Gromov Hyperbolicity of Regular Graphs

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Abstract

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is *δ-hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. Regular graphs are a very interesting class of graphs with many applications. The main aim of this paper is to obtain information about the hyperbolicity constant of regular graphs. We obtain several bounds for this parameter; in particular, we prove that $\delta(G) \le \Delta n/(8(\Delta-1))+1$ for any Δ-regular graph G with n vertices. Furthermore, we show that for each $\Delta \ge 2$ and every possible value t of the hyperbolicity constant, there exists a Δ-regular graph G with $\delta(G) = t$. We also study the regular graphs G with $\delta(G) \le 1$, i.e., the graphs which are like trees (in the Gromov sense). Besides, we prove some inequalities involving the hyperbolicity constant and domination numbers for regular graphs.

Keywords: Regular graphs; Gromov hyperbolicity; Geodesics; Domination numbers; Infinite graphs.

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1 Introduction

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1, 22, 23]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 22, 23]).

The first works on Gromov hyperbolic spaces deal with finitely generated groups (see [23]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, *e.g.*, [34]); indeed, hyperbolic groups are strongly geodesically automatic, *i.e.*, there is an automatic structure on the group [15].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [44] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension; the same holds for many complex networks, see [29]. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see, *e.g.*, [19]). Another important application of these spaces is the study of the spread of viruses through on the internet (see [25, 26]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [25, 26, 33]). The hyperbolicity has also been used extensively in the context of random graphs (see, *e.g.*, [16, 27, 41, 42, 43]).

The study of Gromov hyperbolic graphs is a subject of increasing interest (see, *e.g.*, [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 18, 21, 24, 25, 26, 27, 28, 29, 30, 31, 33, 35, 37, 38, 40, 41, 42, 43, 46, 47] and the references therein).

We say that a curve $\gamma : [a,b] \to X$ in a metric space X is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t),\gamma(s)) = |t-s|$ for every $s,t \in [a,b]$, where L and d denote length and distance, respectively, and $\gamma|_{[t,s]}$ is the restriction of the curve γ to the interval $[t,s]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u,v]$.

Along the paper we just consider graphs with every edge of length 1. In order to consider a graph G as a geodesic metric space, we identify (by an isometry) any edge $[u, v] \in E(G)$ with the real interval $[0,1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0,1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. Throughout this paper, $G = (V, E) = (V(G), E(G))$ denotes a simple (without loops and multiple edges) connected graph such that every edge has length 1 and $V(G) \neq \emptyset$. These properties guarantee that any graph is a geodesic metric space. Note that to exclude multiple edges and loops is not an important loss of generality, since [5, Theorems 8 and 10] reduces the problem of compute the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the vertices of T; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. We say that T is *δ-thin* if any side of T is contained in the δ-neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T, *i.e.* $\delta(T) := \inf\{\delta \geq 0 : T \text{ is } \delta\text{-thin}\}$. The space X is *δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, *i.e.*, $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$.

If we have a triangle with two identical vertices, we call it a *bigon*; note that since this is a special case of the definition, every geodesic bigon in a δ-hyperbolic space is δ-thin.

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, *e.g.*, [17]).

Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic. However, for a general graph or a general geodesic metric space deciding whether or not a space is hyperbolic is usually very difficult. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant and to study the hyperbolicity of a particular class of graphs.

The papers [6, 2, 9, 30, 47, 12, 14, 32, 35, 39, 36, 45] study the hyperbolicity of, respectively, complement of graphs, chordal graphs, line graphs, Cartesian product graphs, cubic graphs, tessellation graphs and median graphs. The main result in [6] states that if the graph G does not have small diameter, then the hyperbolicity constant of its complement satisfies $\delta(\overline{G}) \leq 2$. [2, 9, 30, 47] provide several generalizations of the concept of chordality and give some necessary or sufficient conditions for hyperbolicity. The main results in [12, 14] are that the line graph $L(G)$ (of the graph G) is hyperbolic if and only if G is hyperbolic and $\delta(G) \leq \delta(L(G)) \leq 5\delta(G)+5/2$. In [32] it is proved that the Cartesian product of the graphs G_1 and G_2 is hyperbolic if and only if some G_j is hyperbolic and the other one is bounded. The main result in [36] is that a planar graph $L(G)$ is hyperbolic if and only if its dual graph is hyperbolic. In [45] it is proved that a median graph is hyperbolic if and only if every bigon is δ_0-thin for some constant δ_0. The nature of the results in [35, 39] is very different of the previous ones: these papers obtain inequalities relating the hyperbolicity constant of a cubic graph G with other parameters of G (such as its order, size, Laplacian spectral radius, vertex cover number or algebraic connectivity). Furthermore, [39] studies the complement of cubic graphs.

The main aim of this paper is to obtain results about the hyperbolicity constant of Δ-regular graphs (graphs with all of their vertices of degree Δ), since they are a very interesting class of graphs with many applications, and they are the natural generalization of cubic graphs. We obtain several bounds for this parameter (see Theorems 2.6 and 2.14, and Proposition 2.9); in particular, Theorem 2.6 gives $\delta(G) \leq \Delta n/(8(\Delta-1)) + 1$ for any Δ-regular graph G with n vertices. Furthermore, we show in Theorem 2.19 that for each $\Delta \geq 2$ and every possible value t of the hyperbolicity constant, there exists a Δ-regular graph G with $\delta(G) = t$. We also study in Theorems 3.7 and 3.9 and Corollary 3.8 the regular graphs G with $\delta(G) \leq 1$ (since the hyperbolicity constant of a graph can be viewed as a measure of how "tree-like" the graph is, it is interesting to study the graphs with small hyperbolicity constant). Besides, we prove some inequalities involving the hyperbolicity constant and other parameters for regular graphs (Theorem 2.12 gives that for any regular graph G with n vertices and k-domination numbers $\gamma_k(G)$, we have $\delta(G) + \gamma_k(G) \leq n$ for every $1 \leq k \leq \Delta$). We want to remark that, except for Theorems 2.6, 2.12 and 3.7, the results in this paper are new even in the context of cubic graphs. Although Theorem 3.7 generalizes [39, Theorem 2.4], the proof devised now is different and more sophisticated.

2 Bounds for the hyperbolicity constant of regular graphs

Recall that a graph is Δ-*regular* if every vertex has Δ neighbors. A graph is *regular* if it is Δ-regular for some Δ.

As usual, by *cycle* we mean a simple closed curve, *i.e.*, a path with different vertices, unless the last one, which is equal to the first vertex.

Denote by $J(G)$ the set of vertices and midpoints of edges in G. Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

The following three results, which appear in [4], will be used throughout this paper.

Theorem 2.1. *[4, Theorem 2.5] For every graph G we have $\delta_1(G) = \delta(G)$.*

The next result will narrow the possible values for the hyperbolicity constant δ.

Theorem 2.2. *[4, Theorem 2.6] If G is a hyperbolic graph G, then $\delta(G)$ is a multiple of* $1/4$.

