SUBGRAPHS AND SIMILARITY OF VERTICES

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ABSTRACT

When G and F are graphs, $v \in V(G)$ and φ is an orbit of V(F) under the action of the automorphism group of F, *s(F,G,v,φ)* denotes the number of induced subgraphs of G isomorphic to F such that v lies in orbit φ of F. Vertices $v \in V(G)$ and $w \in V(H)$ are called *k-vertex subgraph equivalent* (k-SE), $2 \leq k < n = |V(G)|$ if for each graph F with k vertices and for every orbit φ of F, $s(F,G,v,\varphi) = s(F,H,w,\varphi)$, and they are called similar if there is an isomorphism from G to H taking v to w. We prove that k-SE vertices are (k−1)-SE and several parameters of (n−1)-SE vertices are equal. It is also proved that in many situations, "(n−1)-SE between vertices is equivalent to their similarity" and it is true always if and only if Ulam's Graph Reconstruction Conjecture is true.

1. INTRODUCTION

We follow the terminology in Harary [7] . Throughout this paper, G stands for a graph on $n \geq 4$ vertices unless stated otherwise. "There is an isomorphism from graph G to graph H taking vertex $v \in V(G)$ to $w \in V(H)$" means that "the graph G *looked at from* v is same as the graph H looked at from w". This is studied using the way in which their vertex proper subgraphs on k vertices are intersecting at v. We abbreviate $v \in V(G)$ as $v \in G$.

1.1. Definition. For a vertex u of a graph G, the *orbit of u* is the subset of V(G) consisting of the images of u under all automorphisms of G. It is also called the *orbit of G* containing u.

Orbits of G partition V(G). When H is a graph isomorphic to G, each isomorphism from G to H maps orbits of G to orbits of H. All the isomorphisms from G to H induce the same mapping from the set of orbits of G to the set of orbits of H. The image of an orbit φ of G under this mapping is called the *orbit φ of H.* When G and F are graphs, $v \in G$ and φ is an orbit of F, *s(F,G,v,φ)* denotes the number of induced subgraphs J of G isomorphic to F such that v is in orbit φ of J.

1.2. Example. Let G be a cycle of length six. Label a vertex of G as v. Let φ denote the orbit of the graph P_3 (the path with three vertices) that contain its two end vertices. There are six induced subgraphs of G isomorphic to P_3. Three of them contain the vertex v. But v occurs as a vertex of orbit φ of P_3 in exactly two of these and hence $s(P_3,G,v,\varphi) = 2$.

1.3. Definition. When G and H are any two graphs, vertices $v \in G$ and $w \in H$ are called *similar* if there exists an isomorphism from G to H taking v to w. Vertices $v \in$

G and $w \in H$ are called *k- vertex subgraph equivalent* (k-SE), $2 \leq k \leq n-1$, if $s(F,G,v,\varphi) = s(F,H,w,\varphi)$ for each graph F with k vertices and for all orbits φ of F; (n−1)-vertex subgraph equivalent vertices are simply called *subgraph equivalent* (or *SE*).

1.4. Definition ([6]). *Deck*(G) is the collection (multiset) of unlabeled graphs that result from deleting one vertex in every possible way from the graph G. The elements of a *deck* are referred to as *cards*. A graph H is called a *hypomorph* of G if *Deck*(H) = *Deck*(G). A graph is called *reconstructible* if it is fixed uniquely (up to isomorphism) by its deck. (i.e., if it is isomorphic to all its hypomorphs).

Ulam's Graph Reconstruction Conjecture (URC): All graphs on at least three vertices are reconstructible. (See [2] for a survey).

The main purpose of this paper is to investigate whether "SE between $v \in G$ and $w \in H$" is equivalent to "the similarity between v and w". We prove that it is true in many situations including the following.

(i) G is regular.

(ii) One among G and G^C is disconnected with positive degree for v. (G^C denotes the complement of G).

(iii) v is a cut vertex of G or G^C.

(iv) A specific 2-vertex coloring of G−v is reconstructible (Theorem 4.10).

(v) G−v is either disconnected or regular or a tree or unicyclic or separable without endvertices.

