On the Crossing Numbers of the Joint Graphs of a Path or a Cycle*

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Abstract

The crossing number problem is in the forefront of topological graph theory. At present, there are only a few results concerning crossing numbers of join of some graphs. In this paper, for the special graph Q on six vertices we give the crossing numbers of its join with n isolated vertices as well as with the path P_n on n vertices and with the cycle C_n.

keywords drawing; crossing numbers; joint graphs; path; cycle.

1 Introduction

Let G be a simple and undirected graph with the vertex set $V{=}V(G)$ and the edge set $E{=}E(G)$. The crossing number $cr(G)$ of the graph G is defined as the minimum number of edge crossings in a drawing of G in the plane. A drawing with the minimum number of crossings(an optimal drawing) must be a good drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D be a good drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G, we denote by $cr_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $cr_D(G_i)$ the number of crossings among edges of G_i in D. Let H be a subgraph of G, the restricted drawing $D|_H$ is said to be a subdrawing of H. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map". The following Theorem and Proposition are trivial observation.

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Jordan Curve Theorem[1]: Any simple closed curve J in the plane partitions the rest of the plane into two disjoint arcwise-connected open sets called the interior and the exterior of J. We denote them by $int(J)$ and $ext(J)$, and their closures by $Int(J)$ and $Ext(J)$, respectively. Clearly $Int(J) \cap Ext(J) = J$. The Jordan Curve Theorem implies that every arc joining a point of $int(J)$ to a point of $ext(J)$ meet J in at least one point.

Proposition 1.1. Let D be a good drawing of a graph G, G_i, G_j and G_k are three mutually disjoint edge subsets of G, then

(1) $cr_D(G_i \cup G_j) = cr_D(G_i) + cr_D(G_j) + cr_D(G_i, G_j)$.

(2) $cr_D(G_i \cup G_j, G_k) = cr_D(G_i, G_k) + cr_D(G_j, G_k)$.

Proposition 1.2. If G_1 is a subgraph of G_2, then $cr(G_1) \leq cr(G_2)$.

Proposition 1.3. Let G_1 be a graph homeomorphic to G_2, then $cr(G_1) = cr(G_2)$.

Computing the crossing number of a given graph is, in general, an elusive problem. In fact, determining the crossing number of a graph is NP-complete[2], and exact values are known only for very restricted classes of graphs. At present the crossing number is not even known exactly for complete or complete bipartite graph. The crossing number of the complete bipartite graphs $K_{m,n}$ was computed by Kleitman[3], More precisely, he proved that

$$cr(K_{m,n}) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor, \quad if \;\; m \leq 6. \qquad (1)$$

Where the number $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is often denoted by $Z(m, n)$(for any real number x, $\lfloor x \rfloor$ denotes the maximum integer that is no more than x). So it is important to study crossing numbers of join product of graphs. Kulli and Muddebihal[4] gave the characterization of all pairs of graphs the join of which is plannar graph. It thus seems natural to inquire about crossing numbers of the join of graphs.

Let G_1 and G_2 be two disjoint graphs, the join product of two graphs of G_1 and G_2, denoted by $G_1 \vee G_2$, has vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{e(u, v) \mid \forall u \in V(G_1)$, and $v \in V(G_2)\}$, (where $e(u, v)$ denotes the edge connecting vertex u and vertex v). Let nK_1 denote the graph on n isolated vertices and let P_n and C_n be the path and the cycle on n vertices, respectively. In [5] M. Klešč gave the exact values of crossing numbers for join of two paths, join of two cycles, and for join of path and cycle. Moreover, the exact values for crossing numbers of $G \vee P_n$ and $G \vee C_n$ for all graphs G of order at most four are given. Recently, M.Klešč[6] proved that the crossing numbers of join of a special graph on six vertices with Path P_n and cycle C_n. In this paper, we determine the crossing number for the join of the graph nK_1 with the special graph Q on six vertices shown in Fig.1.

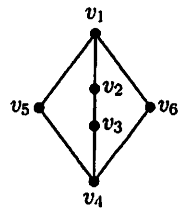

Figure 1: The graph Q on six vertices.

This result enables us, in Section 3 and 4, to give the crossing numbers of $Q \vee P_n$ and $Q \vee C_n$. Our method is simple and differs from M.Klešč. The following theorems are our main results:
Theorem 2.1 $cr(Q \vee nK_1) = Z(6,n) + n$ for $n \geq 1$.
Theorem 3.1 $cr(Q \vee P_n) = Z(6,n) + n + 1$ for $n \geq 2$.
Theorem 4.1 $cr(Q \vee C_n) = Z(6,n) + n + 3$ for $n \geq 3$.