Theorem 2.3. *[4, Theorem 2.7] If G is a hyperbolic graph G, then there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

If G is a graph and $v \in V(G)$, we denote by $\deg(v)$ or $\deg_G(v)$ the degree of v in the graph G.

We will need also the following result appearing in [31, Theorem 30].

Lemma 2.4. *If G has n vertices, then $\delta(G) \le n/4$.*

This inequality can be improved for regular graphs.

Theorem 2.5. *If G is a graph with n vertices, minimum degree $\delta_0 \ge 2$ and maximum degree Δ, then*

$$\delta(G) \le \min\left\{\frac{\Delta n}{8(\delta_0-1)}+1,\ \frac{n}{4}\right\}.$$

Proof. By Lemma 2.4, it suffices to prove

$$\delta(G) \le \frac{\Delta n}{8(\delta_0-1)}+1.$$

Without loss of generality we can assume that $\delta(G) > 1$, since otherwise the inequality is trivial.

By Lemma 2.4, G is hyperbolic and Theorem 2.3 gives that there exist a geodesic triangle $T = \{x,y,z\} = \{\gamma_1,\gamma_2,\gamma_3\}$ in G and $p \in \gamma_1 := [xy]$ with $d(p,\gamma_2 \cup \gamma_3) = \delta(G)$. Let us define the curve $\gamma_1^* \subset \gamma_1$ by $\gamma_1^* := \{z \in \gamma_1 : d(z,p) < \delta(G) - 1\}$. Note that $1 < \delta(G) = d(p,\gamma_2 \cup \gamma_3) \le d(p,\{x,y\})$. Then, $\gamma_1 = [xx'] \cup \gamma_1^* \cup [y'y]$ with $d(x',p) = d(y',p) = \delta(G) - 1$. Note that if $v \in V(G) \cap \gamma_1^*$ and $w \in V(G) \cap (\gamma_2 \cup \gamma_3)$, then $[v,w] \notin E(G)$, since otherwise $d(p,\gamma_2 \cup \gamma_3) < \delta(G)$.

Consider the set of curves joining x and y

$$\begin{aligned} A: &= \{\gamma_2 \cup \gamma_3\} \cup \{([xu_2] \cup [u_2,u_3] \cup [u_3y]) : u_2 \in V(G) \cap \gamma_2, \\ &\qquad u_3 \in V(G) \cap \gamma_3,\ [u_2,u_3] \in E(G)\}, \end{aligned}$$

and $g_0 \in A$ such that $L(g_0) \le L(\eta)$ for every $\eta \in A$. Since g_0 is a shortest curve in A, if α,β are non-adjacent vertices in g_0 (*i.e.*, we have either $[\alpha,\beta] \notin E(G)$ or $[\alpha,\beta] \not\subseteq g_0$), then $[\alpha,\beta] \notin E(G)$.

Given $\alpha,\beta \in V(G) \cap g_0$, we denote by $g_{\alpha,\beta}$ the subcurve of g_0 joining α and β. Consider now the set of curves joining $[xx']$ and $[y'y]$

$$\begin{aligned} B: &= \{g_0\} \cup \{([a,\alpha] \cup g_{\alpha,\beta} \cup [\beta,b]) : a \in V(G) \cap [xx'],\ b \in V(G) \cap [y'y], \\ &\qquad \alpha,\beta \in V(G) \cap g_0,\ [a,\alpha],[\beta,b] \in E(G)\}, \end{aligned}$$

and $g \in B$ such that $L(g) \le L(\eta)$ for every $\eta \in B$. If $g = [a,\alpha] \cup g_{\alpha,\beta} \cup [\beta,b]$, let $\gamma_{a,b}$ be the curve with $\gamma_1^* \subset \gamma_{a,b} \subseteq \gamma_1$ joining a and b. Since g is a shortest curve in B and g_0 is a shortest curve in A, if v,w are non-adjacent vertices in $\sigma := \gamma_{a,b} \cup g$, then $[v,w] \notin E(G)$. Hence, if $v,w \in V(G) \cap \sigma$ and $[v,w] \in E(G)$, then $[v,w] \subset \sigma$.

Recall that $L(\gamma_1^*) = 2\delta(G) - 2$, $\gamma_1^* \subset \gamma_{a,b}$ and g is a shortest curve in B. Thus,

$$L(\sigma) = L(\gamma_{a,b}) + L(g) \ge 2L(\gamma_{a,b}) \ge 2L(\gamma_1^*) = 4\delta(G) - 4.$$

Since G has minimum degree $\delta_0 \ge 2$, for each $v \in V(G) \cap \sigma$ there exist at least $\delta_0 - 2$ edges in $E(G) \setminus \sigma$ adjacent to v. Furthermore, if v,w are different vertices in $V(G) \cap \sigma$ and e_v, e_w are edges in $E(G) \setminus \sigma$ adjacent to v,w (*i.e.*, $v \in e_v$ and $w \in e_w$), respectively, then $e_v \ne e_w$ (recall that if $[v,w] \in E(G)$, then $[v,w] \subset \sigma$). If m is the cardinality of $E(G)$, then

$$m \ge L(\sigma) + (\delta_0 - 2)L(\sigma) \ge (\delta_0 - 1)4(\delta(G) - 1).$$

Since

$$m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \le \frac{\Delta n}{2},$$

we obtain the inequality. □

We have the following direct consequence for regular graphs.

Theorem 2.6. *If $\Delta \geq 2$ and G is a Δ-regular graph with n vertices, then*

$$\delta(G) \leq \min\Big\{ \frac{\Delta n}{8(\Delta - 1)} + 1,\ \frac{n}{4} \Big\}.$$

The inequality in Theorem 2.6 is essentially sharp, as the following example shows: denote by $G_1, \dots, G_r$ graphs isomorphic to the "diamond graph" (the complete graph K_4 with an edge removed); let G be the 3-regular graph obtained by connecting G_j with G_{j+1} by an edge e_j, for $j = 1, \dots, r-1$, and G_r with G_1 by an edge e_r; then $n = 4r$ and $\delta(G) = 3r/4 = 3n/16 = 3n/(8(3-1))$.

From [31, Proposition 5 and Theorem 7] we deduce the following result.

Lemma 2.7. *Let G be any graph with a cycle g. If $L(g) \geq 3$, then $\delta(G) \geq 3/4$. If $L(g) \geq 4$, then $\delta(G) \geq 1$.*

The following lemma is a consequence of [20, Proposition 1.3.1].

Lemma 2.8. *If $m \geq 2$ is a natural number and G is a finite graph with minimum degree $\delta_0 \geq m$, then there exists a cycle η in G with $L(\eta) \geq m+1$.*

Lemmas 2.7 and 2.8 give directly the following proposition.

Proposition 2.9. *If G is a finite Δ-regular graph with $\Delta \geq 3$, then $\delta(G) \geq 1$.*

The equality in Proposition 2.9 is attained in the complete graph $K_{\Delta+1}$ and in the complete bipartite graph $K_{\Delta,\Delta}$.

