Resistance distance and Kirchhoff index in subdivision-vertex and subdivision-edge neighbourhood coronae

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Abstract

In this paper, formulas of the resistance distance for the arbitrary two-vertex resistance of $G_1 \boxdot G_2$ and $G_1 \boxminus G_2$ in the electrical networks are obtained in a much simpler way. Furthermore, $Kf(G_1 \boxdot G_2)$ and $Kf(G_1 \boxminus G_2)$ can be expressed as a combination of $Kf(G_1)$ and $Kf(G_2)$.

Key words: Kirchhoff index, Resistance distance, Laplacian matrix, Group inverse

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1 Introduction

All graphs considered in this paper are simple and undirected. The resistance distance is a distance function on graphs introduced by Klein and Randic [1]. The resistance distance $r_{ij}(G)$ between any two vertices i and j in G is defined to be the effective resistance between them when unit resistors are placed on every edge of G. The Kirchhoff index $Kf(G)$ is the sum of resistance distances between all pairs of vertices of G. The resistance distance and the Kirchhoff index has attracted extensive attention due to its wide applications in physics, chemistry and others. For more information on resistance distance and Kirchhoff index of graphs, the readers are referred to Refs. ([2] – [9]) and the references therein.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let d_i be the degree of vertex i in G and $D_G = diag(d_1, d_2, \cdots d_{|V(G)|})$ the diagonal matrix with all vertex degrees of G as its diagonal entries. For a graph G, let A_G and B_G denote the adjacency matrix and vertex-edge incidence matrix of G, respectively. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of G, where D_G is the diagonal matrix of vertex degrees of G. We use $\mu_1(G) \geq u_2(G) \geq \cdots \mu_n(G) = 0$ to denote the eigenvalues of L_G. The $\{1\}$-inverse of M is a matrix X such that $MXM = M$. If M is singular, then it has infinite $\{1\}$- inverse [10]. We use $M^{(1)}$ to denote any $\{1\}$- inverse of a matrix M, and let $(M)_{uv}$ denote the (u, v)- entry of M. It is known that resistance distances in a connected graph G can be obtained from any $\{1\}$- inverse of G ([11], [17]).

In [18], two graph operations based on $S(G)$ graphs: The subdivision-vertex and the subdivision-edge neighbourhood corona, are introduced, and their A-spectra(resp., L-spectra) are investigated. The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. Let $I(G)$ be the set of newly added vertices, *i.e* $I(G) = V(S(G)) \setminus V(G)$.

Let G_1 and G_2 be two vertex-disjoint graphs.

Definition 1.1 ([18]) The subdivision-vertex neighbourhood corona of G_1 and G_2, denoted by $G_1 \boxdot G_2$ is the graph obtained from $S(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex-disjoint, and joining the neighbours of the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2.

Definition 1.2 ([18]) The subdivision-edge neighbourhood corona of G_1 and G_2, denoted by $G_1 \boxminus G_2$ is the graph obtained from $S(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex-disjoint, and joining the neighbours of the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2.

Bu et al. investigated resistance distance in subdivision-vertex join and subdivision-edge join of graphs [13]. Liu et al. [9] gave the resistance distance and Kirchhoff index of R-vertex join and R-edge join of two graphs. Liu et al. [19] gave the Laplacian generalized inverse of subdivision-vertex and subdivision-edge neighbourhood corona. Motivated by this work, in this paper, we further investigate the resistance distances and Kirchhoff index of $G_1 \boxdot G_2$ and $G_1 \boxminus G_2$. Compared with the paper [19], we compute the resistance distance of $G_1 \boxdot G_2$ and $G_1 \boxminus G_2$ in a much simpler way. Furthermore, we show that $Kf(G_1 \boxdot G_2)$ and $Kf(G_1 \boxminus G_2)$ can be expressed as a combination of $Kf(G_1)$ and $Kf(G_2)$.

2 Preliminaries

At the beginning of this section, we review some concepts in matrix theory. The Kronecker product of matrices $A = (a_{ij})$ and B, denoted by $A \otimes B$, is defined to be the partition matrix $(a_{ij}B)$. See [16]. In cases where each multiplication makes sense, we have$M_1M_2 \otimes M_3M_4 = (M_1 \otimes M_3)(M_2 \otimes M_4)$.

For a square matrix M, the group inverse of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is known that $M^{\#}$ exists if and only if $rank(M) = rank(M^2)$ ([12], [10]). If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric {1}- inverse of M. Actually, $M^{\#}$ is equal to the Moore-Penrose inverse of M since M is symmetric [12].