2 The Crossing Number of $Q \vee nK_1$

Let the six vertices of Q be $v_1, v_2, ..., v_6$, and the 3-degree vertices be v_1 and v_4. The graph $Q \vee nK_1$ consists of one copy of the graph Q and n vertices $t_1, t_2, ..., t_n$, where every vertex $t_i, i = 1, 2, ..., n$, is adjacent to six vertices of Q. For $i = 1, 2, ..., n$, let T_i denote the subgraph induced by six edges incident with the vertex t_i(Fig.3). In Fig.2 one can easily see that

$$Q \vee nK_1 = Q \cup K_{6,n},\ E(Q \vee nK_1) = E(Q) \cup (\bigcup_{i=1}^{n} E(T_i)) \tag{2}$$

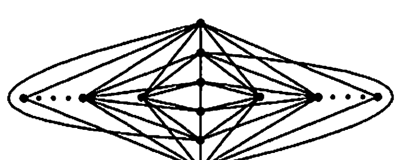

Figure 2: A good drawing of $Q \vee nK_1$.

Lemma 2.1. $cr(Q \vee K_1) = 1, cr(Q \vee 2K_1) = 2$.
Proof. The drawing in Fig.2 shows that $cr(Q \vee K_1) \leq 1$ and $cr(Q \vee 2K_1) \leq 2$. Moreover $Q \vee K_1$ contains a subgraph homeomorphic to complete bipartite graph $K_{3,3}$; $Q \vee 2K_1$ contains a subgraph homeomorphic to complete tripartite graph $K_{2,3,2}$, whose crossing number is 2 (see [7]). By Proposition 1.3, hence, $cr(Q \vee K_1) = 1, cr(Q \vee 2K_1) = 2$.
Lemma 2.2. Let D be a good drawing of $Q \vee 2K_1$, if $cr_D(T_1, T_2) = 0$, then

$cr_D(Q, T_1 \cup T_2) \geq 2$.

Proof. Let $\langle T_1 \cup T_2 \rangle$ be the subgraph induced by the edges of $T_1 \cup T_2$. Since $cr_D(T_1, T_2) = 0$, and in good drawing two edges incident with the same vertex can not cross, the subdrawing of $\langle T_1 \cup T_2 \rangle$ induced by D induces the map in the plane without crossing as shown in Fig.3(b).

As the two 3-degree vertices of Q are v_1 and v_4, let $E(v_i)$ denote the edges incident with v_i in Q, thus, $cr_D(E(v_1), T_1 \cup T_2) \geq 1, cr_D(E(v_4), T_1 \cup T_2) \geq 1$. Moreover, the two vertices v_1 and v_4 are non-adjacent, therefore, $cr_D(Q, T_1 \cup T_2) \geq 2$.

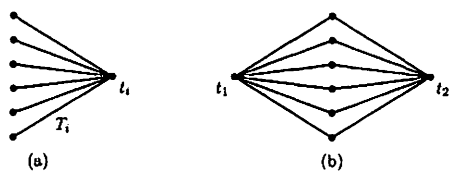

Figure 3: The drawing of T_i and $\langle T_1 \cup T_2 \rangle$.

Theorem 2.1 $cr(Q \vee nK_1) = Z(6, n) + n$ for $n \geq 1$.

Proof. The drawing in Fig.2 shows that $cr(Q \vee nK_1) \leq Z(6, n) + n$ and that the theorem is true if the equality holds. We prove the reverse inequality $cr(Q \vee nK_1) \geq Z(6, n) + n$ by induction on n. By Lemma 2.1, the theorem is true for $n = 1, 2$. Suppose now that for $n \geq 3$

$$cr(Q \vee (n-2)K_1) \geq Z(6, n-2) + (n-2). \tag{3}$$

and consider such a good drawing D of $Q \vee nK_1$ that

$$cr_D(Q \vee nK_1) < Z(6, n) + n. \tag{4}$$

Our next analysis depends on whether or not there are different subgraphs T_i and T_j that do not cross each other in D.

Case 1. Assume that there are two different subgraphs T_i and $T_j, i, j = 1, 2, ...n, i \neq j$, such that $cr_D(T_i, T_j)$=0.

Without loss of generality, let $cr_D(T_n, T_{n-1}) = 0$. When $1 \leq i \leq n-2$, as $\langle T_n \cup T_{n-1} \cup T_i \rangle$ is isomorphic to complete bipartite graph $K_{3,6}$, moreover, by formula (1) $cr(K_{3,6}) = 6$, we have

$$cr_D(T_n \cup T_{n-1}, T_i) = cr_D(K_{3,6}) - cr_D(T_n \cup T_{n-1}) - cr_D(T_i) \geq 6. \tag{5}$$

Since $Q \vee nK_1 = (Q \cup T_n \cup T_{n-1} \cup \bigcup_{i=1}^{n-2} T_i)$ and $\langle Q \cup (\bigcup_{i=1}^{n-2} T_i) \rangle$=$Q \vee (n -$

$2)K_1$, using Proposition 1.1 and Lemma 2.2, we have

$$\begin{aligned} cr_D(Q \vee nK_1) &= cr_D(Q \cup T_n \cup T_{n-1} \cup \bigcup_{i=1}^{n-2} T_i) \\ &= cr_D(T_n \cup T_{n-1}, \bigcup_{i=1}^{n-2} T_i) + cr_D(T_n \cup T_{n-1}, Q) \\ &+ cr_D(Q \cup \bigcup_{i=1}^{n-2} T_i) + cr_D(T_n \cup T_{n-1}) \\ &\geq 6(n-2) + 2 + Z(6, n-2) + (n-2) \\ &\geq Z(6,n) + n. \end{aligned}$$

This contradicts (4).