We say that a subset $A \subset V(G)$ is an *independent set* if $[v,w] \notin E(G)$ for every $v, w \in A$. We denote by $\beta(G)$ the *independence number* of G, *i.e.*, the cardinality of the largest independent set in G.

Given a graph G with maximum degree Δ, a set $X \subset V(G)$ is a *k-dominant set* of G with $1 \leq k \leq \Delta$, if any vertex in $V(G) \setminus X$ is adjacent to at least k vertices of X. The *k-domination number* of a graph G, $\gamma_k(G)$, is the minimum cardinality of a k-dominant set of G.

For any graph G, we define

$$\operatorname{diam} V(G) := \sup\big\{ d_G(v,w) \mid v, w \in V(G) \big\},$$
$$\operatorname{diam} G := \sup\big\{ d_G(x,y) \mid x, y \in G \big\}.$$

Theorem 2.10. *[40, Theorem 8] In any graph G the inequality $\delta(G) \leq (\operatorname{diam} G)/2$ holds.*

We have the following direct consequence.

Corollary 2.11. *In any graph G the inequality $\delta(G) \le (\operatorname{diam} V(G)+1)/2$ holds.*

Denote by $\lfloor t \rfloor$ the *lower integer part* of t, *i.e.*, the largest integer k with $k \le t$.

The following result relates, for regular graphs, the hyperbolicity constant and the k-domination numbers $\gamma_k(G)$ for every $1 \le k \le \Delta$.

Theorem 2.12. *If G is a Δ-regular graph with n vertices, $\Delta \ge 2$ and $1 \le k \le \Delta$, we have*

$$\delta(G) + \gamma_k(G) \le n.$$

Proof. Consider first the case $\Delta = n-1$. Then G is isomorphic to the complete graph K_n, and we have $\delta(G) \le 1$ by [40, Theorem 11]. Furthermore, $\gamma_k(G) = k$ for every $1 \le k \le n-1$, and we conclude $\delta(G) + \gamma_k(G) \le 1 + k \le n$.

Assume now $2 \le \Delta \le n-2$. Let us consider a geodesic γ in G with $L(\gamma) = \operatorname{diam} V(G)$. We have $\operatorname{diam} V(G) \ge 2$ since G is not isomorphic to a complete graph. Since γ is a geodesic, if $v, w \in \gamma \cap V(G)$ and $d(v,w) \ge 2$, then $[v,w] \notin E(G)$. Hence, it is possible to choose a set of vertices $v_{i_1}, v_{i_2}, \dots, v_{i_r} \in \gamma \cap V(G)$ with $d(v_{i_j}, v_{i_{j+1}}) = 2$ and $r = \lfloor \operatorname{diam} V(G)/2 \rfloor + 1$. Thus $\beta(G) \ge r$, and Corollary 2.11 gives

$$\beta(G) \ge \left\lfloor \frac{\operatorname{diam} V(G)}{2} \right\rfloor + 1 \ge \frac{\operatorname{diam} V(G)+1}{2} \ge \delta(G).$$

Since G is a Δ-regular graph, $X \subset V(G)$ is a Δ-dominant set of G if and only if $V(G) \setminus X$ is an independent set. Therefore, $\gamma_\Delta(G) + \beta(G) = n$ and $\gamma_k(G) + \beta(G) \le n$ since $\gamma_k(G) \le \gamma_\Delta(G)$ for every $1 \le k \le \Delta$. Thus we obtain the inequality. □

A cycle C in a graph G is called a *dominating cycle* if $V(G) \setminus V(C)$ is an independent set in G.

We say that a vertex v of a graph G is a *cut-vertex* if $G \setminus \{v\}$ is not connected. A graph is *two-connected* if it does not contain cut-vertices.

It is well-known that Bondy [8] proved the following theorem.

Theorem 2.13. *Let G be a two-connected graph with n vertices such that $\deg(x) + \deg(y) + \deg(z) \ge n+2$ for all independent sets of vertices x, y, z. Then every longest cycle in G is a dominating cycle.*

Given a graph G and $v \in V(G)$, let us denote by $N(v)$ the set of neighbors of v. If $C \subset G$, we denote by $V(C)$ the set $V(G) \cap C$. Given a path h joining x and y, we denote by $\operatorname{int}(h)$ the set $h \setminus \{x, y\}$.

The next result provides another good upper bound of the hyperbolicity constant for a large class of regular graphs.

Theorem 2.14. *Let G be a non-Hamiltonian two-connected Δ-regular graph with n vertices, $\Delta \geq (n+t)/3$ and $t \geq 2$. Then*

$$\delta(G) \leq \frac{2n+21-t}{12}.$$

Proof. Consider a longest cycle C in G. Since $\Delta \geq (n+t)/3 \geq (n+2)/3$, C is a dominating cycle by Theorem 2.13. Since G is not a Hamiltonian graph, there exists $w \in V(G) \setminus V(C)$.

We are going to bound $\operatorname{diam} V(G)$. Let $u, v \in V(G)$.

Case (I). Assume first that $u, v \in V(C)$. Since G is a Δ-regular graph and $V(G) \setminus V(C)$ is an independent set in G, we have

$$\big|(V(C) \setminus N(w)) \cup \{w\}\big| \leq \big|V(G) \setminus N(w)\big| = n - \Delta \leq n - \frac{n+t}{3} = \frac{2n-t}{3}.$$

Case (I.1). Assume that there exists a curve g joining u and v with $\operatorname{int}(g)$ contained in $C \setminus N(w)$. Thus $N(w) \subseteq C_1 := C \setminus \operatorname{int}(g)$. Let $w_1, w_2 \in N(w)$ such that

$$d_{C_1}(w_1, w_2) = \max\big\{ d_{C_1}(w', w'') \mid w', w'' \in C_1 \cap N(w)\big\},$$

and g_1 the geodesic in C_1 joining w_1 and w_2. Consider the cycle

$$C' := (C \setminus g_1) \cup [w_1, w] \cup [w, w_2].$$

Thus $u, v, w \in V(C')$, $g \subset C'$, $|V(C') \cap N(w)| = 2$ and

$$L(C') = \big|V(C')\big| \leq \big|(V(C) \setminus N(w)) \cup \{w, w_1, w_2\}\big| \leq \frac{2n-t}{3} + 2 = \frac{2n+6-t}{3}.$$

Hence,

$$d_G(u,v) \leq d_{C'}(u,v) \leq \frac{1}{2} L(C') \leq \frac{2n+6-t}{6}. \tag{2.1}$$

Case (I.2). Assume that there is no curve g joining u and v with $\operatorname{int}(g)$ contained in $C \setminus N(w)$. Thus there exists two paths g_1, g_2 contained in $C \setminus \{u, v\}$, with endpoints in $N(w)$, $L(g_1 \cup g_2) \geq \Delta - 2$ and $N(w) \subset g_1 \cup g_2$. Therefore,

$$\begin{aligned} d_C(u, N(w)) + d_C(v, N(w)) &\leq \frac{1}{2}\Big(L(C) - L(g_1 \cup g_2)\Big) \leq \frac{1}{2}\Big(L(C) - (\Delta - 2)\Big) \\ &\leq \frac{1}{2}\Big(n - 1 - \frac{n+t}{3} + 2\Big) = \frac{2n+3-t}{6}, \end{aligned}$$

and so

$$d_G(u,v) \le d_C(u,N(w)) + d_C(v,N(w)) + 2 \le \frac{2n+15-t}{6}.$$

Case (II). Assume that $u \in V(C)$ and $v \in V(G)\setminus V(C)$. The argument in Case $(I.1)$ also gives the inequality (2.1) in this case.

