{v-1}{v^2} & \frac{1}{v^2} & 0 \\ \frac{2v^2-3v+1}{v^3} & -\frac{3(v-1)}{v^3} & \frac{1}{v^3} \end{bmatrix} \begin{bmatrix} v & 0 & 0 \\ v(v-1) & v^2 & 0 \\ v(v-1)(v-2) & 3v^2(v-1) & v^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Equation (3.11) can be represented in the matrix form

$$A(v)S(v,\bar{a}) = S_{va}. \tag{4.5}$$

For example, if $n = 3$, then

$$\begin{bmatrix} v & 0 & 0 \\ 0 & v^2 & 0 \\ 0 & 0 & v^3 \end{bmatrix} \begin{bmatrix} \frac{1}{v} & 0 & 0 \\ -\frac{2va-1}{v^2} & \frac{1}{v^2} & 0 \\ \frac{3v^2a^2-3va+1}{v^3} & -\frac{3(va-1)}{v^3} & \frac{1}{v^3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2va+1 & 1 & 0 \\ 3v^2a^2-3va+1 & -3va+3 & 1 \end{bmatrix}.$$

Equation (3.16) is equivalent to

$$S(v,\bar{\alpha})C(v) = S(\bar{\alpha}). \tag{4.6}$$

For example, if $n = 3$, then

$$\begin{bmatrix} \frac{1}{v} & 0 & 0 \\ -\frac{v\alpha_0+v\alpha_1-1}{v^2} & \frac{1}{v^2} & 0 \\ \frac{v^2\alpha_1\alpha_0+v^2\alpha_0\alpha_2-v\alpha_1+v^2\alpha_1\alpha_2+1-v\alpha^2}{v^3} & -\frac{v\alpha_0+v\alpha_1-3+v\alpha_2}{v^3} & \frac{1}{v^3} \end{bmatrix} \begin{bmatrix} v & 0 & 0 \\ v(v-1) & v^2 & 0 \\ v(v-1)(v-2) & 3v^2(v-1) & v^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha_0-\alpha_1 & 1 & 0 \\ \alpha_0\alpha_1-\alpha_0+\alpha_0\alpha_2-\alpha_1+\alpha_1\alpha_2+1-\alpha_2 & -\alpha_0-\alpha_1+3-\alpha_2 & 1 \end{bmatrix}.$$

Equation (3.26) is equivalent to

$$S(-1,\bar{a}) = DS(-a), \tag{4.7}$$

where D is a diagonal matrix defined by $D = diag(-1,1,-1,\cdots,(-1)^n)$.

For example, if $n = 3$, then

$$\begin{bmatrix} -1 & 0 & 0 \\ 2a+1 & 1 & 0 \\ 3a^2-3a-1 & -3a-3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2a+1 & 1 & 0 \\ 3a^2+3a+1 & 3a+3 & 1 \end{bmatrix}.$$

Equation (3.27) is equivalent to

$$s(v,\bar{\alpha})S(v,\bar{\alpha}) = I. \tag{4.8}$$

For example, if $n = 3$, then

$$\begin{bmatrix} v & 0 & 0 \\ v(v\alpha_0+v\alpha_1-1) & v^2 & 0 \\ v(v^2\alpha_0^2-3v\alpha_0+v^2\alpha_1\alpha_0+v^2\alpha_1^2-3v\alpha_1+2) & v^2(v\alpha_0+v\alpha_1+v\alpha_2-3) & v^3 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{v} & 0 & 0 \\ -\frac{v\alpha_0+v\alpha_1-1}{v^2} & \frac{1}{v^2} & 0 \\ \frac{v^2\alpha_1\alpha_0-v\alpha_0+v^2\alpha_0\alpha_2-v\alpha_1+v^2\alpha_1\alpha_2+1-v\alpha_2}{v^3} & -\frac{v\alpha_0+v\alpha_1-3+v\alpha_2}{v^3} & \frac{1}{v^3} \end{bmatrix} = I.$$

Equation (3.28) is equivalent to

$$s(v_1,\bar{\alpha})S(v_2,\bar{\alpha}) = C(v_1)C(v_2) = C(v_1v_2^{-1}). \tag{4.9}$$

For example, if $n = 3$, then

$$\begin{bmatrix} v_1 & 0 & 0 \\ v_1(v_1\alpha_0+v_1\alpha_1-1) & v_1^2 & 0 \\ v_1(v_1^2\alpha_0^2-3v_1\alpha_0+v_1^2\alpha_1\alpha_0+v_1^2\alpha_1^2-3v_1\alpha_1+2) & v_1^2(v_1\alpha_0+v_1\alpha_1+v_1\alpha_2-3) & v_1^3 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{v} & 0 & 0 \\ -\frac{v_2\alpha_0+v\alpha_1-1}{v_2^2} & \frac{1}{v_2^2} & 0 \\ \frac{v_2^2(\alpha_1\alpha_0+\alpha_0\alpha_2+\alpha_1\alpha_2)-v_2(\alpha_0-\alpha_1-\alpha_2)+1}{v_2^3} & -\frac{v_2(\alpha_0+\alpha_1+\alpha_2)-3}{v_2^3} & \frac{1}{v_2^3} \end{bmatrix} =$$
$$\begin{bmatrix} v_1 & 0 & 0 \\ v_1(v_1-1) & v_1^2 & 0 \\ v_1(v_1-1)(v_1-2) & 3v_1^2(v_1-1) & v_1^3 \end{bmatrix} \begin{bmatrix} v_2 & 0 & 0 \\ v_2(v_2-1) & v_2^2 & 0 \\ v_2(v_2-1)(v_2-2) & 3v_2^2(v_2-1) & v_2^3 \end{bmatrix} =$$
$$\begin{bmatrix} v_1v_2^{-1} & 0 & 0 \\ v_1v_2^{-1}(v_1v_2^{-1}-1) & (v_1v_2^{-1})^2 & 0 \\ v_1v_2^{-1}(v_1v_2^{-1}-1)(v_1v_2^{-1}-2) & 3(v_1v_2^{-1})^2(v_1v_2^{-1}-1) & (v_1v_2^{-1})^3 \end{bmatrix}.$$

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