Case 2. For all $i, j = 1, 2, ...n$, $i \neq j$, there holds $cr_D(T_i, T_j) \geq 1$.

Using Proposition 1.1 and (2) together with $cr(K_{6,n}) = Z(6,n)$, we have

$$\begin{aligned} cr_D(Q \vee nK_1) &= cr_D(K_{6,n}) + cr_D(Q) + cr_D(K_{6,n}, Q) \\ &\geq Z(6,n) + cr_D(Q) + cr_D(K_{6,n}, Q). \end{aligned}$$

This, together with the assumption (4), implies that

$$cr_D(Q) + cr_D(K_{6,n}, Q) < n. \tag{6}$$

and hence, in D there is at least one subgraph T_i which does not cross Q.

Without loss of generality, let $cr_D(Q, T_n) = 0$. In D there is at least one subgraph T_i, $i \in \{1, 2, ..., n-1\}$, for which

$$cr_D(Q \cup T_n, T_i) \leq 3. \tag{7}$$

Otherwise, $cr_D(Q \cup T_n, T_i) \geq 4$, as $Q \vee nK_1 = K_{6,n-1} \cup (Q \cup T_n)$, we have

$$\begin{aligned} cr_D(Q \vee nK_1) &= cr_D(K_{6,n-1}) + cr_D(Q \cup T_n) + cr_D(K_{6,n-1}, Q \cup T_n) \\ &\geq Z(6, n-1) + 1 + 4(n-1) \\ &\geq Z(6,n) + n. \end{aligned} \tag{8}$$

This contradicts (4).

Considering now the restricted drawing $D|_Q$, since $cr_D(Q, T_n) = 0$, then, the restricted drawing $D|_Q$ divides the plane in such a way that there is a disk C such that the vertices of Q are all located on the boundary of C, and the edges of Q are all located in the inner of C. Furthermore, as D is a good drawing and the edges of Q can be presented by straight lines, vertex t_n and the edges incident with t_n are all located on the outside of C. Since Q has two 3-degree vertices v_1 and v_4. Regarding to the symmetry of Q, firstly, we have drawn all the six vertices of Q on the boundary of C and all possibilities of the subdrawing of $E(v_1)$ as dotted line as shown in Fig.4.

As the two vertices of v_1,v_4 are non-adjacent, according to the characterization of Q,the vertex v_4 can only be vertex a or b in Fig.5. Connecting the three edges of v_4. It is easy to see that in the inner of C in Fig.5(1-5), there are at most three vertices of Q on the boundary of the arbitrary region. Only in Fig.5(6), there is a region ω with 4 vertices of Q on its boundary. Since $cr_D(Q, T_n) = 0$, it is easy to see that $cr_D(Q) \geq 1$. By (6) we have that

$$cr_D(K_{6,n}, Q) < n - 1. \tag{9}$$

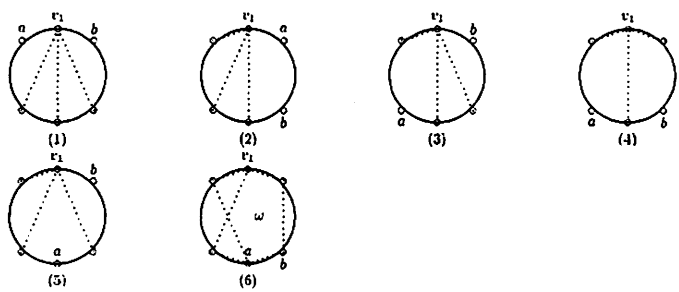

Figure 4: The drawing of $D|_Q$.

Case 2.1. Considering the drawing in Fig.4(1-5). Since $cr_D(Q, T_n) = 0$, adding the edges of T_n to outside of the C, it is easy to see that in these cases there are at most three vertices of $Q \cup T_n$ on the boundary of the arbitrary region in $Q \cup T_n$. Hence, $cr_D(Q \cup T_n, T_i) \geq 4$, according to (8), $cr_D(Q \vee nK_1) \geq Z(6, n) + n$.This contradicts (4).
Case 2.2. Considering the drawing in Fig.4(6). Since $cr_D(Q, T_n) = 0$, adding the edges of T_n to outside of the C, its accordant drawing is Fig.5.

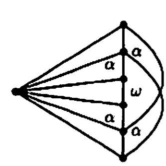

Figure 5: The subdrawing of $Q \cup T_n$ of Fig.4(6).

(1) When vertices $t_i, i \in \{1, 2, ..., n-1\}$, are located in the region labeled as ω. Since the region ω contains four vertices of Q, then $cr_D(Q, T_i) \geq 2$. Using $cr_D(T_i, T_j) \geq 1$, we have $cr_D(Q \cup T_n, T_i) \geq 3$. Let r be the number of vertices $t_i, i \in \{1, 2, ..., n-1\}$, which satisfy $cr_D(Q \cup T_n, T_i) \geq 3$. By (7) at

least one subgraph $T_i, i \in \{1,2,...,n-1\}$, for which $cr_D(Q \cup T_n, T_i) \leq 3$, so we have $r \geq 1$.