Case (III). Finally, assume that $u,v \in V(G)\setminus V(C)$.

Case $(III.1)$. If $N(u)\cap N(v) \neq \emptyset$, then $d_G(u,v)=2$.

Case $(III.2)$. Assume that $N(u)\cap N(v) = \emptyset$. Consider two paths h_1, h_2 contained in C joining $N(u)$ and $N(v)$ so that $\operatorname{int}(h_1)\cap \operatorname{int}(h_2)=\emptyset$ and $\operatorname{int}(h_j)\cap (N(u)\cup N(v)) = \emptyset$ for $j=1,2$. Since $V(G)\setminus V(C)$ is an independent set in G and $N(u)\cap N(v)=\emptyset$, we have

$$\begin{aligned} 2\min\big\{L(h_1),L(h_2)\big\} &\le L(h_1\cup h_2) \le 2 + \big|V(G)\setminus\big(N(u)\cup N(v)\big)\big| \\ &= 2+n-2\Delta \le 2+n-2\frac{n+t}{3} = \frac{n+6-2t}{3}, \end{aligned}$$

and so

$$d_G(u,v) \le 1+\min\big\{L(h_1),L(h_2)\big\}+1 \le 2+\frac{n+6-2t}{6} = \frac{n+18-2t}{6}.$$

Since $3 \le n+t$, we deduce $n+18-2t \le 2n+15-t$. Therefore, we have in any case

$$\operatorname{diam} V(G) \le \max\left\{\frac{2n+6-t}{6}, \frac{2n+15-t}{6}, 2, \frac{n+18-2t}{6}\right\} = \frac{2n+15-t}{6},$$

and Corollary 2.11 gives the result. □

Definition 2.15. *Given any edge in G, let us consider the maximal two-connected subgraph containing it. We call to the set of these maximal two-connected subgraphs $\{G_s\}_s$ the* canonical T-decomposition *of G.*

We will need the following result, which allows to obtain global information about the hyperbolicity constant of a graph from local information (see [5, Theorem 3]).

Theorem 2.16. *Let G be any graph with canonical T-decomposition $\{G_s\}_s$. Then*

$$\delta(G) = \sup_s \delta(G_s).$$

If H is a subgraph of G and $w \in V(H)$, we denote by $\deg_H(w)$ the degree of the vertex w in the subgraph induced by $V(H)$.

Theorem 2.17. *[3, Theorem 3.2] Let G be any graph. Then $\delta(G) \geq 5/4$ if and only if there exist a cycle g in G with length $L(g) \geq 5$ and a vertex $w \in g$ such that* $\deg_g(w) = 2$.

The following result appears in [35, Theorem 3.16].

Theorem 2.18. *For each possible value t of the hyperbolicity constant, there exists a cubic graph G with $\delta(G) = t$. Furthermore, G can be chosen as a finite graph if $t \geq 1$.*

The next result shows that the statement of Theorem 2.18 holds for the set of Δ-regular graphs, for any fixed $\Delta \geq 2$.

Theorem 2.19. *For each $\Delta \geq 2$ and every possible value t of the hyperbolicity constant, there exists a Δ-regular graph G with $\delta(G) = t$. Furthermore, G can be chosen as a finite graph if $t \geq 1$.*

Proof. First of all, recall that t is a multiple of $1/4$ by Theorem 2.2. Furthermore, $t \neq 1/4, 1/2$, by [31, Theorem 11].

Let us start with $\Delta = 2$. If G is the Cayley graph of the group $\mathbb{Z}$ (a 2-regular tree isometric to the real line), then $\delta(G) = 0$. For any $r \geq 3$ the cycle graph C_r satisfies $\delta(C_r) = r/4$ by [40, Theorem 11].

Let us fix any $\Delta \geq 3$. If G is a Δ-regular tree, then $\delta(G) = 0$. Let $T_{\Delta-1}$ be a "rooted tree with $\Delta - 1$ sons" (every vertex has $\Delta - 1$ sons, *i.e.*, the degree of the root is $\Delta - 1$ and the degree of the other vertices is Δ). Fix $r \geq 3$ and consider a graph G_0 obtained by attaching $\Delta - 2$ edges to each vertex of a cycle graph C_r, such that G_0 has r vertices with degree Δ and $r(\Delta - 2)$ vertices $v_1, \dots, v_{r(\Delta-2)}$ with degree one. Consider graphs $T^1_{\Delta-1}, \dots, T^{r(\Delta-2)}_{\Delta-1}$ isomorphic to $T_{\Delta-1}$, with roots $w_1, \dots, w_{r(\Delta-2)}$. Let G be the graph obtained from G_0 and $T^1_{\Delta-1}, \dots, T^{r(\Delta-2)}_{\Delta-1}$ by identifying v_j with w_j for $1 \leq j \leq r(\Delta - 2)$. Thus G is Δ-regular. Since $G_0, T^1_{\Delta-1}, \dots, T^{r(\Delta-2)}_{\Delta-1}$ is the canonical T-decomposition of G, Theorem 2.16 gives

$$\delta(G) = \max\left\{\delta(C_r),\, \delta(T^j_{\Delta-1})\right\} = \max\left\{\delta(C_r),\, \delta(T_{\Delta-1})\right\} = \max\left\{\frac{r}{4},\, 0\right\} = \frac{r}{4}.$$

This finishes the proof of the first statement.

Since the complete graph $K_{\Delta+1}$ is Δ-regular and $\delta(K_{\Delta+1}) = 1$, the second statement holds for $t = 1$.

Fix now $r \geq 5$. By Theorem 2.18, there exists a finite cubic graph G^r with $\delta(G^r) = r/4$. Fix $\Delta \geq 4$.

Consider the complete graph $K_{\Delta+1}$, $u, v \in V(K_{\Delta+1})$ and a point $w \notin K_{\Delta+1}$. Let G_Δ be the graph with $V(G_\Delta) = V(K_{\Delta+1}) \cup \{w\}$ and $E(G_\Delta) = E(K_{\Delta+1}) \cup \{[u, w], [v, w]\} \setminus [u, v]$. Note that $\deg_{G_\Delta}(w) = 2$ and $\deg_{G_\Delta}(x) =$

Δ for every $x \in V(G_\Delta) \setminus \{w\}$. Let us show now that $\delta(G_\Delta) = 5/4$ for every $\Delta \geq 4$. One can check that $\operatorname{diam}(G_\Delta) = 5/2$ and Theorem 2.10 gives $\delta(G_\Delta) \leq 5/4$. If we choose any cycle g in G_Δ with length $L(g) = 5$ and $w \in g$, then $\deg_g(w) = \deg_{G_\Delta}(w) = 2$ and Theorem 2.17 gives $\delta(G_\Delta) \geq 5/4$. Therefore, we conclude $\delta(G_\Delta) = 5/4$.