Lemma 2.1 ([11], [12]) Let G be a connected graph. Then

$$\begin{aligned} r_{uv}(G) &= (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} \\ &= (L_G^{\#})_{uu} + (L_G^{\#})_{vv} - 2(L_G^{\#})_{uv}. \end{aligned}$$

Let 1_n denotes the column vector of dimension n with all the entries equal one. We will often use 1 to denote an all-ones column vector if the dimension can be read from the context.

Lemma 2.2 ([13]) For any graph, we have $L_G^{\#}1 = 0$.

For a vertex i of a graph G, let $T(i)$ denote the set of all neighbors of i in G.

Lemma 2.3 ([13]) Let G be a connected graph. For any $i, j \in V(G)$,

$$r_{ij}(G) = d_i^{-1}(1 + \sum_{k \in T(i)} r_{kj}(G) - d_i^{-1} \sum_{k,l \in T(i)} r_{kl}(G)).$$

For a square matrix M, let $tr(M)$ denote the trace of M.

Lemma 2.4 ([14]) Let G be a connected graph on n vertices. Then

$$Kf(G) = ntr(L_G^{(1)}) - 1^T L_G^{(1)} 1 = ntr(L_G^{\#}).$$

Lemma 2.5 ([15]) Let G be a connected r-regular graph of order n, let $l(G)$ denote the line of a graph G. Then

$$Kf(l(G)) = \frac{r}{2} Kf(G) + \frac{(r-2)n^2}{8}.$$

Lemma 2.6 Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be the Laplacian matrix of a connected graph. If D is nonsingular, then

$$X = \begin{pmatrix} H^{\#} & -H^{\#}BD^{-1} \\ -D^{-1}B^TH^{\#} & D^{-1}+D^{-1}B^TH^{\#}BD^{-1} \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of L, where $H = A - BD^{-1}B^T$.

Proof Since $H = A - BD^{-1}B^T$ is symmetric, $H^{\#}$ exists and is symmetric. Since

$$L = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix},$$

we know that

$$X = \begin{pmatrix} I & 0 \\ -D^{-1}B^T & I \end{pmatrix} \begin{pmatrix} H^{\#} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}$$

is a symmetric $\{1\}$- inverse of L.

Remarks: The above result is similar to Lemma 2.8 in [14], which can be viewed as another form of Lemma 2.8.

3 Resistance distance in $G_1 \boxdot G_2$ and $G_1 \boxminus G_2$

Theorem 3.1 Let G_1 be an r_1- regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices and m_2 edges. Then the following holds:

(a) For any $e, f \in E(G_1)$, we have

$$r_{ef}(G_1 \boxdot_Q G_2) = \frac{r_1}{1+n_2} r_{ef}(l(G_1)).$$

(b) For any $u, v \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G_1 \boxdot G_2) &= (I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})^{-1})_{ii} + (I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})^{-1})_{jj} \\ &\quad -2(I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})^{-1})_{ij}. \end{aligned}$$

(c) For any $u \in V(G_1)$, let $e_1, e_2, ..., e_r$ be r edges incident to u in G_1. For any $u \in V(G_1), f \in E(G_1)$, we have

$$\begin{aligned} r_{uf}(G_1 \boxdot G_2) &= \left(\frac{1}{r_1} I_{n_1} + \frac{1}{r_1(1+n_2)} R L^{\#}_{l(G_1)} R^T\right)_{uu} + \left(\frac{r_1}{1+n_2} L^{\#}_{l(G_1)}\right)_{ff} \\ &\quad -\frac{2}{1+n_2} (R L^{\#}_{l(G_1)})_{uf}. \end{aligned}$$

(d) For any $u, v \in V(G_1)uv$, let $e_1, e_2, ..., e_r (resp. f_1, f_2, ..., f_r)$ be r edges incident to $u(resp. v)$ in G_1, we have

$$\begin{aligned} r_{ij}(G_1 \boxdot G_2) &= \frac{2}{r_1} + \frac{1}{r_1(n_2+1)}(RL^{\#}_{l(G_1)}R^T)_{uu} + \frac{1}{r_1(n_2+1)} \\ & (RL^{\#}_{l(G_1)}R^T)_{vv} - \frac{2}{r_1(n_2+1)}(RL^{\#}_{l(G_1)}R^T)_{uv}. \end{aligned}$$