We also prove that it is true for all graphs if and only if URC is true. Moreover, $v \in G$ and $w \in H$ are SE implies $G \cong H$ when $n \leq 11$. A "deck" form of k-SE is given and it is proved that k-SE vertices are (k−1)-SE when $3 \leq k \leq n-1$. We define a subclass of reconstructible graphs called *strongly reconstructible graphs* (Definition 4.21) and use them to prove some results on SE. This study is expected to give new insights on graph isomorphism and Ulam's Reconstruction Conjecture. Isomorphisms between the (n−1)-vertex subgraphs of G and H and their relation to the existence of isomorphisms from G to H are studied in [9] and [15] also.

2. FUNDAMENTAL RESULTS

A graph G in which a single vertex v is labeled is denoted by (G,v). (G,v) is said to be isomorphic to (H,w) and written $(G,v) \cong (H,w)$ if there is an isomorphism from G to H taking v to w. Study of k-SE between vertices $v \in G$ and $w \in H$ becomes easier with the concepts "k-vertex card" and "k-vertex deck" of G. If v is a vertex of G, then the k-vertex cards of G at v are the subgraphs G−W, where W are (n−k)-subsets of V(G)−{v}, with v labeled in each; k-vertex deck of G at v is the collection of k-vertex cards of G at v.

2.1. Definition. Let G be a graph on n vertices in which a single vertex v is labeled and others are unlabeled. For each k, $2 \leq k \leq n-1$, consider the k-subsets of V(G), each containing v. There are C(n−1,k−1) of them. The collection (multiset) of k-subgraphs of G induced by these k-subsets of V(G) in each of which the label v is retained, is called the *k-vertex deck of G at v* and is denoted by (k,G,v). A member of (k,G,v) is called a *k- vertex card of G at v.*

2.2. Example. For the graphs G and H in Figure 1, (3,G,v) = (3,H,w). (We write (k,G,v) = (k,H,w) if there is a pairing between the cards in (k,G,v) and (k,H,w) such that cards in each pair are are isomorphic as graphs rooted at the label).

2.3. Observation. "$v \in G$ and $w \in H$ are k-SE" and "(k,G,v) = (k,H,w)" are two ways of representing the same thing, since $s(F,G,v,\varphi)$ is the number of cards in (k,G,v) isomorphic to F in which v occurs as a vertex of orbit φ of F. □

The graph obtained from G by replacing edges incident with a chosen vertex v by nonedges and nonedges incident with v by edges is called the *switching* of G at v and is denoted by G_v.

2.4. Lemma. The statements "$v \in G$ and $w \in H$ are k-SE", "$v \in G^C$ and $w \in H^C$ are k-SE" and "$v \in G_v$ and $w \in H_w$ are k-SE" are equivalent. □

2.5. Theorem. For $3 \leq k \leq n-1$, k-SE vertices are (k–1)-SE.

Proof. We can derive (k–1,G,v) from (k,G,v). Now the theorem follows because of Observation 2.3. □

2.6. Example. For the 5-vertex graphs G and H in Figure 1, $v \in G$ and $w \in H$ are (n–2)-SE, but are not (n–1)-SE. Also $(G,v) \not\cong (H,w)$.

The following two results are proved using subgraph counting arguments.

2.7. Lemma. Let F be a graph with $|F| \leq n-1$ with one of its vertices labeled v_1.

(1) The number of subgraphs (induced subgraphs) of (G,v) isomorphic to (F,v_1) such that v_1 coincides with v can be found from (n–1,G,v).

(2) The number among them that contain u can also be found for each card (G,v)–u of (n–1,G,v) from (n–1,G,v). □

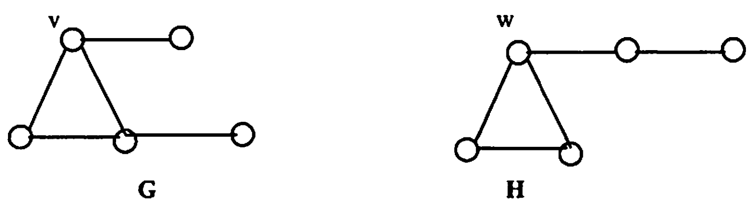


Figure 1. Graphs G and H with (n–2,G,v) = (n–2,H,w) but (n–1,G,v) ≠ (n–1,H,w) and $(G,v) \not\cong (H,w)$.

2.8. Corollary. Let G be a graph on $n \geq 4$ vertices and F be any graph with $2 \leq |F| \leq n-1$. The number of subgraphs (induced subgraphs) of G isomorphic to F of G having v as a vertex of a given orbit φ of F can be found from (n–1,G,v). The number of such subgraphs containing a vertex u of G can also be found for each card (G,v) – u in (n–1,G,v).