(2) When the vertices $t_i, i \in \{1,2,...,n-1\}$, are located in the region labeled as α. We have $cr_D(Q \cup T_n, T_i) \geq 5$, moreover when $cr_D(Q \cup T_n, T_i) = 5$ we have $cr_D(Q, T_i) \geq 1$. Let s_1 be the number of vertices $t_i, i \in \{1,2,...,n-1\}$, which satisfies $cr_D(Q \cup T_n, T_i) = 5$; s_2 be the number of vertices $t_i, i \in \{1,2,...,n-1\}$, which satisfies $cr_D(Q \cup T_n, T_i) > 5$.

(3) When the vertices $t_i, i \in \{1,2,...,n-1\}$, are located in the other regions. We have $cr_D(Q \cup T_n, T_i) \geq 6$. Then, $n-r-s_1-s_2-1$ be the number of vertices $t_i, i \in \{1,2,...,n-1\}$, which satisfies $cr_D(Q \cup T_n, T_i) \geq 6$. Hence, we have

$$\begin{aligned} cr_D(Q \vee nK_1) &= cr_D(K_{6,n-1}) + cr_D(Q \cup T_n) + cr_D(K_{6,n-1}, Q \cup T_n) \\ &\geq Z(6,n-1) + 1 + 3r + 5s_1 + 6s_2 \\ &+ 6(n-r-s_1-s_2-1) \\ &= Z(6,n) - 6\lfloor\tfrac{n-1}{2}\rfloor + 1 + 6(n-1) - 3r - s_1. \end{aligned}$$

This, together with the assumption (4), gives

$$3r + s_1 > 6(n-1) - 6\lfloor\frac{n-1}{2}\rfloor - n + 1 = 6\lfloor\frac{n}{2}\rfloor - (n-1).$$

When n is odd: $3r+s_1 > 3(n-1)-(n-1) = 2(n-1)$; On the other hand, by (9) we have $2r + s_1 < n-1$ and the inequality

$$2(2r + s_1) < 2(n-1) < 3r + s_1$$

implies that

$$r + s_1 < 0.$$

This contradicts $r \geq 1$ and $s_1 \geq 0$.
When n is even: $3r + s_1 > 3n - (n-1) = 2n + 1$, which implies that $3r+s_1-1 > 2n$; On the other hand, $2r+s_1 < n-1$, implies that $2r+s_1+1 < n$ and the inequality

$$2(2r + s_1 + 1) < 2n < 3r + s_1 - 1$$

implies that

$$r + s_1 + 3 < 0.$$

This contradicts $r \geq 1$ and $s_1 \geq 0$. Hence, we have

$$\begin{aligned} cr_D(Q \vee nK_1) &\geq Z(6,n) - 6\lfloor\tfrac{n-1}{2}\rfloor + 1 + 6(n-1) - 3r - s_1 \\ &\geq Z(6,n) - 6\lfloor\tfrac{n-1}{2}\rfloor + 1 + 6(n-1) - (6\lfloor\tfrac{n}{2}\rfloor - (n-1)) \\ &\geq Z(6,n) + n. \end{aligned}$$

This contradicts (4).

Thus formula (4) doesn't hold. So, we have shown that $cr(Q \vee nK_1) \geq Z(6,n) + n$. Hence, $cr(Q \vee nK_1) = Z(6,n) + n$. This completes the proof.□

3 The Crossing Number of $Q \vee P_n$

Let P_n denote the path on n vertices of $Q \vee P_n$ which does not belong to the subgraph Q, we will use the same notation as above. Obviously, The graph $Q \vee P_n$ contains $Q \vee nK_1$ as a subgraph. One can easily see that

$$Q \vee P_n = Q \cup K_{6,n} \cup P_n, \quad E(Q \vee P_n) = E(Q) \cup (\bigcup_{i=1}^{n} E(T_i)) \cup E(P_n).$$

Lemma 3.1[4] Let D be a good drawing of $mK_1 \vee C_n, m \geq 2, n \geq 3$, in which no edge of C_n is crossed, and C_n does not separate the other vertices of the graph. Then, for all $1 \leq i \neq j \leq m$, two subgraphs T_i and T_j cross each other in D at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times.

As the graph $Q \vee P_1$ is $Q \vee K_1$. So $cr(Q \vee P_1) = cr(Q \vee K_1) = 1$, the case $n = 1$ is trivial. For $n \geq 2$ we have the next result.

Theorem 3.1 $cr(Q \vee P_n) = Z(6, n) + n + 1$ for $n \geq 2$.

Proof. Fig.2 shows the drawing of the graph $Q \vee nK_1$ with $Z(6, n) + n$ crossings. One can easily see that in this drawing it is possible to add $n - 1$ edges which form the path P_n on the vertices of nK_1 in such a way that only one edge of P_n is crossed by an edge of Q. Hence, $cr(Q \vee P_n) \leq Z(6, n) + n + 1$. To prove the reverse inequality $cr(Q \vee P_n) \geq Z(6, n) + n + 1$, we assume that there is a good drawing D of the graph $Q \vee P_n$ such that

$$cr_D(Q \vee P_n) < Z(6, n) + n + 1. \tag{10}$$

As the graph $Q \vee P_n$ contains $Q \vee nK_1$ as a subgraph. By Theorem 2.1, $cr(Q \vee P_n) \geq Z(6, n) + n$ and therefore, no edge of the path P_n is crossed in D.