(e) For any $u \in V(G_1)$, let $e_1, e_2, ..., e_r$ be r edges incident to u in G_1. For any $u \in V(G_1)$, $v \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G_1 \boxdot G_2) &= \frac{1}{r_1} + \frac{1}{r_1(n_2+1)}(RL^{\#}_{l(G_1)}R^T)_{uu} + (I_{n_1} \otimes (L_{G_2} + \\ & r_1 I_{n_2})^{-1})_{vv} - 2(I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})^{-1})_{uv}. \end{aligned}$$

(f) For any $e \in E(G_1)$, $v \in V(G_2)$, we have

$$\begin{aligned} r_{ev}(G_1 \boxdot G_2) &= \frac{r_1}{1+n_2}(L^{\#}_{l(G_1)})_{ee} + (I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})^{-1})_{vv} \\ & -\frac{2}{1+n_2}(RL^{\#}_{l(G_1)})_{ev}. \end{aligned}$$

Proof Let R be the incidence matrix of G_1. Then with a proper labeling of vertices , the Laplacian matrix of $G_1 \boxdot G_2$ can be written as

$$L(G_1 \boxdot G_2) = \left(\begin{array}{c|cc} (2+2n_2)I_{m_1} & -R^T & -R^T \otimes 1^T_{n_2} \\ \hline -R & r_1 I_{n_1} & 0_{n_1 \times n_1} \otimes 1^T_{n_2} \\ -R \otimes 1_{n_2} & 0_{n_1 \times n_1} \otimes 1_{n_2} & I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2}) \end{array} \right)$$

where $0_{s,t}$ denotes the $s \times t$ matrix with all entries equal to zero.

By Lemma 2.6, we are ready to calculate H.

Let $K = R^T \otimes 1^T_{n_2}$, $Q = I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})$, then

$H = (2+2n_2)I_{m_1} -$

$$\begin{aligned} & \begin{pmatrix} -R^T & -K \end{pmatrix} \begin{pmatrix} r_1 I_{n_1} & 0_{n_1 \times n_1} \otimes 1^T_{n_2} \\ 0_{n_1 \times n_1} \otimes 1_{n_2} & Q \end{pmatrix}^{-1} \begin{pmatrix} -R \\ -K^T \end{pmatrix} \\ = & \frac{1+n_2}{r_1} L_{l(G_1)}, \end{aligned}$$

so, we have $H^{\#} = \frac{r_1}{1+n_2} L^{\#}_{l(G_1)}$.

According to Lemma 2.6, we calculate $-H^{\#}BD^{-1}$ and $-D^{-1}B^T H^{\#}$.

$-H^{\#}BD^{-1}$

$$\begin{aligned} &= -H^{\#} \begin{pmatrix} -R^T & -K \end{pmatrix} \begin{pmatrix} \frac{1}{r_1} I_{n_1} & 0_{n_1 \times n_1} \otimes 1^T_{n_2} \\ 0_{n_1 \times n_1} \otimes 1_{n_2} & Q^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{r_1} H^{\#} R^T & \frac{1}{r_1} H^{\#} (R^T \otimes 1^T_{n_2}) \end{pmatrix} \end{aligned}$$

and

$$-D^{-1}B^TH^{\#} = (-H^{\#}BD^{-1})^T = \begin{pmatrix} \frac{1}{r_1}RH^{\#} \\ \frac{1}{r_1}(R\otimes 1_{n_2})H^{\#} \end{pmatrix}.$$

Next we are ready to compute the $D^{-1}B^TH^{\#}BD^{-1}$.

$$\begin{aligned} &D^{-1}B^TH^{\#}BD^{-1} \\ = &\begin{pmatrix} -\frac{1}{r_1}RH^{\#} \\ -\frac{1}{r_1}K^TH^{\#} \end{pmatrix}\begin{pmatrix} -R^T & -K \end{pmatrix} \\ &\begin{pmatrix} \frac{1}{r_1}I_{n_1} & 0_{n_1\times n_1}\otimes 1_{n_2}^T \\ 0_{n_1\times n_1}\otimes 1_{n_2} & Q^{-1} \end{pmatrix} \\ = &\begin{pmatrix} \frac{1}{r_1^2}RH^{\#}R^T & \frac{1}{r_1^2}RH^{\#}(R^T\otimes 1_{n_2}^T) \\ \frac{1}{r_1^2}(R\otimes 1_{n_2})H^{\#}R^T & \frac{1}{r_1^2}(R\otimes 1_{n_2})H^{\#}(R^T\otimes 1_{n_2}^T) \end{pmatrix}. \end{aligned}$$

