$P_m \cup P_k$-equipackable paths and cycles *

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Abstract

A graph G is called H-equipackable if every maximal H-packing in G is also a maximum H-packing in G. In 2012, P_k-equipackable paths and cycles, M_k-equipackable paths and cycles are characterized. In this paper, $P_m \cup P_k$-equipackable paths and cycles are characterized.

Keywords: Equipackable, path, cycle.

1 Introduction

The problem that we study stems from the research of H-decomposable graphs, randomly packable graphs. For further definitions and results, we refer the reader to [1]. The path and cycle on n vertices are denoted by P_n and C_n, respectively. In this paper, $P_m \cup P_k$ denote the union of P_m and P_k which are vertex-disjoint. Without loss of generality, we shall assume $k \geq m$. The edge set of P_n is denoted by $E(P_n) = \{e_1, e_2, \cdots, e_{n-1}\}$. The edge set of C_n is denoted by $E(C_n) = \{e_1, e_2, \cdots, e_n\}$. A vertex with degree 1 of a path is called an end vertex of the path. Let H be a subgraph of G. By $G - H$, we denote the graph left after we delete from G the edges of H and any resulting isolated vertices.

A collection of edge disjoint copies of H, say $H_1, H_2, \cdots, H_l$, where each $H_i (i = 1, 2, \cdots, l)$ is a subgraph of G, is called an H-packing in G. A graph G is called H-packable if there exists an H-packing of G. An H-packing in G with l copies $H_1, H_2, \cdots, H_l$ of H is called maximal if $G - \bigcup_{i=1}^{l} E(H_i)$ contains no subgraph isomorphic to H. An H-packing in G with l copies $H_1, H_2, \cdots, H_l$ of H is called maximum if no more than l edge disjoint copies of H can be packed into G. Let $p(G; H)$ denote the

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‡Foundation items: This research was supported by Natural Science Foundation of Hebei province(No.A20152023010), the Natural Science Foundation of Tianjin(No.15JCYBJC19100), National Natural Science Foundation of China (No.11401430).

number of H in the maximum H-packing of G. A graph G is called H-decomposable if there exists an H-packing of G which uses all edges in G. A graph G is called H-equipackable if every maximal H-packing in G is also a maximum H-packing in G. In 2006, Zhang and Fan([2]) characterized M_2-equipackable graphs. In 2010, B. Randerath and P. D. Vestergaard([3]) characterized all P_3-equipackable graphs. In [4], P_k-equipackable paths and cycles, M_k-equipackable paths and cycles are characterized.

In this paper, we investigate $P_m \cup P_k$-equipackable paths and cycles.

We first give a lemma which is important to our work:

Lemma 1. *([4]) Let G be an F-packable graph and H be an F-packable subgraph of G which satisfy: (1) H is not F-equipackable; (2) $G - H$ is F-decomposable. Then G is not F-equipackable.*

2 Main results

In 2006, Zhang and Fan([2]) characterized all M_2-equipackable graphs. The path P_n is M_2-equipackable if and only if $n = 2t (t \in \mathcal{Z}, t \geq 2)$. The cycle C_n is M_2-equipackable if and only if $n = 2t + 1 (t \in \mathcal{Z}, t \geq 2)$. In the following, we discuss the case when $k > m = 2$ and $k \geq m > 2$.

2.1 $P_m \cup P_k$-equipackable paths

Theorem 2. *A path P_n is $P_2 \cup P_k$-equipackable $(k > 2)$ if and only if*
$$\begin{cases} n = 5, 6, 8, 9, & \text{when } k = 3 \\ n = 6, 7, 8, 12, & \text{when } k = 4 \\ k + 2 \leq n \leq 2k, & \text{when } k > 4 \end{cases}.$$

Proof. 1. $n \leq k + 1$, since P_n contains no copy of $P_2 \cup P_k$, P_n can not be $P_2 \cup P_k$-equipackable.

2. $k + 2 \leq n \leq 2k$, it is easy to see $p(P_n; P_2 \cup P_k) = 1$. And P_n is $P_2 \cup P_k$-packable, so each maximal $P_2 \cup P_k$-packing is also a maximum $P_2 \cup P_k$-packing. Thus P_n must be $P_2 \cup P_k$-equipackable.