Assume that Δ is even. Consider the cycle graph C_r with vertices $v_1, \dots, v_r$ and $r(\Delta/2 - 1)$ isomorphic graphs $\{G_\Delta^{i,j}\}$ to G_Δ with $w_{i,j} \in V(G_\Delta^{i,j})$ and $\deg_{G_\Delta^{i,j}}(w_{i,j}) = 2$ for $i = 1, \dots, r$ and $j = 1, \dots, \Delta/2 - 1$. Let G be the graph obtained from C_r and $\{G_\Delta^{i,j}\}$ by identifying v_i and $w_{i,j}$ ($j = 1, \dots, \Delta/2 - 1$) for each $i = 1, \dots, r$. Thus G is a Δ-regular graph and $\{C_r\}, \{G_\Delta^{i,j}\}_{i=1,\dots,r,\, j=1,\dots,\Delta/2-1}$ is its canonical T-decomposition, and so Theorem 2.16 gives

$$\delta(G) = \max\left\{\delta(C_r),\, \delta(G_\Delta^{i,j})\right\} = \max\left\{\delta(C_r),\, \delta(G_\Delta)\right\} = \max\left\{\frac{r}{4},\, \frac{5}{4}\right\} = \frac{r}{4}.$$

Finally, assume that Δ is odd and consider a finite cubic graph G^r with $\delta(G^r) = r/4$ and vertices $v_1, \dots, v_k$. Consider $k(\Delta-3)/2$ isomorphic graphs $\{G_\Delta^{i,j}\}$ to G_Δ with $w_{i,j} \in V(G_\Delta^{i,j})$ and $\deg_{G_\Delta^{i,j}}(w_{i,j}) = 2$ for $i = 1, \dots, k$ and $j = 1, \dots, (\Delta - 3)/2$. Let G be the graph obtained from G^r and $\{G_\Delta^{i,j}\}$ by identifying v_i and $w_{i,j}$ ($j = 1, \dots, (\Delta - 3)/2$) for each $i = 1, \dots, k$. Thus G is a Δ-regular graph and $\{G^r\}, \{G_\Delta^{i,j}\}_{i=1,\dots,k,\, j=1,\dots,(\Delta-3)/2}$ is its canonical T-decomposition, and so Theorem 2.16 gives

$$\delta(G) = \max\left\{\delta(G^r),\, \delta(G_\Delta^{i,j})\right\} = \max\left\{\delta(G^r),\, \delta(G_\Delta)\right\} = \max\left\{\frac{r}{4},\, \frac{5}{4}\right\} = \frac{r}{4}.$$

This finishes the proof. □

3 Regular graphs with small hyperbolicity constant

Definition 3.1. *Given a graph G and its canonical T-decomposition $\{G_s\}_s$, we define the* effective diameter *as*

$$\operatorname{effdiam} V(G) := \sup_s \operatorname{diam} V(G_s),$$
$$\operatorname{effdiam}(G) := \sup_s \operatorname{diam}(G_s).$$

Note that if G is a two-connected graph, then $\operatorname{effdiam} V(G) = \operatorname{diam} V(G)$ and $\operatorname{effdiam}(G) = \operatorname{diam}(G)$.

The hyperbolicity constant $\delta(X)$ of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces with $\delta(X) = 0$ are precisely the metric trees. Hence, it is interesting to study the regular graphs with small hyperbolicity constant.

It is not difficult to characterize the graphs with $\delta(G) < 1$ (see [31, Theorem 11]). The next result provides a characterization of the graphs with $\delta(G) = 1$ (see [3, Proposition 4.5 and Theorem 4.14]).

Theorem 3.2. *A graph G verifies $\delta(G) = 1$ if and only if* $\operatorname{effdiam}(G) = 2$. *Furthermore, $\delta(G) \le 1$ if and only if* $\operatorname{effdiam}(G) \le 2$.

Given two graphs G, Γ, we write $G = \Gamma$ if G and Γ are isomorphic.

The following result characterizes the finite cubic graphs with hyperbolicity constant 1 (see [39, Theorem 2.4]).

Theorem 3.3. *If G is a finite cubic graph with $\delta(G) = 1$, then $G = K_4$ or $G = K_{3,3}$.*

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. The *graph join* $G_1 \uplus G_2$ of G_1 and G_2 has $V(G_1 \uplus G_2) = V(G_1) \cup V(G_2)$ and two different vertices u and v of $G_1 \uplus G_2$ are adjacent (*i.e.*, $[u, v] \in E(G_1 \uplus G_2)$) if $u \in V(G_1)$ and $v \in V(G_2)$, or $[u, v] \in E(G_1)$ or $[u, v] \in E(G_2)$.

We say that a graph F is an *empty graph* if $E(F) = \emptyset$. Denote by E_r the empty graph with r vertices.

The following result is the main tool in order to generalize Theorem 3.3.

Lemma 3.4. *If G is a Δ-regular graph with n vertices, $\Delta \ge 1$ and $\delta(G) \le 1$, then G is two-connected,* $\operatorname{diam}(G) \le 2$ *and $\Delta + 1 \le n \le 2\Delta$.*

Furthermore, $n = \Delta + 1$ if and only if $G = K_{\Delta+1}$, and $n = 2\Delta$ if and only if $G = K_{\Delta,\Delta}$.

Given any fixed vertex v in $V(G)$, denote by G^* the subgraph of G induced by $N(v)$ and by F the subgraph induced by $V(G) \setminus N(v)$. Then F is an empty graph with $n - \Delta$ vertices, G^* is a $(2\Delta - n)$-regular graph with Δ vertices, $N(u) = N(v)$ for every $u \in V(G) \setminus N(v)$, $G = G^* \uplus F = G^* \uplus E_{n-\Delta}$, and G^* either is an empty graph or it verifies $\operatorname{diam}(G^*) \le 2$.

Proof. Every finite 1-regular graph is isomorphic to the path graph P_2, $n = 2$ and $\delta(P_2) = 0$. Hence, the statements for $\Delta = 1$ follow from these facts.

Every finite 2-regular graph with n vertices is isomorphic to the cycle graph C_n, and $\delta(C_n) = n/4$. Thus we have either $n = 3$ or $n = 4$ and the statements for $\Delta = 2$ follow from these facts.

Assume now $\Delta \ge 3$.