3. PROPERTIES COMMON FOR SUBGRAPH EQUIVALENT VERTICES

Properties of G and v that can be determined from (k,G,v) will be common for k-SE vertices.

3.1. Theorem. If $|G| = n \geq 4$ and $v \in V(G)$, the following can be determined from (n–1,G,v).

(i) Degree of v in G.

(ii) Degree sequence of G.

(iii) Neighborhood degree sequence of v.

(iv) For a card A of (n–1,G,v) with corresponding deleted vertex u, the degree of u, adjacency between u and v, the neighborhood degree sequence of u in G, and the degrees of the neighbors of u in G which are also adjacent to v.

Proof. Each edge incident with v occurs in (n–2) members of (n–1,G,v). Hence the sum of the degrees of v in the cards of (n–1,G,v) is (n–2)(d_G(v)), and so d_G(v) is known. Using similar arguments, (ii), (iii) and (iv) can be proved. □

3.2. Lemma. G is connected if and only if there is a connected card in (n–1,G,v) with the corresponding deleted vertex having positive degree. When G is a connected graph, v ∈ V(G) is a cutvertex if and only if at most one card in (n–1,G,v) gives a connected graph when vertex v is removed from it. □

3.3.Theorem. Connectivity κ(G) of G can be determined from (n–1,G,v). □

Distance is a very tricky and evasive concept in reconstruction. However, we have some good results on distance in subgraph equivalence.

3.4. Theorem. For each card (G,v)–u in (n–1,G,v), the distance d(v,u) in G is known from (n–1,G,v).

Proof. Let $n \geq 4$ and (n–1,G,v) be given. If $deg_G(v) = 0$, then d(v,u) is infinity. Hence let $deg_G(v) > 0$. Whether G is connected or not is known by Lemma 3.2. If G is disconnected, (G,v) itself can be determined from (n–1,G,v) (by first finding the component containing v and then other components) and comparing it with G–u, d(v,u) is known. Now let G be connected. If the degree sequence of G is 1, 1, 2, ..., 2, (so that G is P_n) and $deg_G(v) = deg_G(u) = 1$, then d(v,u) = n–1. Otherwise, d(v,u) = *m*–1 where *m* is the minimum value of k such that P_k is an induced subgraph of G having v as an endvertex and containing u and is known by Lemma 2.7. □

4. SUBGRAPH EQUIVALENCE PROBLEM

The main purpose for which "subgraph equivalence" between v ∈ G and w ∈ H is introduced is dealt with in this section.

4.1. Problem. G is a graph on at least four vertices and *v* ∈ G; *w* is a vertex of an arbitrary graph H such that *v* ∈ G and *w* ∈ H are SE. Are *v* and *w* similar ?

4.2. Notation. The above problem is called *subgraph equivalence problem (G,v)* or *SEP(G,v)*. If the answer for SEP(G,v) is "yes", then we say that *SEP(G,v) holds*. If SEP(G,v) holds for all v ∈ V(G), we say that *SEP holds for G.*

4.3. Observation. The following statements are equivalent.

(i) SEP(G,v) holds.

(ii) "SE between v ∈ G and w ∈ H" is a necessary and sufficient condition for the "similarity of v and w".

(iii) (n–1,G,v) determines "G with v labeled" up to isomorphism.

SEP holds for G when G is regular by Theorem 3.1(ii). A conjecture (as reported in [2, page 250]), proposed by Harary and Manvel [8] while studying the reconstruction of partially labeled graphs includes the following.

4.4. Conjecture ([8]). SEP holds for all graphs.

Giles has proved it for outerplanar graphs while proving its reconstructibility.

4.5. Theorem ([5, Lemma 3.2]). SEP holds for G when G is an outerplanar graph. □

We give some results on SEP(G,v) below.

4.6. Theorem. The statements "SEP(G,v) holds", "SEP(G^C,v) holds" and "SEP(G_v,v) holds" are equivalent. □

SEP(G,v) using the deck of G–v

Deck(G–v) is known from (n–1,G,v). We prove that SEP(G,v) holds in some cases using the reconstructibility of an appropriate vertex coloring of G–v.

4.7. Definition. A *k-vertex coloring* of a graph G is a function f from V(G) to a set of k colors, k > 0. The *vertex colored graph* obtained is denoted by (G,f).