3. $2k + 1 \leq n \leq 3k$, it is easy to see $p(P_n; P_2 \cup P_k) = 2$. In the following, there are two subcases:

   (a) $2k + 1 \leq n \leq 3k - 2$, there exists a $P_2 \cup P_k$-packing H with only one copy of $P_2 \cup P_k$ such that $G - H$ has three components denoted by H_1, H_2 and H_3, with $|E(H_1)| = 1$, $|E(H_2)| = k - 2$, $1 \leq |E(H_3)| \leq k - 2$. P_n is not $P_2 \cup P_k$-equipackable.

(b) $3k-1 \le n \le 3k$, for $k=3$, it's easy to verify P_n is $P_2 \cup P_3$-equipackable. For $k=4$, P_{11} is not $P_2 \cup P_4$-equipackable and P_{12} is $P_2 \cup P_4$-equipackable. For $k>4$, there exists a $P_2 \cup P_k$-packing H with only one copy of $P_2 \cup P_k$ such that $G-H$ has three components denoted by H_1, H_2 and H_3, with $|E(H_1)|=2$ or 3, $|E(H_2)|=|E(H_3)|=k-2$, P_n is not $P_2 \cup P_k$-equipackable.

4. $3k+1 \le n \le 4k$, P_n is not $P_2 \cup P_k$-equipackable. It is easy to see $p(P_n; P_2 \cup P_k)=3$, there are two subcases:

   (a) $n=3k+1$, there exists a $P_2 \cup P_k$-packing H with two copies of $P_2 \cup P_k$, such that $G-H$ has three components denoted by H_1, H_2 and H_3 with $|E(H_1)|=k-2$, $|E(H_2)|=|E(H_3)|=1$. So H is a maximal $P_2 \cup P_k$-packing which is not maximum. By the definition, P_n is not $P_2 \cup P_k$-equipackable.

   (b) $3k+2 \le n \le 4k$, there exists a $P_2 \cup P_k$-packing H with two copies of $P_2 \cup P_k$, such that $G-H$ has five components denoted by H_1, H_2, H_3, H_4 and H_5 with $|E(H_1)|=k-2$, $|E(H_2)|=|E(H_3)|=|E(H_4)|=1$, $0 \le |E(H_5)| \le k-2$. So H is a maximal $P_2 \cup P_k$-packing which is not maximum. By the definition, P_n is not $P_2 \cup P_k$-equipackable.

5. $n \ge 4k+1$, there are two subcases:

   (a) $n-(2k+1) \equiv r \pmod{k}$ $(r=0,1,\cdots,k-3)$, $P_n - P_{2k+1+r}$ has kt $(t \in \mathcal{Z}, t \ge 2)$ edges, so $P_n - P_{2k+1+r}$ is $P_2 \cup P_k$-decomposable. By Lemma 1, P_n is not $P_2 \cup P_k$-equipackable.

   (b) $n-(2k+1) \equiv s \pmod{k}$ $(s=k-2, k-1)$, there are two possibilities:

      - $5k-1 \le n \le 5k$, it is easy to see $p(P_n; P_2 \cup P_k)$ is 4. And the number of every maximal $P_2 \cup P_k$-packing of P_n is 3 or 4. So P_n is not $P_2 \cup P_k$-equipackable.
      - $n \ge 6k-1$, $P_n - P_{3k+1+s}$ is $P_2 \cup P_k$-decomposable. By Lemma 1, P_n is not $P_2 \cup P_k$-equipackable.

□

Theorem 3. *A path P_n is $P_m \cup P_k$-equipackable $(k \ge m > 2)$ if and only if*
$$m+k \le n \le 2m+2k-4 \text{ or } \begin{cases} 4m+2k-6 \le n \le 3m+3k-6, & m \le k \le \frac{3}{2}m \\ m+4k-6 \le n \le 3m+3k-6, & \frac{3m}{2} < k \le 2m \end{cases}.$$

Proof. 1. $n \le m+k-1$, since P_n contains no copy of $P_m \cup P_k$, P_n can not be $P_m \cup P_k$-equipackable.

2. $m+k \le n \le 2m+2k-4$, it is easy to see $p(P_n; P_m \cup P_k) = 1$. And P_n is $P_m \cup P_k$-packable, so each maximal $P_m \cup P_k$-packing is also a maximum $P_m \cup P_k$-packing. Thus P_n must be $P_m \cup P_k$-equipackable.

3. $2m+2k-3 \le n \le 3m+3k-6$, it is easy to see $p(P_n; P_m \cup P_k) = 2$.