Let $P = L_{l(G_1)}^{\#}$, based on Lemma 2.6, the following matrix

$N =$

$$\begin{pmatrix} \frac{r_1}{1+n_2}P & \frac{1}{1+n_2}PR^T & \frac{1}{1+n_2}PK \\ \frac{1}{1+n_2}RP & \frac{1}{r_1}I_{n_1} + \frac{1}{r_1(1+n_2)}RPR^T & \frac{1}{r_1(1+n_2)}RPK \\ \frac{1}{1+n_2}K^TP & \frac{1}{r_1(1+n_2)}K^TPR^T & Q^{-1} + \frac{1}{r_1(1+n_2)}K^TPK \end{pmatrix} \quad (1)$$

is a symmetric $\{1\}$- inverse of $L(G_1 \Box G_2)$, where $Q = I_{n_1}\otimes(L_{G_2} + r_1I_{n_2})$, $K = R^T\otimes 1_{n_2}^T$.

For any $e, f \in E(G_1)$, by Lemma 2.1 and the Equation (1), we have

$$r_{ef}(G_1 \Box_Q G_2) = \frac{r_1}{1+n_2}r_{ef}(l(G_1)).$$

For any $u, v \in V(G_2)$, by Lemma 2.1 and the Equation (1), we have

$$\begin{aligned} r_{ij}(G_1 \Box G_2) = &(I_{n_1}\otimes(L_{G_2}+r_1I_{n_2})^{-1})_{ii} + (I_{n_1}\otimes(L_{G_2}+r_1I_{n_2})^{-1})_{jj} \\ &-2(I_{n_1}\otimes(L_{G_2}+r_1I_{n_2})^{-1})_{ij}. \end{aligned}$$

For any $u \in V(G_1)$, let $e_1, e_2, ..., e_r$ be r edges incident to u in G_1. For any $u \in V(G_1), f \in E(G_1)$, by Lemma 2.1 and the Equation (1), we have

$$\begin{aligned} r_{uf}(G_1 \Box G_2) = &(\frac{1}{r_1}I_{n_1} + \frac{1}{r_1(1+n_2)}RL_{l(G_1)}^{\#}R^T)_{uu} + (\frac{r_1}{1+n_2}L_{l(G_1)}^{\#})_{ff} \\ &-\frac{2}{1+n_2}(RL_{l(G_1)}^{\#})_{uf}. \end{aligned}$$

For any $u, v \in V(G_1)(u \neq v)$, let $e_1, e_2, ..., e_r(resp. f_1, f_2, ..., f_r)$ be r edges incident to $u(resp. v)$ in G_1, by Lemma 2.1 and the Equation (1), we have

$$\begin{aligned} r_{ij}(G_1 \Box G_2) = &\frac{2}{r_1} + \frac{1}{r_1(n_2+1)}(RL_{l(G_1)}^{\#}R^T)_{uu} + \frac{1}{r_1(n_2+1)} \\ &(RL_{l(G_1)}^{\#}R^T)_{vv} - \frac{2}{r_1(n_2+1)}(RL_{l(G_1)}^{\#}R^T)_{uv}. \end{aligned}$$

For any $u \in V(G_1)$, let $e_1, e_2, ..., e_r$ be r edges incident to u in G_1. For any $u \in V(G_1)$, $v \in V(G_2)$, by Lemma 2.1 and the Equation (1), we have

$$\begin{aligned} r_{ij}(G_1 \boxdot G_2) &= \frac{1}{r_1} + \frac{1}{r_1(n_2+1)}(RL^{\#}_{l(G_1)}R^T)_{uu} + (I_{n_1} \otimes \\ &\quad (L_{G_2} + r_1 I_{n_2})^{-1})_{vv} - 2(I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})^{-1})_{uv}. \end{aligned}$$

For any $e \in E(G_1)$, $v \in V(G_2)$, by Lemma 2.1 and the Equation (1), we have

$$\begin{aligned} r_{ev}(G_1 \boxdot G_2) &= \frac{r_1}{1+n_2}(L^{\#}_{l(G_1)})_{ee} + (I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})^{-1})_{vv} \\ &\quad - \frac{2}{1+n_2}(RL^{\#}_{l(G_1)})_{ev}. \end{aligned}$$

Next we will give the formulae for resistance distance in $G_1 \boxminus G_2$ as follows.