Seeking for a contradiction assume that G has some cut-vertex. By Theorem 3.2, G verifies $\operatorname{effdiam}(G) \le 2$. Let $\{G_s\}$ be the canonical T-decomposition of G. Since G is a finite graph, there exists G_s in the canonical T-decomposition of G containing just one cut-vertex of G. We have $\operatorname{diam}(G_s) \le \operatorname{effdiam}(G) = 2$. Let $u \in V(G)$ be the cut-vertex of G contained in G_s. Since u is the unique cut-vertex in G_s, we have $N(w) \subseteq V(G_s)$ for every $w \in V(G_s) \setminus \{u\}$. Since $|N(u) \cap V(G_s)| < \Delta$, we have $N(w) \not\subseteq (\{u\} \cup N(u)) \cap V(G_s)$ for every $w \in N(u) \cap V(G_s)$ and there exists $z \in V(G_s) \setminus (\{u\} \cup N(u))$. Since $u \notin N(z)$ and $|N(u) \cap V(G_s)| < \Delta$, there exists $z' \in V(G_s) \setminus (\{u\} \cup N(u))$ with $[z, z'] \in E(G_s)$. Since $d_{G_s}(u, z) = d_{G_s}(u, z') = 2$, if p is the midpoint of $[z, z']$, then $d_{G_s}(u, p) = 5/2 > 2$. This contradicts $\operatorname{diam}(G_s) \le 2$, and we conclude that G is two-connected.

If $v \in V(G)$, then $n \ge |N(v)| + |\{v\}| = \Delta + 1$. Let us prove now $n \le 2\Delta$. By Theorem 3.2, G verifies $\operatorname{diam}(G) = \operatorname{effdiam}(G) \le 2$, since G is two-connected. Let $v \in V(G)$ be any fixed vertex in $V(G)$. If $V(G) = N(v) \cup \{v\}$, then $n = |N(v)| + |\{v\}| = \Delta + 1 \le 2\Delta$. If $V(G) \neq N(v) \cup \{v\}$, then let us fix $u \in V(G) \setminus (N(v) \cup \{v\})$. Seeking for a contradiction assume that there exists $z \in V(G) \setminus (N(v) \cup \{v\})$ with $[u, z] \in E(G)$. Since $d_G(v, u) = d_G(v, z) = 2$, if p is the midpoint of $[u, z]$, then $d_G(v, p) = 5/2 > 2$. This contradicts $\operatorname{diam}(G) \le 2$, and we conclude that the subgraph F induced by $V(G) \setminus N(v)$ is an empty graph and $N(u) \subseteq N(v)$. Since G is a regular graph, $N(u) = N(v)$ for every $u \in V(G) \setminus N(v)$. Hence, $|V(G) \setminus N(v)| \le \Delta$ and $n = |V(G) \setminus N(v)| + |N(v)| \le 2\Delta$. Furthermore, if we denote by G^* the subgraph induced by $N(v)$, then $G = G^* \uplus F$.

We have $n = 2\Delta$ if and only if $|V(G) \setminus N(v)| = \Delta$, *i.e.*, G^* is an empty graph (G is isomorphic to the complete bipartite graph $K_{\Delta,\Delta}$).

Note that $n = \Delta + 1$ if and only if $V(G) \setminus N(v)$ is a single vertex and G^* is isomorphic to the complete graph K_Δ, *i.e.*, G is isomorphic to the complete graph $K_{\Delta+1}$.

Assume that G^* is not an empty graph ($n < 2\Delta$). Let us prove now that $\operatorname{diam}(G^*) \le 2$.

Let $v_1 \in V(G^*)$ and $p_1 \in J(G^*) \setminus V(G^*)$. Since $\operatorname{diam}(G) \le 2$, we have $d_G(v_1, p_1) \le 3/2$. If a curve g joins v_1 and p_1 and contains some vertex in $V(G) \setminus V(G^*) = V(G) \setminus N(v)$, then $L(g) > 2$; therefore, $d_{G^*}(v_1, p_1) = d_G(v_1, p_1) \le 3/2$. Hence, $d_{G^*}(w_1, w_2) \le 2$ for every $w_1, w_2 \in J(G^*)$, and $\operatorname{diam}(G^*) \le 2$. In particular, G^* is connected.

Since $N(u) = N(v) = V(G^*)$ for every $u \in V(G) \setminus N(v)$ and

$$\Delta - |V(F)| = \Delta - |V(G) \setminus N(v)| = \Delta - (n - |N(v)|) = 2\Delta - n,$$

the graph G^* is $(2\Delta - n)$-regular. Furthermore, $|V(G^*)| = |N(v)| = \Delta$.

We have seen $|V(F)| = n - \Delta$. Since F is an empty graph, we have $F = E_{n-\Delta}$ and $G = G^* \uplus E_{n-\Delta}$. □

The following result is a kind of converse of Lemma 3.4, and it is interesting by itself (note that we do not require regularity).

Theorem 3.5. *Let G^* be a two-connected graph with* $\operatorname{diam}(G^*) \le 2$ *and F an empty graph with $V(G^*) \cap V(F) = \emptyset$. If G is the graph $G := G^* \uplus F$, then* $\operatorname{diam}(G^*) \le 2$ *and $\delta(G) \le 1$. Furthermore, $\delta(G) = 1$ if and only if $|V(G^*)| \ge 2$ and $|V(G)| \ge 4$.*

Proof. Since $\operatorname{diam}(G^*) \le 2$, we have $d_{G^*}(x,y) \le 2$ for every $x, y \in V(G^*)$ and $d_{G^*}(x,y) \le 3/2$ for every $x \in J(G^*) \setminus V(G^*)$ and $y \in V(G^*)$.

If $x, y \in V(G)$, then it is clear that $d_G(x,y) \le 2$.

If $x \in J(G^*) \setminus V(G^*)$ and $y \in V(F)$, then $d_G(x,y) \le 3/2$; therefore, $d_G(x,y) \le 2$ for every $x \in J(G^*) \setminus V(G^*)$ and $y \in J(G)$.

If $x \in J(G) \setminus (V(G) \cup J(G^*))$ and $y \in V(G)$, then $d_G(x,y) \le 3/2$; thus, $d_G(x,y) \le 2$ for every $x \in J(G) \setminus (V(G) \cup J(G^*))$ and $y \in J(G)$.

Hence, $\operatorname{diam}(G) \le 2$ and Theorem 3.2 gives $\delta(G) \le 1$.

Note that $|V(G^*)| \ge 2$ and $|V(G)| \ge 4$ if and only if G contains a cycle C with $L(C) \ge 4$, and by [31, Theorem 11], this is equivalent to $\delta(G) \ge 1$. Since $\delta(G) \le 1$ in this context, the previous requisite is fulfilled if and only if $\delta(G) = 1$. □

We need the following result (see [40, Theorem 11]).

Theorem 3.6. *The complete graph K_n verifies $\delta(K_n) = 1$ if and only if $n \ge 4$. The complete bipartite graph K_{n_1,n_2} verifies $\delta(K_{n_1,n_2}) = 1$ if and only if $n_1, n_2 \ge 2$.*

Given two graphs Γ_1, Γ_2, we define $\Gamma_1(\uplus\Gamma_2)^r$ inductively as $\Gamma_1(\uplus\Gamma_2)^0 := \Gamma_1$, $\Gamma_1(\uplus\Gamma_2)^1 := \Gamma_1 \uplus \Gamma_2$ and $\Gamma_1(\uplus\Gamma_2)^r := (\Gamma_1(\uplus\Gamma_2)^{r-1}) \uplus \Gamma_2$ for $r \ge 2$.