   To get the maximal packing with only one copy $H = P_m \cup P_k$ which satisfies that $|E(G-H)| = |E(H_1) \cup E(H_2) \cup E(H_3)|$ is maximum, see Fig.1. There are two possibilities:

   (i) $|E(H_1)| = |E(H_2)| = m-2$, $|E(H_3)| = m+k-2$. So $|E(G-H)| = 3m+k-6$;

   (ii) $|E(H_1)| = |E(H_2)| = |E(H_3)| = k-2$. So $|E(G-H)| = 3k-6$.

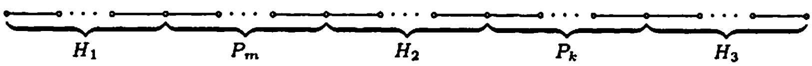

Figure 1:

   When $3m+k-6 \ge 3k-6$, that is, $k \le \frac{3}{2}m$, $|E(H_1)| = |E(H_2)| = m-2$ and $|E(H_3)| = m+k-2$ which makes $|E(G-H)|$ get to the maximum.

   When $3m+k-6 < 3k-6$, that is, $k > \frac{3}{2}m$, $|E(H_1)| = |E(H_2)| = |E(H_3)| = k-2$ which makes $|E(G-H)|$ get to the maximum.

   Thus we have the following two cases:

   (a) $m \le k \le \frac{3}{2}m$, there are two subcases:

   - $2m+2k-3 \le n \le 4m+2k-7$, there exists a $P_m \cup P_k$-packing H with only one copy of $P_m \cup P_k$ such that $G-H$ has three components denoted by H_1, H_2 and H_3, with $|E(H_1)| = |E(H_2)| = m-2$, $k-m+2 \le |E(H_3)| \le m+k-2$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, P_n is not $P_m \cup P_k$-equipackable. From above, when $|E(H_1)| = |E(H_2)| = m-2$, $|E(H_3)| = m+k-2$, the maximal $P_m \cup P_k$-packing of P_n with one copy of $P_m \cup P_k$ makes $|E(G-H)|$ get to the maximum.
   - $4m+2k-6 \le n \le 3m+3k-6$, the number of every maximal $P_m \cup P_k$-packing of P_n is 2 by the Pigeonhole Principle. So P_n is $P_m \cup P_k$-equipackable.

   (b) $k > \frac{3m}{2}$, there are two subcases:

- $2m+2k-3 \le n \le m+3k-6$, similarly, $|E(H_1)| = m-2$, $|E(H_2)| = k-2$, $2 \le |E(H_3)| \le k-m-1 < k-1$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, P_n is not $P_m \cup P_k$-equipackable.
- $m+3k-5 \le n \le 3m+3k-6$.
  i. $k \le 2m$, then $m+4k-6 \le 3m+3k-6$. There are the following two possibilities:
  When $m+3k-5 \le n \le m+4k-7$, similarly, $|E(H_1)| = |E(H_2)| = k-2$, $0 \le |E(H_3)| \le k-2$. Thus P_n is not $P_m \cup P_k$-equipackable. When $|E(H_1)| = |E(H_2)| = |E(H_3)| = k-2$, the maximal $P_m \cup P_k$-packing of P_n with one copy of $P_m \cup P_k$ makes $|E(G-H)|$ maximum. When $m+4k-6 \le n \le 3m+3k-6$, the number of every maximal $P_m \cup P_k$-packing of P_n is 2 by the Pigeonhole Principle. So P_n is $P_m \cup P_k$-equipackable.
  ii. $k > 2m$.
  When $m+3k-5 \le n \le 3m+3k-6$, similarly, $|E(H_1)| = |E(H_2)| = k-2$, $0 \le |E(H_3)| \le 2m-1 < k-1$. So P_n is not $P_m \cup P_k$-equipackable.

4. $3m+3k-5 \le n \le 4m+4k-8$, P_n is not $P_m \cup P_k$-equipackable. It is easy to see $p(P_n; P_m \cup P_k) = 3$, there are two subcases:

   (a) $3m+3k-5 \le n \le 3m+4k-7$, there exists a $P_m \cup P_k$-packing H with two copies of $P_m \cup P_k$, such that $G-H$ has five components denoted by H_1, H_2, H_3, H_4 and H_5 with $|E(H_1)| = m-2$, $|E(H_2)| = k-2$, $|E(H_3)| = |E(H_4)| = 1$, $0 \le |E(H_5)| \le k-2$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, P_n is not $P_m \cup P_k$-equipackable.