4.8. Lemma. For each card G–u of a vertex colored graph G, the color of the deleted vertex u, its degree and the degrees of the neighbors of u along with their colors are known from *Deck*(G).

Proof. Number of vertices of each color in G is found first and then the color of the vertex missing from each card. Now comparing the degrees of the vertices of each color in G and in G–u, the required information is obtained. □

4.9. Definition. Let v be a vertex of G. Color a vertex $u \in V(G)$, $u \neq v$ with c_1 if u is adjacent to v and with c_2 if u is not adjacent to v. The resulting vertex coloring of G–v is called the *coloring of G–v induced by v* and is denoted by f_v.

The following theorem gives a fundamental relationship between SEP(G,v) and the reconstructibility of a vertex coloring of G–v.

4.10. Theorem. SEP(G,v) holds if and only if (G–v, f_v) is reconstructible, where f_v is the coloring of G–v induced by v.

Proof. Let SEP(G,v) hold. Hence SEP(G_v,v) also holds. ... (1)

Let *Deck* (G–v, f_v) be given and c_1 and c_2 be the two colors used. To each card of *Deck* (G–v, f_v), annex a vertex labeled v and join it with all vertices of color c_1 of that card. From the graphs thus obtained, remove the colors c_1 and c_2. The resulting graphs together give (n–1,H,v), where (H,v) is (G,v) or (G_v,v)). Hence this deck fixes (H,v) uniquely by (1). Now by coloring the vertices adjacent to v in H with color c_1 and others with color c_2 and deleting v, we get (G–v, f_v).

Conversely, let (G–v, f_v) be reconstructible. From (n–1,G,v), *Deck* ((G–v, f_v)) can be derived using Definition 4.9 and it gives (G–v, f_v) by hypothesis. By applying the reverse process to (G–v, f_v) to locate the vertices adjacent to v, we get (G,v). Thus (n–1,G,v) gives (G,v) and so SEP(G,v) holds. □

As given in [2], Manvel [11] has verified that all vertex colored graphs on at most seven vertices are reconstructible. Hence Theorem 4.10 gives the following.

4.11. Theorem. SEP holds for all graphs on at most eight vertices. □

We give some more results on SEP below.

4.12. Theorem. SEP(G,v) holds when G–v or $(G\text{–}v)^C$ is a disconnected graph or a tree or a separable graph without endvertices or a unicyclic graph.

Proof. From (n–1,G,v), the deck of G–v and the deck of $(G\text{–}v)^C$ are known and from them, we can decide whether G–v or $(G\text{–}v)^C$ is disconnected or a tree or separable without endvertices or unicyclic (using standard reconstruction results).

Weinstein [18] has verified that vertex colored disconnected graphs, vertex colored trees and vertex colored separable graphs without endvertices are reconstructible. Manvel's proof [10] for the reconstructibility of unicyclic graphs can be extended to the vertex colored case also (by "choosing" the "base b" for the "number" as 1+ Max $f(T_i)$). Moreover, a vertex colored graph is reconstructible if and only if its complement is reconstructible. Hence by Theorem 4.10, the present theorem follows. □

4.13. Corollary. SEP(G,v) holds if v is a cutvertex or G is disconnected and deg(v) > 0. □

4.14. Corollary. SEP holds for G when it is one of the following.

(i) G is a tree.

(ii) G is separable and $\delta(G) \geq 3$.

(iii) G is a critical block with $\delta(G) \geq 3$. □

We now proceed to define a subclass of reconstructible graphs and use it to show that SEP(G,v) holds for some more classes.

4.15. Definition. Let J be a card in *Deck*(G) and $S \subseteq V(J)$. The graph obtained from J by adding a vertex u and joining it to the vertices in S is called the *completion of J using S.*

Each hypomorph of G can be obtained as a completion of J using a suitable subset of V(J). We define such subsets below.

4.16. Definition. Let J be a card of *Deck*(G). A subset W of V(J) such that the completion of J using W is a hypomorph of G is called *a set of wounded vertices of J.* The hypomorph so obtained is called the *hypomorph of G using W.*

If there is a card J in Deck(G) such that J has a unique set of wounded vertices, then there is only one hypomorph of G (namely G) and so G is reconstructible. If there is more than one set of wounded vertices of J, and the completions of J using them are all isomorphic, then again G is reconstructible.