The following result generalizes Theorem 3.3 to Δ-regular graphs.

Theorem 3.7. *Let G be a Δ-regular graph with n vertices and $\Delta \ge 3$. Then $\delta(G) = 1$ if and only if we have either*

$$n = \Delta + 1 \quad \text{or} \quad n = \frac{j+2}{j+1}\Delta \quad \text{for some } 0 \le j \le J := \left\lfloor \frac{\Delta}{2} - 1 \right\rfloor, \tag{3.2}$$

and $G = K_{\Delta+1}$ if $n = \Delta + 1$, $G = K_{\Delta,\Delta}$ if $n = 2\Delta$ ($j = 0$), and $G = K_{n-\Delta,n-\Delta}(\uplus E_{n-\Delta})^j$ if $n = \Delta(j+2)/(j+1)$ for some $1 \le j \le J$.

Proof. Let us define $n_0 := n$, $\Delta_0 := \Delta$,

$$\Delta_{j+1} := 2\Delta_j - n_j, \quad n_{j+1} := \Delta_j, \quad \text{for } j \ge 0.$$

A direct computation gives $n_{j+1} - \Delta_{j+1} = \Delta_j - (2\Delta_j - n_j) = n_j - \Delta_j$ for every $j \ge 0$. Hence, $n_j - \Delta_j = n - \Delta$ for every $j \ge 0$. Since $\Delta_{j+1} = 2\Delta_j - n_j = \Delta_j - (n_j - \Delta_j) = \Delta_j - (n - \Delta)$, we conclude

$$\Delta_j = \Delta - j(n - \Delta), \quad n_j = \Delta_{j-1} = \Delta - (j-1)(n - \Delta), \quad \text{for } j \ge 0.$$

We have $n_{j+1} = n_j - (n-\Delta) \le n_j - 1$ and $\Delta_{j+1} = \Delta_j - (n-\Delta) \le \Delta_j - 1$.

Note that J is the greatest integer j satisfying the inequality

$$\Delta + 2 \le \frac{j+2}{j+1}\Delta,$$

and thus

$$\frac{J+3}{J+2}\Delta < \Delta + 2. \tag{3.3}$$

Assume first that $\delta(G) = 1$.

Given any fixed vertex v in $V(G)$, denote by $G^{*,1} = G^*$ the subgraph of G induced by $N(v)$. Lemma 3.4 gives that $G^{*,1}$ is a $(2\Delta - n)$-regular graph with Δ vertices (*i.e.*, a Δ_1-regular graph with n_1 vertices), $G = G^{*,1} \uplus E_{n-\Delta}$, and $G^{*,1}$ either is an empty graph or it verifies $\operatorname{diam}(G^{*,1}) \le 2$. Furthermore, $\Delta + 1 \le n \le 2\Delta$, $n = \Delta + 1$ if and only if $G = K_{\Delta+1}$, and $n = 2\Delta$ (*i.e.*, $\Delta_1 = 0$) if and only if $G^{*,1}$ is an empty graph, and then $G = K_{\Delta,\Delta}$. If $n < 2\Delta$, then $\operatorname{diam}(G^{*,1}) \le 2$ and $\delta(G^{*,1}) \le 1$ by Theorem 3.2.

Let us define inductively $G^{*,j}$. Assume that $G^{*,j}$ is defined and it is a Δ_j-regular graph with n_j vertices, $\Delta_j \ge 1$ and $\delta(G^{*,j}) \le 1$. Given any $v_j \in V(G^{*,j})$, denote by $G^{*,j+1}$ the subgraph of $G^{*,j}$ induced by $N_{G^{*,j}}(v_j)$. Lemma 3.4 gives that $G^{*,j} = G^{*,j+1} \uplus E_{n_j - \Delta_j} = G^{*,j+1} \uplus E_{n-\Delta}$, and $G^{*,j+1}$ either is an empty graph or it verifies $\operatorname{diam}(G^{*,j+1}) \le 2$. Furthermore, $\Delta_j + 1 \le n_j \le 2\Delta_j$, and $n_j = 2\Delta_j$ (*i.e.*, $\Delta_{j+1} = 0$) if and only if $G^{*,j+1}$ is an empty graph, and thus $G^{*,j}$ is isomorphic to the complete bipartite graph K_{Δ_j,Δ_j}; since $\Delta_{j+1} = 0$, we deduce $\Delta_j - (n - \Delta) = \Delta_{j+1} = 0$, $\Delta_j = n - \Delta$ and $K_{\Delta_j,\Delta_j} = K_{n-\Delta,n-\Delta}$. If $n_j = 2\Delta_j$, then the sequence stops for this value of j. If $n_j < 2\Delta_j$, then $\Delta_{j+1} \ge 1$, $\operatorname{diam}(G^{*,j+1}) \le 2$ and $\delta(G^{*,j+1}) \le 1$ by Theorem 3.2.

We have $n_j < 2\Delta_j$ if and only if $\Delta - (j-1)(n-\Delta) < 2\Delta - 2j(n-\Delta)$, *i.e.*, $n < \Delta(j+2)/(j+1)$.

The same argument gives that $n_j = 2\Delta_j$ if and only if $n = \Delta(j+2)/(j+1)$.

Hence, if $n = \Delta(j+2)/(j+1)$ for some $1 \le j \le J$, then $G = G^{*,1} \uplus E_{n-\Delta}$, $G^{*,i} = G^{*,i+1} \uplus E_{n_i-\Delta_i} = G^{*,i+1} \uplus E_{n-\Delta}$ for $1 \le i < j$ and $G^{*,j} = K_{\Delta_j,\Delta_j} = K_{n-\Delta,n-\Delta}$. Thus $G = K_{n-\Delta,n-\Delta}(\uplus E_{n-\Delta})^j$.

Now, if $n \ne \Delta(j+2)/(j+1)$ for every $0 \le j \le J$, then $n_J < 2\Delta_J$ and $n < \Delta(J+2)/(J+1)$. Seeking for a contradiction, assume that $n \ne \Delta(j+2)/(j+1)$ for every $j \ge 0$; therefore, $\Delta + 1 \le n < \Delta(j+2)/(j+1)$ for every $j \ge 0$ and thus $\Delta + 1 \le \Delta$, which is a contradiction. Hence, $n = \Delta(j+2)/(j+1)$ for some $j > J$, and so $n = \Delta(j+2)/(j+1) \le \Delta(J+3)/(J+2)$. Therefore, (3.3) gives $\Delta + 1 \le n \le \Delta(J+3)/(J+2) < \Delta + 2$ and we conclude $n = \Delta + 1$.

Finally assume that we have $G = K_{\Delta+1}$ if $n = \Delta + 1$, $G = K_{\Delta,\Delta}$ if $n = 2\Delta$ ($j = 0$), or $G = K_{n-\Delta,n-\Delta}(\uplus E_{n-\Delta})^j$ if $n = \Delta(j+2)/(j+1)$ for some $1 \le j \le J$. Theorem 3.5 gives that $\delta(G) \le 1$.