Theorem 3.2 Let G_1 be a graph on n_1 vertices and m_1 edges and G_2 be a graph on n_2 vertices and m_2 edges. Then the following holds:

(a) For any $i, j \in V(G_1)$, we have

$$r_{ij}(G_1 \boxminus G_2) = \frac{2}{1+n_2} r_{ij}(G_1).$$

(b) For any $i, j \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G_1 \boxminus G_2) &= (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})_{ii} + (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})_{jj} \\ &\quad -2(I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})_{ij}. \end{aligned}$$

(c) For any $i \in V(G_1)$, $j \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G_1 \boxminus G_2) &= \frac{2}{1+n_2}(L^{\#}_{G_1})_{ii} + (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})_{jj} \\ &\quad - \frac{1}{2(1+n_2)}((R^T \otimes 1_{n_2}) L^{\#}_{G_1} (R \otimes 1^T_{n_2}))_{ij}. \end{aligned}$$

(d) For any $i = uv \in E(G_1)$, $j \in V(G_1) \cup V(G_2)$, we have

$$\begin{aligned} r_{ij}(G_1 \boxminus G_2) &= \frac{1}{2} + \frac{1}{2} r_{u_i j}(G_1 \boxminus G_2) + \frac{1}{2} r_{v_i j}(G_1 \boxminus G_2) \\ &\quad - \frac{1}{4} r_{u_i v_i}(G_1 \boxminus G_2). \end{aligned}$$

(e) For any $i = u_1 v_1, j = u_2 v_2 \in E(G_1) (i \neq j)$, we have

$$\begin{aligned} r_{ij}(G_1 \boxminus G_2) &= 1 + \frac{1}{4}\left(r_{u_i u_j}(G_1 \boxminus G_2) + r_{u_i v_j}(G_1 \boxminus G_2) + r_{v_i u_j}\right. \\ &\quad (G_1 \boxminus G_2) + r_{v_i v_j}(G_1 \boxminus G_2) - r_{u_i v_i}(G_1 \boxminus G_2) \\ &\quad \left. - r_{u_j v_j}(G_1 \boxminus G_2)\right). \end{aligned}$$

Proof Let R be the incidence matrix of G_1. Then with a proper labeling of vertices , the Laplacian matrix of $G_1 \boxminus G_2$ can be written as

$$L(G_1 \boxminus G_2) = \left(\begin{array}{c|cc} D_{G_1} + n_2 D_{G_1} & -R & -R \otimes 1_{n_2}^T \\ \hline -R^T & 2I_{m_1} & 0_{m_1 \times m_1} \otimes 1_{n_2}^T \\ -R^T \otimes 1_{n_2} & 0_{m_1 \times m_1} \otimes 1_{n_2} & I_{m_1} \otimes (L_{G_2} + 2I_{n_2}) \end{array} \right),$$

where $0_{s,t}$ denotes the $s \times t$ matrix with all entries equal to zero.

By Lemma 2.6, we are ready to calculate H.

Let $K = R^T \otimes 1_{n_2}$, $M = I_{m_1} \otimes (L_{G_2} + 2I_{n_2})$, then

$H = D_{G_1} + n_2 D_{G_1} -$

$$\begin{pmatrix} -R & -K^T \end{pmatrix} \begin{pmatrix} 2I_{m_1} & 0_{m_1 \times m_1} \otimes 1_{n_2}^T \\ 0_{m_1 \times m_1} \otimes 1_{n_2} & M \end{pmatrix}^{-1} \begin{pmatrix} -R^T \\ -K \end{pmatrix} = \frac{1+n_2}{2} L_{G_1},$$

so, we have $H^{\#} = \frac{2}{1+n_2} L_{G_1}^{\#}$.

According to Lemma 2.6, we compute $-H^{\#}BD^{-1}$ and $-D^{-1}B^T H^{\#}$.

$-H^{\#}BD^{-1}$

$$= -H^{\#} \begin{pmatrix} -R & -K \end{pmatrix} \begin{pmatrix} \frac{1}{2}I_{m_1} & 0_{m_1 \times m_1} \otimes 1_{n_2}^T \\ 0_{m_1 \times m_1} \otimes 1_{n_2} & M^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}H^{\#}R & \frac{1}{2}H^{\#}(R \otimes 1_{n_2}^T) \end{pmatrix}$$

and

$$-D^{-1}B^T H^{\#} = (-H^{\#}BD^{-1})^T = \begin{pmatrix} \frac{1}{2}R^T H^{\#} \\ \frac{1}{2}(R^T \otimes 1_{n_2})H^{\#} \end{pmatrix}.$$

Next we are ready to compute the $D^{-1}B^T H^{\#} BD^{-1}$.