For $n = 2$, in Fig.2 it is easy to see that $cr(Q \vee P_2) \leq 3$. Moreover $Q \vee P_2$ contains a subgraph homeomorphic to complete tripartite graph $K_{3,3,1}$, whose crossing number is 3(see [8]). Hence, $cr(Q \vee P_2) = 3$. The theorem is true for $n = 2$.

Assume $n \geq 3$, we divide the problem into several cases to prove that $cr_D(Q \vee P_n) \geq Z(6, n) + n + 1$. Consider the subdrawing D^* of Q induced by D. Since $cr_D(P_n) = 0$, the subdrawing D^* divides the plane in such a way that all of P_n is located in one region.

Case 1. Assume that all of P_n is located in the region with at most 4 vertices of Q in the subdrawing D^*. So, the edges of every $T_i, i \in \{1, 2, ..., n\}$, cross the edges of Q at least twice, thus, $\sum_{i=1}^{n} cr_D(T_i, Q) \geq 2n$. Hence

$$\begin{aligned} cr_D(Q \vee P_n) &\geq cr_D(K_{6,n}) + \sum_{i=1}^{n} cr_D(T_i, Q) \geq Z(6, n) + 2n \\ &\geq Z(6, n) + n + 1. \end{aligned}$$

Case 2. Assume that all of P_n is located in the region with 5 vertices of Q in the subdrawing D^*. So, the edges of every $T_i, i \in \{1, 2, ..., n\}$, cross the edges of Q at least once. Moreover all of P_n is located in the region the boundary of which forms a 5-cycle. As there is no crossing between the edges of the 5 vertices cycle and P_n, which possesses the qualifications of Lemma 3.1 and therefore, in D there are at least $C_n^2\lfloor\frac{5}{2}\rfloor\lfloor\frac{4}{2}\rfloor$ crossings. Hence

$$cr_D(Q \vee P_n) \geq C_n^2\lfloor\tfrac{5}{2}\rfloor\lfloor\tfrac{4}{2}\rfloor + n \geq Z(6, n) + n + 1.$$

Case 3. Assume that all of P_n is located in the region with 6 vertices of Q in the subdrawing D^*, in this case it is easy to see that $cr_D(Q) \geq 1$(see Fig.6).

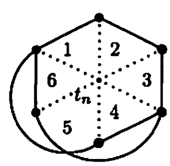

Figure 6: The possible placements of T_n inside D(Q) with 6 vertices.

So, the $n-1$ vertices $t_i, i \in \{1, 2, ..., n-1\}$, can be located only in one region, and there are exactly two vertices of Q on the boundary of each region. No matter which region does t_i locate in, it is easy to see that there are two vertices of Q that haven't locate in the region with the common boundary. By Jordan Curve Theorem, we have $cr_D(Q \cup T_n, T_i) \geq 5$. Hence

$$\begin{aligned} cr_D(Q \vee P_n) &= cr_D(K_{6,n-1} \cup (Q \cup T_n) \cup P_n) \\ &\geq cr_D(K_{6,n-1}) + cr_D(Q \cup T_n) + cr_D(\bigcup_{i=1}^{n-1} T_i, Q \cup T_n) \\ &\geq Z(6, n-1) + 1 + 5(n-1) \\ &\geq Z(6, n) + n + 1. \end{aligned}$$

This contradicts (10).

Thus, formula (10) doesn't hold. So, we have shown that $cr(Q \vee P_n) \geq Z(6, n) + n + 1$. Hence, $cr(Q \vee P_n) = Z(6, n) + n + 1$. This completes the proof. □

4 The Crossing Number of $Q \vee C_n$

The graph $Q \vee C_n$ contains both $Q \vee nK_1$ and $Q \vee P_n$ as subgraphs. Let C_n denote the subgraph of $Q \vee C_n$ induced on the vertices not belonging to the subgraph Q. One can easily see that

$$Q \vee C_n = Q \cup K_{6,n} \cup C_n, \quad E(Q \vee C_n) = E(Q) \cup (\bigcup_{k=1}^{n} E(T_k)) \cup E(C_n).$$

On the other hand, the graph $Q \vee C_n$ contains the graph $6K_1 \vee C_n$ as a subgraph and

$$Q \vee C_n = Q \cup (\bigcup_{k=1}^{6} T_k) \cup C_n, \quad E(Q \vee C_n) = E(Q) \cup (\bigcup_{k=1}^{6} E(T_k)) \cup E(C_n),$$

where T_k denotes the subgraph induced by n edges of $K_{6,n}$ incident with the ith vertex of Q. The proof of the main result of this section is based on the Lemma 3.1 and Lemma 4.1.

Lemma 4.1[9] Let ϕ be an optimal drawing of $Q \vee C_n$, then $cr_\phi(E(C_n)) = 0$.