Theorem 3.6 gives $\delta(G) = 1$ if $n = \Delta + 1$ or $n = 2\Delta$, since $\Delta \ge 3$. Assume now that $G = K_{n-\Delta,n-\Delta}(\uplus E_{n-\Delta})^j$ and $n = \Delta(j+2)/(j+1)$ for some $1 \le j \le J$. Thus $\Delta_{j+1} = 0$ and $\Delta - j(n - \Delta) = \Delta_j \ge \Delta_{j+1} + 1 = 1$. Since

$$\left|V\big(K_{n-\Delta,n-\Delta}(\uplus E_{n-\Delta})^{j-1}\big)\right| \ge |V(K_{n-\Delta,n-\Delta})| = |V(K_{\Delta_j,\Delta_j})| = 2\Delta_j \ge 2,$$

$$|V(G)| = n \ge \Delta + 1 \ge 4,$$

Theorem 3.5 gives $\delta(G) = 1$. □

The hypothesis $\Delta \ge 3$ in Theorem 3.7 is not a real restriction, since the cases $\Delta = 1$ and $\Delta = 2$ are very simple, as Corollary 3.8 below shows. Furthermore, it shows explicitly the graphs with $\delta(G) = 1$ for small values of Δ.

Corollary 3.8. *Assume that G is a finite Δ-regular graph.*

If $\Delta = 1$, then $\delta(G) = 0$.

If $\Delta = 2$, then $\delta(G) = 1$ if and only if $G = C_4 = K_{2,2}$.

If $\Delta = 3$, then $\delta(G) = 1$ if and only if $G = K_4$ or $G = K_{3,3}$.

If $\Delta = 4$, then $\delta(G) = 1$ if and only if $G = K_5$, $G = K_{4,4}$ or $G = K_{2,2} \uplus E_2$.

If $\Delta = 5$, then $\delta(G) = 1$ if and only if $G = K_6$ or $G = K_{5,5}$.

If $\Delta = 6$, then $\delta(G) = 1$ if and only if $G = K_7$, $G = K_{6,6}$, $G = K_{3,3} \uplus E_3$ or $G = (K_{2,2} \uplus E_2) \uplus E_2$.

If $\Delta = 7$, then $\delta(G) = 1$ if and only if $G = K_8$ or $G = K_{7,7}$.

Proof. If $\Delta = 1$, then $G = P_2$ and $\delta(G) = \delta(P_2) = 0$.

Every finite 2-regular graph G is isomorphic to the cycle graph C_n with $n = |V(G)| \ge 3$, and $\delta(C_n) = n/4$. Thus $\delta(G) = 1$ if and only if G is isomorphic to the cycle graph $C_4 = K_{2,2}$.

If $\Delta = 3$, $\Delta = 5$ or $\Delta = 7$, then Theorem 3.7 gives the result.

If $\Delta = 4$, then $J = 1$ and Theorem 3.7 gives that $\delta(G) = 1$ if and only if we have either $G = K_5$ (if $n = 5$), $G = K_{4,4}$ (if $n = 8$), or $G = K_{2,2} \uplus E_2$ (if $j = 1$ and $n = (1+2)\Delta/(1+1) = 6$).

If $\Delta = 6$, then $J = 2$ and Theorem 3.7 gives that $\delta(G) = 1$ if and only if we have either $G = K_7$ (if $n = 7$), $G = K_{6,6}$ (if $n = 12$), $G = K_{3,3} \uplus E_3$ (if $j = 1$ and $n = (1+2)\Delta/(1+1) = 9$) or $G = (K_{2,2} \uplus E_2) \uplus E_2$ (if $j = 2$ and $n = (2+2)\Delta/(2+1) = 8$). □

Theorem 3.7 and Corollary 3.8 have the following beautiful and unexpected consequence (it provides a sophisticated characterization of prime numbers in terms of hyperbolic graphs).

Theorem 3.9. *Let $\Delta \geq 3$. The following statements are equivalent:*

- *Δ is a prime number.*
- *We have $\delta(G) = 1$ with G a finite Δ-regular graph if and only if $G = K_{\Delta+1}$ or $G = K_{\Delta,\Delta}$.*

Proof. Assume first that Δ is a prime number. Since $j + 1 \leq J + 1 = \lfloor \Delta/2 \rfloor < \Delta$, we have

$$\gcd\{\Delta, j+1\} = 1.$$

Furthermore,

$$\gcd\{j+2, j+1\} = 1$$

if $j \geq 1$, and hence,

$$\frac{j+2}{j+1}\Delta \notin \mathbb{N}$$

for every $j \geq 1$, and Theorem 3.7 gives the result.

Assume now that $\delta(G) = 1$ with G a finite Δ-regular graph if and only if $G = K_{\Delta+1}$ or $G = K_{\Delta,\Delta}$. By Theorem 3.7, we have that $\Delta(j+2)/(j+1)$ is not an integer number for every $1 \leq j \leq J$. By Corollary 3.8, we can assume $\Delta \geq 8$. Seeking for a contradiction assume that Δ is not a prime number, and then $\Delta = m_1 m_2$ with integers m_1 and m_2 verifying $m_1, m_2 \geq 2$. By symmetry, we can assume $2 \leq m_1 \leq m_2$ and, therefore, $m_1 \leq \lfloor \sqrt{\Delta} \rfloor$. Since

$$\sqrt{\Delta} \leq \frac{\Delta}{2} - 1$$

for every $\Delta \geq 8$, we have that $m_1 \leq J$ and $\Delta(m_1+1)/m_1 = m_2(m_1+1)$ is an integer number, which is a contradiction. Hence, Δ is a prime number. □

4 Conclusion

The main aim of this paper is to obtain results about the hyperbolicity constant of Δ-regular graphs. We obtain several bounds for this parameter (see Theorems 2.6 and 2.14, and Proposition 2.9); in particular, Theorem 2.6 gives $\delta(G) \leq \Delta n/(8(\Delta - 1)) + 1$ for any Δ-regular graph G with n vertices. Furthermore, we show in Theorem 2.19 that for each $\Delta \geq 2$ and every possible value t of the hyperbolicity constant, there exists a Δ-regular graph G with $\delta(G) = t$. We also study in Theorems 3.7 and 3.9 and Corollary 3.8 the regular graphs G with $\delta(G) \leq 1$. Besides, we prove some inequalities involving the hyperbolicity constant and other parameters for regular graphs (Theorem 2.12 gives that for any regular graph G with n vertices and k-domination numbers $\gamma_k(G)$, we have $\delta(G) + \gamma_k(G) \leq n$ for every $1 \leq k \leq \Delta$).

A natural open problem is to obtain a generalization of the inequalities for the hyperbolicity constant of cubic graphs in [35, 39] to the context of regular graphs. Also, it would be desirable to relate the hyperbolicity constant of regular graphs with other parameters (such as its girth, circumference or the first eigenvalue of the adjacency matrix). For future works, it would be interesting to study how to improve the results in [6, 2, 9, 30, 47, 12, 14, 32, 36, 45] when we just consider regular graphs.

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