$D^{-1}B^T H^{\#} BD^{-1}$

$$= \frac{\begin{pmatrix} -\frac{1}{2}R^T H^{\#} \\ -\frac{1}{2}KH^{\#} \end{pmatrix} \begin{pmatrix} -R & -K^T \end{pmatrix}}{\begin{pmatrix} \frac{1}{2}I_{m_1} & 0_{m_1 \times m_1} \otimes 1_{n_2}^T \\ 0_{m_1 \times m_1} \otimes 1_{n_2} & -1 \end{pmatrix}} = \begin{pmatrix} \frac{1}{4}R^T H^{\#} R & \frac{1}{4}R^T H^{\#} K^T \\ \frac{1}{4}KH^{\#}R & \frac{1}{4}KH^{\#}K^T \end{pmatrix}.$$

Let $P = L_{l(G_1)}^{\#}$, based on Lemma 2.6, the following matrix

$N =$

$$\begin{pmatrix} \frac{2}{1+n_2}P & \frac{1}{1+n_2}PR & \frac{1}{1+n_2}PK^T \\ \frac{1}{1+n_2}R^T P & \frac{1}{2}I_{m_1} + \frac{1}{2(1+n_2)}R^T PR & \frac{1}{2(1+n_2)}R^T PK^T \\ \frac{1}{1+n_2}KP & \frac{1}{2(1+n_2)}KPR & M^{-1} + \frac{1}{2(1+n_2)}KPK^T \end{pmatrix} \quad (2)$$

is a symmetric {1}- inverse of $L(G_1 \boxminus G_2)$, where $K = R^T \otimes 1_{n_2}$, $M = I_{m_1} \otimes (L_{G_2} + 2I_{n_2})$.

For any $i, j \in V(G_1)$, by Lemma 2.1 and the Equation (2), we have

$$r_{ij}(G_1 \boxminus G_2) = \frac{2}{1+n_2} r_{ij}(G_1),$$

as stated in (a).

For any $i, j \in V(G_2)$, by Lemma 2.1 and the Equation (2), we have

$$\begin{aligned} r_{ij}(G_1 \boxminus G_2) &= \left(I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1}\right)_{ii} + (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})_{jj} \\ &\quad -2(I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})_{ij}, \end{aligned}$$

as stated in (b).

For any $i \in V(G_1)$, $j \in V(G_2)$, by Lemma 2.1 and the Equation (2), we have

$$\begin{aligned} r_{ij}(G_1 \boxminus G_2) &= \frac{2}{1+n_2}(L_{G_1}^{\#})_{ii} + (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})_{jj} \\ &\quad -\frac{1}{1+n_2}(K L_{G_1}^{\#} K^T)_{ij}, \end{aligned}$$

as stated in (c).

For any $i = uv \in E(G_1)$, $j \in V(G_1) \cup V(G_2)$, by Lemma 2.3, we have

$$\begin{aligned} r_{ij}(G_1 \boxminus G_2) &= \frac{1}{2} + \frac{1}{2} r_{u_i j}(G_1 \boxminus G_2) + \frac{1}{2} r_{v_i j}(G_1 \boxminus G_2) \\ &\quad -\frac{1}{4} r_{u_i v_i}(G_1 \boxminus G_2), \end{aligned}$$

as stated in (d).

For any $i = u_1 v_1, j = u_2 v_2 \in E(G_1) (i \neq j)$, by Lemma 2.3, we have $r_{ij}(G_1 \boxminus G_2)$

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{2} r_{u_i j}(G_1 \boxminus G_2) + \frac{1}{2} r_{v_i j}(G_1 \boxminus G_2) - \frac{1}{4} r_{u_i v_i}(G_1 \boxminus G_2) \\ &= 1 + \frac{1}{4}\left(r_{u_i u_j}(G_1 \boxminus G_2) + r_{u_i v_j}(G_1 \boxminus G_2) + r_{v_i u_j}(G_1 \boxminus G_2)\right. \\ &\quad \left. + r_{v_i v_j}(G_1 \boxminus G_2) - r_{u_i v_i}(G_1 \boxminus G_2) - r_{u_j v_j}(G_1 \boxminus G_2)\right), \end{aligned}$$

as stated in (e).