   (b) $3m+4k-6 \le n \le 4m+4k-8$, similar to the subcase (a), $|E(H_1)| = m-1$, $|E(H_2)| = |E(H_3)| = k-2$, $|E(H_4)| = 1$, $1 \le |E(H_5)| \le m-1$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, P_n is not $P_m \cup P_k$-equipackable.

5. $n \ge 4m+4k-7$, we only discuss the case when $m \le k \le \frac{3}{2}m$, there are two subcases:

   (a) $n-(2m+2k-3) \equiv r \pmod{m+k-2}$ $(r = 0, 1, \cdots, 2m-4)$, $P_n - P_{2m+2k-3+r}$ has $(k+m-2)t$ $(t \in \mathcal{Z}, t \ge 2)$ edges, so $P_n - P_{2m+2k-3+r}$ is $P_m \cup P_k$-decomposable. By case 3, P_n is not $P_m \cup P_k$-equipackable.

(b) $n-(2m+2k-3) \equiv s \pmod{m+k-2}$ $(s=2m-3, 2m-2, \cdots, m+k-3)$, there are two possibilities:

- $6m+4k-10 \leq n \leq 5m+5k-10$, it is easy to see $p(P_n; P_m \cup P_k)$ is 4. There exists a $P_m \cup P_k$-packing H with three copies of $P_m \cup P_k$, such that $G-H$ has seven components denoted by $H_1, H_2, H_3, H_4, H_5, H_6$ and H_7 with $|E(H_1)| = |E(H_2)| = |E(H_3)| = m-2$, $|E(H_4)| = k-2$, $|E(H_5)| = 2$, $|E(H_6)| = 1$, $0 \leq |E(H_7)| \leq k-m$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, P_n is not $P_m \cup P_k$-equipackable.
- $n \geq 7m+5k-12$, $P_n - P_{3m+3k-5+s}$ is $P_m \cup P_k$-decomposable. By Lemma 1, P_n is not $P_m \cup P_k$-equipackable.

When $k > \frac{3}{2}m$, we discuss similarly, P_n is also not $P_m \cup P_k$-equipackable. □

2.2 $P_m \cup P_k$-equipackable cycles

Theorem 4. *A cycle C_n is $P_2 \cup P_k$-equipackable ($k > 2$) if and only if*

$$\begin{cases} n = 5, 6, 7, 8, 11, & \text{when } k = 3 \\ k+2 \leq n \leq 2k-1 \text{ or } 3k-3 \leq n \leq 3k-1, & \text{when } k > 3 \end{cases}.$$

Proof. 1. $n \leq k+1$, since C_n contains no copy of $P_2 \cup P_k$, C_n can not be $P_2 \cup P_k$-equipackable.

2. $k+2 \leq n \leq 2k-1$, it is easy to see $p(C_n; P_2 \cup P_k) = 1$. And C_n is $P_2 \cup P_k$-packable, so each maximal $P_2 \cup P_k$-packing is also a maximum $P_2 \cup P_k$-packing. Thus C_n must be $P_2 \cup P_k$-equipackable.

3. $2k \leq n \leq 3k-1$, it is easy to see $p(C_n; P_2 \cup P_k) = 2$. For $k=3$, C_n is $P_2 \cup P_3$-equipackable. For $k \geq 4$, there are two subcases:

   (a) $2k \leq n \leq 3k-4$, there exists a $P_2 \cup P_k$-packing H with only one copy of $P_2 \cup P_k$ such that $G-H$ has two components denoted by H_1 and H_2, with $|E(H_1)| = k-2$, $2 \leq |E(H_2)| \leq k-2$. So H is a maximal $P_2 \cup P_k$-packing which is not maximum. By the definition, C_n is not $P_2 \cup P_k$-equipackable. When $|E(H_1)| = |E(H_2)| = k-2$, the maximal $P_2 \cup P_k$-packing of C_n with one copy of $P_2 \cup P_k$ makes $|E(G-H)|$ get to the maximum.

   (b) $3k-3 \leq n \leq 3k-1$, the number of every maximal $P_2 \cup P_k$-packing of P_n is 2 by the Pigeonhole Principle. So C_n is $P_2 \cup P_k$-equipackable.