Theorem 4.1 $cr(Q \vee C_n) = Z(6,n) + n + 3$ for $n \geq 3$.

Proof. In the drawing Fig.2 it is possible to add n edges in such a way that they form the cycle C_n and that the edges of C_n are crossed only three times. Hence, $cr(Q \vee C_n) \leq Z(6,n) + n + 3$. To prove the reverse inequality $cr(Q \vee C_n) \geq Z(6,n) + n + 3$, we assume that there is an optimal drawing of the graph $Q \vee C_n$ with at most $Z(6,n) + n + 2$ crossings and let ϕ be such a drawing. As the graph $Q \vee C_n$ contains $Q \vee P_n$ as a subgraph, by Proposition 1.2 and Theorem 3.1

$$Z(6,n) + n + 1 \leq cr_\phi(Q \vee C_n) \leq Z(6,n) + n + 2. \tag{11}$$

Claim. $cr_\phi(Q \vee C_n) > Z(6,n) + n + 2$.

Proof. Firstly by Lemma 4.1, in the optimal drawing ϕ, $cr_\phi(E(C_n)) = 0$ (no edge of C_n has a self-intersection). Our next analysis depends on whether or not the edges of C_n is crossed by other edges. We divide the problem into several cases to prove that $cr_\phi(Q \vee C_n) > Z(6,n) + n + 2$.

Case 1. Assume that there is at least one edge of C_n which is crossed in ϕ.

Subcase 1.1 Assume now that the edges of C_n is crossed by the edges of Q. By Theorem 2.1, $cr_\phi(Q \vee nK_1) = Z(6,n) + n$ and therefore, C_n can not be crossed more than twice in ϕ, since the graph $Q \vee C_n$ contains a subgraph isomorphic to $Q \vee nK_1$. By Proposition 1.2 and Theorem 2.1

$$\begin{aligned} cr_\phi(Q \vee C_n) &\geq cr(Q \vee nK_1) + 3 = Z(6,n) + n + 3 \\ &> Z(6,n) + n + 2. \end{aligned}$$

Thus, the edges of C_n can be crossed at most twice. Moreover C_n and Q are 2-connected graphs, when Q is crossed by C_n, there are at least two crossings on the edges of C_n. So, when C_n is crossed by Q, it must only produce two crossings and no edge of C_n is crossed by edges of $T_k, k = 1, 2, ..., 6$. On this condition, the two 3-degree vertices v_1 and v_4 of Q either locate in the inner or on the outside of C_n. Otherwise there are more than two crossings on the edges of C_n. Without loss of generality, let the two 3-degree vertices locate in the inner of C_n.

Subcase 1.1.1. Assume that one 2-degree vertex of Q locate on the outside of C_n, other five vertices of Q locate in the inner of C_n. Because no crossing is crossed in the edge of C_n by T_k, every subgraph $T_k, k = 1, 2, ..., 6$, induced on the edges of incident with a vertex of Q possesses the qualifications of Lemma 3.1 and therefore, in ϕ there are at least $C_5^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ crossings, hence

$$cr_\phi(Q \vee C_n) \geq C_5^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2 > Z(6, n) + n + 2.$$

Subcase 1.1.2. Assume that two 2-degree vertices of Q locate on the outside of C_n. According to the characterization of Q only are the vertices v_2, v_3 locate on the outside of C_n, other four vertices of Q locate in the inner of C_n. In this case its accordant drawing is Fig.7.

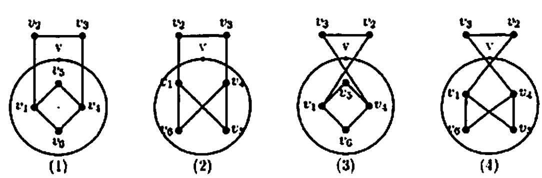

Figure 7: The possible v_2,v_3 of Q locate on the outside of C_n.

Considering the case in Fig.7(1-2), the case in Fig.7(3-4) are similar. Firstly, the two crossings can not be in the same edge of C_n. Otherwise deleting the edge from C_n results in the drawing of the graph $Q \vee C_n$ containing a subgraph isomorphic to $Q \vee P_n$. By Proposition 1.2 and Theorem 3.1

$$\begin{aligned} cr_\phi(Q \vee C_n) &\geq cr_\phi(Q \vee P_n) + 2 = Z(6, n) + n + 3 \\ &> Z(6, n) + n + 2. \end{aligned}$$

This implies that there are at least one vertex of C_n inside the cycle of $v_1v_2v_3v_4v_5v_1$ in C_n, denoting it with v. When the edge v_1v_5 does not cross the edge v_4v_6(Fig.7(1)). Using Lemma 3.1, then, we have

$$\begin{aligned} cr_\phi(Q \vee C_n) &= cr_\phi(Q \cup C_n \cup (\bigcup_{k=1}^{6} T_k)) \\ &= cr_\phi(\bigcup_{k=1}^{6} T_k) + cr_\phi(Q \cup C_n) + cr_\phi(Q \cup C_n, \bigcup_{k=1}^{6} T_k) \\ &= cr_\phi((T_2 \cup T_3) \cup (T_1 \cup T_4 \cup T_5 \cup T_6)) + cr_\phi(Q \cup C_n) \\ &+ cr_\phi(Q \cup C_n, \bigcup_{k=1}^{6} T_k) \\ &= \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor + C_4^2 \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor + 2 + 1 + (n-1) \\ &> Z(6,n) + n + 2. \end{aligned}$$