4 Kirchhoff index in $G_1 \boxdot G_2$ and $G_1 \boxminus G_2$

Theorem 4.1 Let G_1 be an r_1- regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices and m_2 edges. Then

$$
\begin{aligned}
&Kf(G_1 \square G_2) \\
&= (n_1 + m_1 + n_1 n_2)\left(\frac{r_1^2}{2m_1(1+n_2)} Kf(G_1) + \frac{1}{r_1(1+n_2)} tr\left(RL^{\#}_{l(G_1)}R^T\right)\right. \\
&\quad + n_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2) + r_1} + \frac{1}{r_1(1+n_2)} tr\left[(R \otimes 1_{n_2}) L^{\#}_{l(G_1)} (R^T \otimes 1^T_{n_2})\right] + \\
&\quad \left.\frac{n_1(n_1 r_1^2 - 2n_1 r_1 + 8m_1 + 8m_1 n_2}{8m_1 r_1(1+n_2)}\right) - \frac{n_1 + n_1 n_2}{r_1}.
\end{aligned}
$$

Proof Let $L^{(1)}_{G_1 \square G_2}$ be the symmetric $\{1\}$-inverse of $L_{G_1 \square G_2}$. Then

$$
\begin{aligned}
&tr\left(L^{(1)}_{G_1 \square G_2}\right) \\
&= \frac{r_1}{1+n_2} tr\left(L^{\#}_{l(G_1)}\right) + tr\left(\frac{1}{r_1} I_{n_1} + \frac{1}{r_1(1+n_2)} RL^{\#}_{l(G_1)}R^T\right) \\
&\quad + tr\left(I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})^{-1}\right) + \\
&\quad \frac{1}{r_1(1+n_2)} tr\left[(R \otimes 1_{n_2}) L^{\#}_{l(G_1)} (R^T \otimes 1^T_{n_2})\right].
\end{aligned}
$$

Note that the eigenvalues of $(L_{G_2} + r_1 I_{n_2})$ are $\mu_1(G_2) + r_1, \mu_2(G_2) + r_1, \ldots, \mu_{n_2}(G_2) + r_1$. Then

$$
tr\left(I_{n_1} \otimes (L_{G_2} + r_1 I_{n_2})\right)^{-1} = n_1 \sum_{i=1}^{n_2} (\mu_i(G_2) + r_1)^{-1} = n_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2) + r_1}.
$$

By Lemma 2.5, we have

$$
\begin{aligned}
&tr(L^{(1)}_{G_1 \square G_2}) \\
&= \frac{r_1}{m_1(1+n_2)}\left(\frac{r_1}{2} Kf(G_1) + \frac{(r_1 - 2)n_1^2}{8}\right) + \frac{n_1}{r_1} + \frac{1}{r_1(1+n_2)} \\
&\quad tr\left(RL^{\#}_{l(G_1)}R^T\right) + n_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2) + r_1} + \frac{1}{r_1(1+n_2)} \\
&\quad tr\left[(R \otimes 1_{n_2}) L^{\#}_{l(G_1)} (R^T \otimes 1^T_{n_2})\right].
\end{aligned}
$$

Next, we calculate the $1^T(L^{(1)}_{G_1 \square G_2})1$. By Lemma 2.1, we have

$$
\begin{aligned}
&1^T(L^{(1)}_{G_1 \square G_2})1 \\
&= 1^T\left(\frac{1}{r_1} I_{n_1} + \frac{1}{r_1(1+n_2)} RL^{\#}_{l(G_1)}R^T\right) 1 + \frac{1}{r_1(1+n_2)} 1^T RL^{\#}_{l(G_1)} \\
&\quad (R^T \otimes 1^T_{n_2})1 + \frac{1}{r_1(1+n_2)} 1^T (R \otimes 1_{n_2}) L^{\#}_{l(G_1)} R^T 1 + 1^T (I_{n_1}
\end{aligned}
$$

$$\otimes(L_{G_2}+I_{n_2})^{-1})\,1+\frac{1}{r_1(1+n_2)}1^T(R\otimes 1_{n_2})L^{\#}_{l(G_1)}\left(R^T\otimes 1^T_{n_2}\right)1.$$

Note that $1^T R = R^T 1 = 2\cdot 1$, then

$$1^T R L^{\#}_{l(G_1)} R^T 1 = 1^T(R\otimes 1_{n_2})L^{\#}_{l(G_1)} R^T 1 = 1^T R L^{\#}_{l(G_1)}(R^T\otimes 1_{n_2})1 = 0. \quad (3)$$

According to the operation of Kronecker product, we have

$$1^T(R\otimes 1_{n_2})L^{\#}_{l(G_1)}(R^T\otimes 1^T_{n_2})1 = (n_2 r_1)^2 1^T L^{\#}_{l(G_1)}1 = 0. \quad (4)$$