4.17. Example. Let G be a graph in which the degrees of any two vertices are either equal or differ by at least two and J be any card in *Deck*(G). All hypomorphs of G have the same degree sequence, and it is known from *Deck*(G). Hence by the hypothesis on G, there is a unique subset of vertices of J to whose members the vertex annexed to J must be joined in order to get a hypomorph of G. Thus J has a unique set of wounded vertices.

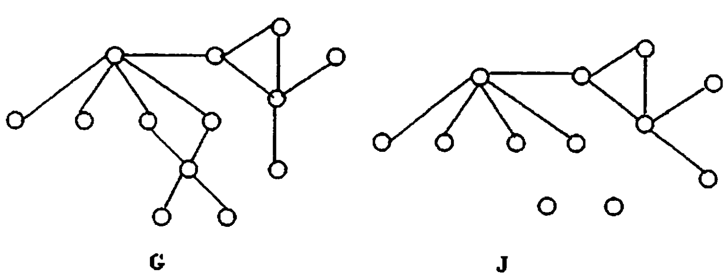
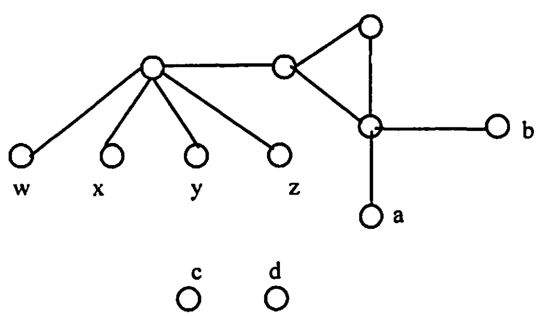


Figure 2. A graph G and a card J of *Deck*(G) with six sets of wounded vertices

4.18. Example. Consider the graph G given in Figure 2. The graph J of Figure 2 is a card of *Deck*(G). Let u be the vertex annexed to J to get hypomorphs of G. By simple calculations based on *Deck*(G) and J, we know that the neighbors of u have degrees 2, 2, 1 and 1 in all hypomorphs of G. So a set of wounded vertices of J must have four vertices whose degrees in J are 1, 1, 0 and 0 respectively. Since J has exactly two vertices of degree zero, both of them must be in every set of wounded vertices of J.

There are six vertices, each of degree one in J and each set of wounded vertices must contain two of them. (For easy description, let us label some of the vertices of J resulting in the partially labeled graph J_1 of Figure 3). So a set of wounded vertices of J must be one among the fifteen sets {c, d}∪X , where X is a 2-subset of {w, x, y, z, a, b}.

As calculated from Deck(G) using Kelly's lemma, every hypomorph of G has exactly one induced C_4 (cycle with four vertices). When the above fifteen sets are checked for this property of the resulting hypomorphs, the eight sets that intersect both {w, x, y, z} and {a, b} get disqualified leaving seven sets behind. Again as calculated from *Deck*(G) using Kelly's lemma, every hypomorph of G has "a vertex of degree at least five lying on an induced C_4". When the seven remaining sets are checked for this property of the resulting hypomorphs, the set {c, d, a, b} fails to give a hypomorph of G. Hence the wounded sets of J must be among the six sets {c, d}∪Y, where Y is a 2-subset of {w, x, y, z}. However, the completion of J using any one among these six sets is isomorphic to the completion of J using any other and so all are hypomorphs of G, as G must be one among them. Hence J has exactly six sets of wounded vertices and G is reconstructible.



J_1

Figure 3. A partial labeling of card J in Example 4.18

4.19. Example. Let G be a graph with Deck(G) = ⟨ G_1, G_2, G_3, G_4, G_5, G_6, G_7 ⟩, where $G_1 \cong G_2 \cong C_6$ (the cycle with six vertices), and $G_3 \cong G_4$, $G_5 \cong G_6$ where G_4, G_5 and G_7 are as in Figure 4. Let us consider the card G_1 ($\cong C_6$). Call it J for uniformity. As calculated from *Deck*(G) and J by Kelly's lemma, we know that a set of wounded vertices of J has two vertices, each of degree two in J. Also each hypomorph of G must have an induced C_4 (the cycle with four vertices). Since J does not have any C_4, the vertex to be annexed to J must lie on a C_4 in the hypomorph and so the two vertices in the wounded set must be at distance two in card J. There are six subsets of vertices of J satisfying these properties. The completions of J based on any two of these are isomorphic and so these six are the sets of wounded vertices of card J and G is reconstructible. However, if we consider the card G_5, then it has only one set of wounded vertices.