4. $3k \leq n \leq 4k-1$, C_n is not $P_2 \cup P_k$-equipackable. It is easy to see $p(C_n; P_2 \cup P_k) = 3$, there are two subcases:

(a) $3k \leq n \leq 4k-2$, there exists a $P_2 \cup P_k$-packing H with two copies of $P_2 \cup P_k$, such that $G - H$ has four components denoted by H_1, H_2, H_3 and H_4 with $|E(H_1)| = k-2$, $|E(H_2)| = |E(H_3)| = 1$, $0 \leq |E(H_4)| \leq k-2$. So H is a maximal $P_2 \cup P_k$-packing which is not maximum. By the definition, C_n is not $P_2 \cup P_k$-equipackable.

(b) $n = 4k - 1$, for $k = 3$, it's easy to verify C_n is $P_2 \cup P_k$-equipackable. For $k > 3$, there exists a $P_2 \cup P_k$-packing H with two copies of $P_2 \cup P_k$ such that $G - H$ has four components denoted by H_1, H_2, H_3 and H_4, with $|E(H_1)| = k-2$, $|E(H_2)| = k-3$, $|E(H_3)| = |E(H_4)| = 2$, C_n is not $P_2 \cup P_k$-equipackable.

5. $n \geq 4k$, there are two subcases:

(a) $n - 2k \equiv r \pmod{k}$ $(r = 0, 1, \cdots, k-3)$, $C_n - P_{2k+1+r}$ has kt $(t \in \mathcal{Z}, t \geq 2)$ edges, so $C_n - P_{2k+1+r}$ is $P_2 \cup P_k$-decomposable. By Lemma 1, C_n is not $P_2 \cup P_k$-equipackable.

(b) $n - 2k \equiv s \pmod{k}$ $(s = k-2, k-1)$, there are two possibilities:

- $5k - 2 \leq n \leq 5k - 1$, it is easy to see $p(C_n; P_2 \cup P_k)$ is 4. And the number of every maximal $P_2 \cup P_k$-packing of P_n is 3 or 4. So C_n is not $P_2 \cup P_k$-equipackable.
- $n \geq 6k - 2$, $C_n - P_{3k+1+s}$ is $P_2 \cup P_k$-decomposable. By Lemma 1, C_n is not $P_2 \cup P_k$-equipackable.

□

Theorem 5. *A cycle C_n is $P_m \cup P_k$-equipackable $(k \geq m > 2)$ if and only if $m + k \leq n \leq 2m + 2k - 5$ or*

$$\begin{cases} 3m + 2k - 5 \leq n \leq 3m + 3k - 7, & \text{when } m \leq k \leq 2m \\ m + 3k - 5 \leq n \leq 3m + 3k - 7, & \text{when } k > 2m \end{cases}$$

Proof. 1. $n \leq m + k - 1$, since C_n contains no copy of $P_m \cup P_k$, C_n can not be $P_m \cup P_k$-equipackable.

2. $m+k \leq n \leq 2m+2k-5$, it's easy to see that $p(C_n; P_m \cup P_k)$ is 1. And C_n is $P_m \cup P_k$-packable, so each maximal $P_m \cup P_k$-packing is also a maximum $P_m \cup P_k$-packing. Thus C_n must be $P_m \cup P_k$-equipackable.

3. $2m + 2k - 4 \leq n \leq 3m + 3k - 7$, it's easy to see $p(C_n; P_m \cup P_k) = 2$. To get the maximal packing with only one copy $H = P_m \cup P_k$ which satisfies that $|E(G - H)| = |E(H_1) \cup E(H_2)|$ is maximum. There are two possibilities:

(i) $|E(H_1)| = m-2$, $|E(H_2)| = m+k-2$. So $|E(G - H)| = 2m+k-4$;

(ii) $|E(H_1)| = |E(H_2)| = k-2$. So $|E(G-H)| = 2k-4$.

When $2m+k-4 \geq 2k-4$, that is, $k \leq 2m$, $|E(H_1)| = m-2$, $|E(H_2)| = m+k-2$ which makes $|E(G-H)|$ get to the maximum.

When $2m+k-4 < 2k-4$, that is, $k > 2m$, $|E(H_1)| = |E(H_2)| = k-2$ which makes $|E(G-H)|$ get to the maximum.