Let $T = 1^T_{n_1 n_2}(I_{n_1}\otimes(L_{G_2}+r_1 I_{n_2})^{-1})1_{n_1 n_2}$, $Q = I_{n_1}\otimes(L_{G_2}+r_1 I_{n_2})$, then

$$\begin{aligned} T &= \begin{pmatrix} 1^T_{n_2} & 1^T_{n_2} & \cdots & 1^T_{n_2}\end{pmatrix}\begin{pmatrix} Q^{-1} & & & \\ & Q^{-1} & & \\ & & \ddots & \\ & & & Q^{-1}\end{pmatrix}\begin{pmatrix} 1_{n_2} \\ 1_{n_2} \\ \cdots \\ 1_{n_2}\end{pmatrix} \\ &= n_1 1^T_{n_2}(L_{G_2}+r_1 I_{n_2})^{-1}1_{n_2} = \frac{n_1 n_2}{r_1} \quad (5)\end{aligned}$$

Plugging (3), (4) and (5) into $1^T(L^{(1)}_{G_1\Box G_2})1$, we get

$$1^T(L^{(1)}_{G_1\Box G_2})1 = \frac{n_1+n_1 n_2}{r_1}.$$

Lemma 2.4 implies that

$$Kf(L^{(1)}_{G_1\Box G_2}) = (n_1+m_1+n_1 n_2)tr(L^{\#}_{G_1\Box G_2}) - 1^T(L^{(1)}_{G_1\Box G_2})1.$$

Then plugging $tr(L^{(1)}_{G_1\Box G_2})$ and $1^T(L^{(1)}_{G_1\Box G_2})1$ into the equation above, we obtain the required result.

Theorem 4.2 Let G_1 be a graph on n_1 vertices and m_1 edges and G_2 be a graph on n_2 vertices and m_2 edges. Then
$Kf(G_1 \boxminus G_2)$

$$\begin{aligned} &= (n_1+m_1+m_1 n_2)\left(\frac{2}{n_1(1+n_2)}Kf(G_1)+\frac{1}{(1+n_2)}tr\left(D_{G_1}L^{\#}_{G_1}\right)\right. \\ &\quad +m_1\sum_{i=1}^{n_2}\frac{1}{\mu_i(G_2)+2}+\frac{1}{2(1+n_2)}tr\left((R\otimes 1^T_{n_2})L^{\#}_{G_1}(R^T\otimes 1_{n_2})\right) - \\ &\quad \left.\frac{m_1+m_1 n_2-n_1+1}{2(1+n_2)}\right) - \frac{n_2+1}{2}\pi^T L^{\#}_{G_1}\pi - \frac{m_1+m_1 n_2}{2},\end{aligned}$$

where $\pi = (d_1, d_2, \cdots, d_{n_2})^T$.

Proof The proof is similar to those of Theorems 4.1, omitted.

Corollary 4.3 Let G_1 be an r_1- regular graph on n_1 vertices and m_1 edges and G_2 be a graph on n_2 vertices and m_2 edges. Then

$$\begin{aligned} & Kf(G_1 \boxminus G_2) \\ = & (n_1 + m_1 + m_1 n_2)\left(\frac{2+r_1}{n_1(1+n_2)} Kf(G_1) + m_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2)+2} + \right. \\ & \left. \frac{1}{2(1+n_2)} tr\left((R \otimes 1_{n_2}^T) L_{G_1}^{\#} (R^T \otimes 1_{n_2})\right) - \frac{m_1 + m_1 n_2 - n_1 + 1}{2(1+n_2)}\right) \\ & - \frac{m_1 + m_1 n_2}{2}. \end{aligned}$$

Proof Since $R \cdot 1 = r_1 1$ and $L_G^{\#} 1 = 0$, then $tr(D_{G_1} L_{G_1}^{\#}) = r_1 tr(L_{G_1}^{\#}) = \frac{r_1}{n_1} Kf(G_1)$ and $\pi^T L_{G_1}^{\#} \pi = r_1^2 1^T L_{G_1}^{\#} 1 = 0$. By Theorem 4.2, the required result is obtained.

Remark: In [18], the authors investigated the Laplacian spectrum of the subdivision-vertex and the subdivision-edge neighbourhood corona of G_1 and G_2 when G_1 is an r-regular graph. Morever, the expression of the Kirchhoff index is complex according to the definition of Kirchhoff index on eigenvalues. Thus we think the proposed method is better than that in [18].

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