4.20. Note. If P is a property of a set of wounded vertices of a card J, then all sets of wounded vertices of J are members of "the set B of subsets of V(J) satisfying property P". Hence, members of B such that "the completion of J using them are hypomorphs of G" are precisely the sets of wounded vertices of J. Further, every member of such a set B is a set of wounded vertices of J if the completion of J using any member of B is isomorphic to the completion using any other member of B. In fat we have refined the initially obtained set B repeatedly using additional properties of the hypomorphs of G till we arrived at a B such that completions of J using all members of B are isomorphic. Thus the technique for the listing of all sets of wounded vertices of card J used in the above two examples includes a proof that G is reconstructible. Depending on the property P of a set of wounded vertices that is initially used for forming B, the exact number of sets of wounded vertices of J will be near or much lower than |B|.

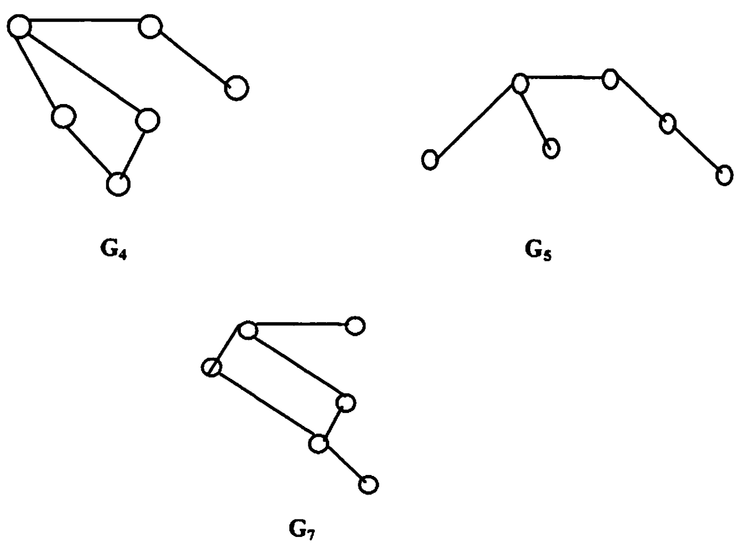


Figure 4. Some cards of *Deck*(G) in Example 4.19

We now define strongly reconstructible graphs in such a way that all their vertex colorings are reconstructible (as proved in Theorem 4.27 below).

4.21. Definition. A graph G is called *strongly reconstructible* if it has a card J such that J has either a unique set of wounded vertices or $C_1, \ldots, C_s$, $s \geq 2$ are the sets of wounded vertices of J and if G* and G** are the hypomorphs using C_i and C_j respectively, then every bijection of V(G*) to V(G**) such that

$$f(C_i - C_j) = C_j - C_i \qquad \ldots \text{(A)}$$

$$f(C_j - C_i) = C_i - C_j \qquad \ldots \text{(B)}$$

$$\text{and } f(x) = x \text{ when } x \notin (C_i - C_j) \cup (C_j - C_i) \qquad \ldots \text{(C)}$$

is an isomorphism from G* to G**. (C_i's have the same cardinality and so there are bijections satisfying (A), (B) and (C) above).

We proceed to give some families of strongly reconstructible graphs. The first one is a generalization of graphs in which the difference between any two distinct degrees is at least two.

4.22. Definition. In a graph G, a vertex w of degree d is called a *good vertex with respect to a vertex u* if there is no vertex of degree d–1 in G other than *u* and its neighbors. The *neighborhood degree sum* ([14]) of a vertex *y* in G is the sum of the degrees of the neighbors of *y* in G.

4.23. Lemma. G is strongly reconstructible if G has a vertex u such that all its neighbors are good vertices with respect to u. □

As the neighborhood degree sequence of the vertex corresponding to each card of Deck(G) is known from Deck(G), the neighborhood degree sums of the vertices of G are known from Deck(G). Comparing them with those of a card J, it is possible to determine the loss incurred by the vertices of J in their neighborhood degree sums (and consequently the sets of wounded vertices of J) in some situations.