When the edges v_1v_5 is crossed by the edge v_4v_6(Fig.7(2)). Using Lemma 3.1, then, we have

$$\begin{aligned} cr_\phi(Q \vee C_n) &= cr_\phi(Q \cup C_n \cup (\bigcup_{k=1}^{6} T_k)) \\ &= cr_\phi(\bigcup_{k=1}^{6} T_k) + cr_\phi(Q \cup C_n) + cr_\phi(Q \cup C_n, \bigcup_{k=1}^{6} T_k) \\ &= cr_\phi((T_2 \cup T_3) \cup (T_1 \cup T_4 \cup T_5 \cup T_6)) + cr_\phi(Q \cup C_n) \\ &+ cr_\phi(Q \cup C_n, \bigcup_{k=1}^{6} T_k) \\ &= \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor + C_4^2 \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor + 1 + 0 + 2 + 2 \\ &> Z(6,n) + n + 2. \end{aligned}$$

Subcase 1.1.3. Assume that the six vertices of Q are all located in the inner of C_n. As C_n does not cross the edges of T_k, every subgraph T_k, $k = 1, 2, ..., 6$, induced on the edges incident with a vertex of Q possesses the qualifications of Lemma 3.1 and therefore, in ϕ there are at least $C_6^2 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings, hence

$$cr_\phi(Q \vee C_n) \geq C_6^2 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2 > Z(6,n) + n + 2.$$

Subcase 1.2. Assume that C_n is crossed by the edges of T_k, $k = 1, 2, ..., 6$. Then C_n is crossed by the edges of T_k, $k = 1, 2, ..., 6$, at most twice, and C_n does not cross the edges of Q. Otherwise, according to Subcase 1.1, $cr_\phi(Q \vee C_n) > Z(6,n) + n + 2$. Thus, the Q is either located in the inner or on the outside of C_n, without loss of generality, let Q locate in the inner of C_n.

Subcase 1.2.1. Assume that C_n is crossed by the edges of T_k, $k = 1, 2, ..., 6$, once.

When $cr_\phi(Q) = 0$(Fig.8(1)): Obviously, there exists one T_k, $k = 1, 2, ..., 6$, satisfying $cr_\phi(T_k, Q) \geq n$, using Lemma 3.1, then, we have

$$\begin{aligned} cr_\phi(Q \vee C_n) &\geq C_5^2 \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor + 5 \lfloor \tfrac{n-1}{2} \rfloor \lfloor \tfrac{n-2}{2} \rfloor + n + 1 \\ &> Z(6,n) + n + 2. \end{aligned}$$

When $cr_\phi(Q) \geq 1$(Fig.8(2)): Using Lemma 3.1, then, we have

$$\begin{aligned} cr_\phi(Q \vee C_n) &\geq C_5^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 5\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + 1 + 1 \\ &> Z(6,n) + n + 2. \end{aligned}$$

Figure 8: The drawing of $cr_\phi(Q) = 0$ and $cr_\phi(Q) \geq 1$ of Q locate in the inner of C_n.

Subcase 1.2.2. Assume that C_n is crossed by the edges of $T_k, k = 1, 2, ..., 6$, twice. Firstly, C_n can not be crossed by the edges of any one T_k twice, or by Lemma 3.1

$$cr_\phi(Q \vee C_n) \geq C_5^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2 > Z(6,n) + n + 2.$$

Thus, without loss of generality, let there is one crossing caused by C_n and the edges of T_1; one crossing caused by C_n and the edges of T_2, and C_n does not cross the edges of $T_k, k = 3, 4, 5, 6$.

Subcase 1.2.2.1 Considering the case $n = 3$ at first: Because the two crossings caused by C_3 and the edges of T_1, T_2 can not happen in the same edge of C_3, in this case its accordant drawing is Fig.9, it is easy to see that $cr_\phi(T_1, T_2) \geq 1$.

When $cr_\phi(Q) = 0$(Fig.8(1)): Obviously there exits $T_k, k = 1, 2, ..., 6$, satisfies $cr_\phi(T_k, Q) \geq 3$, using Lemma 3.1, then, we have

$$cr_\phi(Q \vee C_3) \geq C_4^2\lfloor\frac{3}{2}\rfloor\lfloor\frac{2}{2}\rfloor + 3 + 2 + 1 > Z(6,3) + 3 + 2.$$

When $cr_\phi(Q) \geq 1$(Fig.8(2)): Because the diameter of Q is 3. Denoting $P_{i,j}$ is the shortest path from vertex v_i to v_j, then there are four vertices of Q on $P_{1,2}$ at most. Without loss of generality, let v_5 and v_6 be not on $P_{1,2}$. Because $T_1 \cup T_2 \cup C_3 \cup P_{1,2}$ divides the inner of C_3 into several regions,and the two crossings of C_3 can not happen in the same edge of C_3, thus there are at most two vertices of C_3 on the boundary of every region in these regions(Fig.9).