Thus we have the following two cases:

(a) $m \leq k \leq 2m$. There are two subcases:

- When $2m+2k-4 \leq n \leq 3m+2k-6$, there exists a $P_m \cup P_k$-packing H with only one copy of $P_m \cup P_k$ such that $G-H$ has two components denoted by H_1 and H_2 with $|E(H_1)| = m-2$, $k \leq |E(H_2)| \leq m+k-2$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, C_n is not $P_m \cup P_k$-equipackable. When $|E(H_1)| = m-2$, $|E(H_2)| = m+k-2$, the maximal $P_m \cup P_k$-packing of C_n with one copy of $P_m \cup P_k$ makes $|E(G-H)|$ get to the maximum.
- When $3m+2k-5 \leq n \leq 3m+3k-7$, the number of every maximal $P_m \cup P_k$-packing of P_n is 2 by the Pigeonhole Principle. So C_n is $P_m \cup P_k$-equipackable.

(b) $k > 2m$. There are two subcases:

- When $2m+2k-4 \leq n \leq m+3k-6$, similarly, $|E(H_1)| = k-2$, $m \leq |E(H_2)| \leq k-2$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, C_n is not $P_m \cup P_k$-equipackable. When $|E(H_1)| = k-2$, $|E(H_2)| = k-2$, the maximal $P_m \cup P_k$-packing of C_n with one copy of $P_m \cup P_k$ makes $|E(G-H)|$ get to the maximum.
- When $m+3k-5 \leq n \leq 3m+3k-7$, the number of every maximal $P_m \cup P_k$-packing of P_n is 2 by the Pigeonhole Principle. So C_n is $P_m \cup P_k$-equipackable.

4. $3m+3k-6 \leq n \leq 4m+4k-9$, C_n is not $P_m \cup P_k$-equipackable. It is easy to see $p(C_n; P_m \cup P_k)$ is 3, there are two subcases:

(a) $3m+3k-6 \leq n \leq 3m+4k-8$, there exists a $P_m \cup P_k$-packing H with two copies of $P_m \cup P_k$, such that $G-H$ has four components denoted by H_1, H_2, H_3 and H_4 with $|E(H_1)| = m-1$, $|E(H_2)| = k-2$, $|E(H_3)| = 1$, $0 \leq |E(H_4)| \leq k-2$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, C_n is not $P_m \cup P_k$-equipackable.

(b) $3m+4k-7 \leq n \leq 4m+4k-9$, similarly, $|E(H_1)| = 1$, $|E(H_2)| = |E(H_3)| = k-2$, $m \leq |E(H_4)| \leq 2m-2 \leq m+k-2$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, C_n is not $P_m \cup P_k$-equipackable.

5. $n \geq 4m+4k-8$, we only discuss the case when $m \leq k \leq \frac{3}{2}m$, there are two subcases:

   (a) $n-(2m+2k-4) \equiv r \pmod{m+k-2}$, $(r = 0,1,2,\cdots,2m-4)$. $C_n - P_{2m+2k-3+r}$ has $(k+m-2)t$ $(t \in Z, t \geq 2)$ edges, so $C_n - P_{2m+2k-3+r}$ is $P_m \cup P_k$-decomposable. By Lemma 1, C_n is not $P_m \cup P_k$-equipackable.

   (b) $n-(2m+2k-4) \equiv s \pmod{m+k-2}$, $(s = 2m-3, 2m-2, \cdots, m+k-3)$, there are two possibilities:

   - $6m+4k-11 \leq n \leq 5m+5k-11$, it is easy to see $p(C_n; P_m \cup P_k)$ is 4. There exists a $P_m \cup P_k$-packing H with three copies of $P_m \cup P_k$, such that $G-H$ has six components denoted by H_1, H_2, H_3, H_4, H_5 and H_6 with $|E(H_1)| = |E(H_2)| = |E(H_3)| = m-2$, $|E(H_4)| = k-2$, $|E(H_5)| = 2$, $1 \leq |E(H_6)| \leq k-m+1$. So H is a maximal $P_m \cup P_k$-packing which is not maximum. By the definition, C_n is not $P_m \cup P_k$-equipackable.
   - $n \geq 7m+5k-13$, $C_n - P_{3m+3k-5+s}$ is $P_m \cup P_k$-decomposible. By Lemma 1, C_n is not $P_m \cup P_k$-equipackable.

Then a similar argument shows that When $k > \frac{3}{2}m$, C_n is also not $P_m \cup P_k$-equipackable.

□

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