4.24. Lemma ([14]). If all the vertices of G have the same neighborhood degree sum, then it is strongly reconstructible. □

4.25. Lemma (14, Theorem 6]). If G has an endvertex and the difference between any two distinct neighborhood degree sums is at least two, then G is uniquely determined by the set of its cards (set reconstructible).

4.26. Lemma. If G has an endvertex u and the difference between any two distinct neighborhood degree sums is at least two, then G is strongly reconstructible.

Proof. Each set of wounded vertices of a card J which is isomorphic to G – u is a singleton set. Moreover, $\{w\} \subset V(J)$ is a wounded set if and only if the loss in the neighborhood degree sum of w is one and every other vertex *y* of J satisfies the condition "*y* is adjacent to w if and only if the loss in the neighborhood degree sum of *y* is one". Hence the sets of wounded vertices of J satisfiy Definition 2.21. □

4.27. Theorem. If G is strongly reconstructible then every vertex coloring C(G) of G is reconstructible.

Proof. Let J be a card of *Deck*(G) satisfying Definition 4.21 and J* be a card of C(G) whose uncolored form is J.

Case 1. J has a unique set of wounded vertices.

As J has a unique set of wounded vertices, J* also has the same set as its unique set of wounded vertices (because, a set of wounded vertices of J* must satisfy all the properties of a set of wounded verties of J and some more properties arising out of the coloring). Now the completion of J* using this unique set of wounded vertices by annexing a vertex of appropriate color is the unique hypomorph of C(G) and so C(G) is reconstructible.

Case 2. $C_1, \ldots, C_s, s \geq 2$ are the sets of wounded vertices of J.

A set of wounded vertices of J* must satisfy all the properties satisfied by a set of wounded vertices of J together with additional restrictions based on colors of vertices. So the sets of wounded vertices of J* must be among $C_1, \ldots, C_s$. Let $C_{c1}, \ldots, C_{cs'}$, $1 \leq s' \leq s$ be the sets of wounded vertices of J*. These sets must have the same color composition in J* by Lemma 4.8. . . . (1)

If s' = 1, then there is only one hyomorph of C(G) and so C(G) is reconstructible. If $s' \geq 2$ and C(G)* and C(G)** are the hypomorphs of C(G) based on C_i and C_j among $C_{c1}, \ldots, C_{cs'}$, then by the choice of J, *all* bijections *f* from V(C(G)*) to

V(C(G)**) that satisfy (A), (B) and (C) of Definition 4.21 *preserve all adjacencies.* (There are such bijections). These bijections include bijections *g* which

1. take C_i to C_j preserving colors (because of (A) and (B) of Definition 4.21 and (1) above),
2. preserve colors when restricted to $C_i \cup C_j$ (because, each *f* and hence *g* takes $C_i \cup C_j$ to itself by Definition 4.21(B) and the chosen g preserves colors while taking C_i to C_j), and
3. fix x when $x \notin C_i \cup C_j$.

Obviously, such bijections from V(C(G)*) to V(C(G)**) *are color preserving also* and hence are isomorphisms from C(G)* to C(G)**. Thus C(G) is reconstructible. □

Example 4.17 and the three lemmas above give the following.

4.28. Theorem. Every vertex coloring of G is reconstructible if G is one of the following.

1. G is regular.
2. The degree of any two vertices in G are either equal or differ by at least two.
3. G has a vertex u such that all its neighbors are good vertices with respect to u.
4. G is a graph such that all its vertices have the same neighborhood degree sum.
5. G has an endvertex and the difference between any two distinct neighborhood degree sums is at least two. □

The above theorem together with Theorem 4.10 give the following.

4.29. Theorem. SEP(G,v) holds when any one of the following is true.

1. G–v is regular.
2. In the graph G–v, all vertices have the same neighborhood degree sum.
3. The degrees of any two vertices of G–v are either equal or differ by at least two.
4. G–v has a vertex which is adjacent to all or none of the other vertices of G–v.
5. G–v is a graph having a vertex u such that all its neighbors are good vertices with respect to u.
6. G–v is a graph having an end vertex and the difference between any two distinct neighborhood degree sums is at least two.

Proof. Follows by Theorem 4.10, since G–v is strongly reconstructible by the above theorem. □

We are able to prove that "G is strongly reconstructible implies every vertex coloring of G is reconstructible", whereas in general, it is not known whether the "reconstructibility of G" implies the "reconstructibility of all vertex colorings of G".