Because v_5, v_6 aren't on $P_{1,2}$, then v_5, v_6 must be located in these regions, hence $cr_\phi(T_k, T_1 \cup T_2 \cup P_{1,2}) \geq 1, k = 5, 6$. Moreover C_3 does not cross the edges of $T_k, k = 3,4,5,6$. By Lemma 3.1, $cr_\phi(T_3 \cup T_4 \cup T_5 \cup T_6) \geq C_4^2\lfloor\frac{3}{2}\rfloor\lfloor\frac{2}{2}\rfloor$, $cr_\phi(T_k, C_3) = 1, k = 1,2$. Hence,we have

$$\begin{aligned} cr_\phi(Q \vee C_3) &= cr_\phi(T_3 \cup T_4 \cup T_5 \cup T_6) + cr_\phi(T_5, T_1 \cup T_2 \cup P_{1,2}) \\ &+ cr_\phi(T_6, T_1 \cup T_2 \cup P_{1,2}) + cr_\phi(T_1, C_3) + cr_\phi(T_2, C_3) \\ &+ cr_\phi(T_1, T_2) + cr_\phi(Q) \\ &\geq C_4^2\lfloor\tfrac{3}{2}\rfloor\lfloor\tfrac{2}{2}\rfloor + 1 + 1 + 1 + 1 + 1 + 1 \\ &> Z(6,3) + 3 + 2. \end{aligned}$$

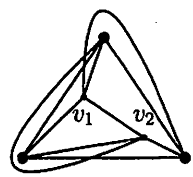

Figure 9: The drawing of Q locate in the inner of C_3.

Subcase 1.2.2.2. Considering now the case $n \geq 4$. By Lemma 3.1, $cr_\phi(T_3 \cup T_4 \cup T_5 \cup T_6) \geq C_4^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$; $cr_\phi(T_1, T_2) \geq \lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$; $cr_\phi(T_k, T_3 \cup T_4 \cup T_5 \cup T_6) \geq 4\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor (k = 1,2)$. Hence, we have

$$\begin{aligned} cr_\phi(Q \vee C_n) &= cr_\phi(Q \cup C_n \cup (\bigcup_{k=1}^{6} T_k)) \\ &= cr_\phi(\bigcup_{k=1}^{6} T_k) + cr_\phi(Q \cup C_n) + cr_\phi(Q \cup C_n, \bigcup_{k=1}^{6} T_k) \\ &= cr_\phi((T_1 \cup T_2) \cup (T_3 \cup T_4 \cup T_5 \cup T_6)) + cr_\phi(Q \cup C_n) \\ &+ cr_\phi(Q \cup C_n, \bigcup_{k=1}^{6} T_k) \\ &\geq \lfloor\tfrac{n-2}{2}\rfloor\lfloor\tfrac{n-3}{2}\rfloor + C_4^2\lfloor\tfrac{n}{2}\rfloor\lfloor\tfrac{n-1}{2}\rfloor + 2 \times 4\lfloor\tfrac{n-1}{2}\rfloor\lfloor\tfrac{n-2}{2}\rfloor + 2 \\ &> Z(6,n) + n + 2 \ (n \geq 4). \end{aligned}$$

Hence, subcase 1.2.2 for $n \geq 3$, $cr_\phi(Q \vee C_n) > Z(6,n) + n + 2$.

Subcase 1.3. Assume that C_n is crossed by the edges of Q and $T_k, k = 1,2,...,6$. By Subcase 1.1, if Q is crossed by C_n there are at least two crossings on the edges of C_n, and C_n is crossed by the edges of $T_k, k = 1,2,...,6$, then there are at least three crossings in C_n, then deleting the crossed edges from C_n results in the drawing of the graph $Q \vee C_n$ contains a subgraph isomorphic to $Q \vee nK_1$, hence

$$\begin{aligned} cr_\phi(Q \vee C_n) &\geq cr(Q \vee nK_1) + 3 = Z(6,n) + n + 3 \\ &> Z(6,n) + n + 2. \end{aligned}$$

Case 2. Assume that no edges of C_n is crossed in ϕ. Since $cr_\phi(C_n) = 0$, it implies Q is either located in the inner or on the outside of C_n, and $T_k, k =$

$1, 2, ..., 6,$ does not cross the edges of C_n, without loss of generality, let Q be located in inner of C_n. Every subgraph $T_k, k = 1, 2, ..., 6,$ induced on the edges of incident with a vertex of Q possesses the qualifications of Lemma 3.1 and therefore, in ϕ there are at least $C_6^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ crossings, hence

$$cr_{\phi}(Q \vee C_n) \geq C_6^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor > Z(6, n) + n + 2.$$

In conclusion, by $cr(Q \vee C_n) \leq Z(6, n)+n$ and the proofs of Claim, $cr_{\phi}(Q \vee C_n) \geq Z(6, n) + n + 3$, formula (11) doesn't hold. So, we have shown that $cr(Q \vee C_n) \geq Z(6, n) + n + 3$. Hence, $cr(Q \vee C_n) = Z(6, n) + n + 3$. This completes the proof. □

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