We now proceed to prove that Ulam's reconstruction conjecture and the conjecture that "SEP holds for all graphs" (Conjecture 4.4) *are equivalent*

4.30. Theorem (Taylor [17]). All graphs are reconstructible if and only if all vertex colored graphs are reconstructible. □

4.31. Thoerem. SEP holds for all graphs G on at least four vertices if and only if Ulam's reconstruction conjecture is true.

Proof. Only if part. SEP holds in all situations. Hence by Theorem 4.10, all 2-vertex colored graphs on at least three vertices are reconstructible. Hence all graphs are reconstructible.

If part: Let URC be true. Now by the above theorem, all 2-vertex colored graphs are reconstructible and so by Theorem 4.10, SEP holds for all graphs. □

The following weaker forms of SEP can also be proposed.

4.32. Problem. G is a graph on at least four vertices and $v \in G$. w is another vertex of G such that *v* and *w* are subgraph equivalent. Are *v* and *w* similar?

4.33. Problem. G is a graph on at least four vertices. Are any two SE vertices of G similar?

Results already proved for SEP(G,v) confirm the truth of the above problems in many situations. In particular, Theorem 4.31 gives that Problem 4.33 holds for all graphs if URC is true. Hence if Problem 4.33 fails for a graph, then URC is false. Thus, if a graph with identity automorphism group has a pair of SE vertices, then URC is false.

5. A WEAKER FORM OF SUBGRAPH EQUIVALENCE PROBLEM

We can relax our demand in SEP and consider also the following problem.

5.1. Problem. G and H are graphs on $n \geq 4$ vertices such that $v \in G$ and $w \in H$ are subgraph equivalent. Are G and H isomorphic?

Obviously, the above problem holds for a pair of graphs G and H if SEP holds for G and so Problem 5.1 holds for all graphs on at most eight vertices and for all trees. If $v \in G$ and $w \in H$ are SE, then (i) *Deck*(G) and *Deck*(H) have n–1 cards in common and (ii) G and H have the same degree sequence. Using these facts and the reconstruction from partial decks already studied, we get some results.

5.2. Definition ([12]). The *adversary reconstruction number* of a graph G is the smallest number such that all $S \subseteq$ *Deck*(G) of that cardinality identify G uniquely. This parameter is denoted by $\forall rn$(G).

Thus if $\forall rn(G) \leq n-1$, then by (i), Problem 5.1 holds for the pair of graphs G and H, where H is arbitrary.

Most of the work done on $\forall rn$ and on the maximum number of common cards in the decks of nonisomorphic graphs can be found in [12], [1], [3] and [4]. In [16], $\forall rn$(G) is calculated for all graphs on up to 11 vertices using computers and the number of nonisomorphic graphs on n vertices that share n–1 cards with a nonisomorphic graph is reported to be zero when $7 \leq n \leq 11$, and 9, 8 and 2 when n = 4, 5 and 6 respectively. This together with our verifications show that for $4 \leq n \leq 11$, there is no pair of nonisomorphic graphs on n vertices satisfying (i) and (ii) above, and give the following.

5.3. Theorem. Problem 5.1 is true for graphs on at most 11 vertices.

The above computer based results on $\forall rn$ indicate that Problem 5.1 is most likely to be true.

6. CONCLUSION

We see that SEP(G,v) holds in many situations and that (n–2,G,v) does not give (G,v) up to isomorphism in general. We have also seen that if Problem 4.33 fails in a graph (possibly on more that eight vertices), then URC is false. Also (n–2)-SE does not imply (n–1)-SE as seen for the graphs in Figure 1. If (n–1)-SE does not imply "n-SE", then URC will be false. Apart from Problems 4.1, 4.32, 4.33 and 5.1, the following are the immediate open questions identified.

1. Vertices $v \in G$ and $w \in H$ are SE. Is $G \cong H$?
2. Vertices $v \in G$ and $w \in H$ are SE. Is $(G,v) \cong (H,w)$?
3. Vertices $v \in G$ and $w \in H$ are SE and $G \cong H$. Is $(G,v) \cong (H,w)$?
4. Vertices $v \in G$ and $w \in H$ are SE and $n \geq 4$. Is $G-v \cong H-w$?
5. If SEP holds for all 2-connected graphs, then is it true that it holds for all graphs ?

We have extended the above concepts to digraphs in a subsequent paper.

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