Extremal Laplacian Estrada Index of Threshold Graphs with Given Size

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Abstract

Let G be a simple graph on n vertices. The Laplacian Estrada index of G is defined as $LEE(G) = \sum_{i=1}^{n} e^{\mu_i}$, where $\mu_1, \mu_2, \ldots, \mu_n$ are the Laplacian eigenvalues of G. In this paper, threshold graphs on n vertices and m edges having maximal and minimal Laplacian Estrada index are determined, respectively.

1 Introduction

All graphs considered here are simple and undirected. Let G be a connected graph with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$. The Laplacian matrix of G is $L = D - A$ where A is the adjacency matrix of G and D is the diagonal degree matrix.

The Estrada index of G, which is introduced by Estrada [6, 7], is defined as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. This graph invariant has already found a remarkable variety of applications. It was shown that $EE(G)$ can be used as a measure of the degree of folding of long chain polymeric molecules [6, 7]. It was also pointed out in [8] that the Estrada index provides a measure of the centrality of complex networks, while a

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connection between the Estrada index and the concept of extended atomic branching was found in [9]. For further properties of the Estrada index we refer the reader to [6–8, 10, 12, 13, 23].

In analogy to Estrada index, the Laplacian Estrada index of G is defined as [10]

$$LEE(G) = \sum_{i=1}^{n} e^{\mu_i}$$

where $\mu_1, \mu_2, \cdots, \mu_n$ are the eigenvalues of L. Also, the Laplacian Estrada index is used as a molecular descriptor [19]. By the power series expansion of the exponential function, we have

$$LEE(G) = \sum_{k=0}^{\infty} \frac{M_k(G)}{k!}$$

where $M_k(G) = \sum_{i=1}^{n} \mu_i^k$ is the k-th Laplacian spectral moment of G, which reflects the structural features of networks [20, 21] and molecular graphs [19]. Bamdad et al. [11] gave a lower bound for Laplacian Estrada index of a graph using the numbers of vertices and edges, which was improved in [2]. Du and Liu [23] determined the unique trees with minimum and maximum Laplacian Estrada indices with some given parameters. Zhou [3] gave lower bounds for Laplacian Estrada index using the degree sequence. Ilić and Zhou [1] proved that the path P_n has minimal, while the star S_n has maximal Laplacian Estrada index among trees on n vertices. Li and Zhang [14] showed that S_n^3 is the unique unicyclic graph on n vertices with maximal Laplacian Estrada index, where S_n^3 is the unicyclic graph obtained by adding an edge to the star graph S_n. For more results on Laplacian Estrada index, we refer the reader to [1–4, 11, 14, 22].

Recently, Li and Zhang [15] gave sharp upper bounds for Laplacian Estrada index of graphs with n vertices and m edges. For a connected graph G of order $n \geq 7$ and $n+1 \leq m \leq \frac{3n-5}{2}$, they showed that $LEE(G) \leq LEE(S_n^m)$, where $S_n^m \cong K_s \vee (S_{x+1} \cup (n-s-x-1)K_1)$. Note that S_n, S_n^3 and S_n^m are all connected threshold graphs, see [1, 14, 15], respectively, so it is natural to consider that if the graphs having maximal Laplacian Estrada index among threshold graphs are those having maximal Laplacian Estrada index among all graphs.

We will see from our results that the answer is positive. In this paper we determined connected threshold graphs on n vertices and m edges having maximal and minimal Laplacian Estrada index, respectively. And we also investigated the case of disconnected threshold graphs. The paper is organized as follows. Next section introduces some notions about threshold graph. Then in Section 3 and 4, we determined connected threshold graphs on n vertices and m edges having maximal and minimal Laplacian

Estrada index, respectively. In the last section, we investigated the case of disconnected threshold graphs.

2 Threshold graph and its Ferrers diagram

In this section, we introduce some notions about threshold graph. Denote by $m = |E|$ the number of edges and by d_i the degree of vertex i. We assume throughout that the vertex numbering is such that degree sequences are non-increasing, *i.e.*, $d_1 \geq \cdots \geq d_n$. Any degree sequence d arising this way is an n-partition of $2m$. For $i \in \{1, \cdots, n\}$ the conjugate degree sequence is defined as $d_i^* := |\{j \in V : d_j \geq i\}|$, so $d_1^* = n$ for connected graphs and $d_n^* = 0$. The conjugate degree sequence is easily visualized by means of Ferrers diagrams, see [18]. For degree sequence d it consists of n left justified rows of boxes where row i holds d_i boxes. In this diagram, the conjugate degree d_i^* counts the number of boxes in column i. The diagonal width of the degree sequence $f = max\{i \in V : d_i \geq i\}$ is called the trace of the partition. The square of f^2 boxes in this diagram is called the Durfee square of the partition. For a given n-partition d of $2m$ one can construct a graph having this degree sequence if and only if $\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k}(d_i^* - 1)$ for $k \in \{1, \ldots, f\}$ (Ruch-Gutman Theorem, cf. [18]).

G is called a threshold graph if $d_i = d_i^* - 1$ for $i \in \{1, \cdots, f\}$. Threshold graphs have found numerous applications in diverse areas which include computer science and psychology [16]. Geometrically, G is a threshold graph of trace f if and only if its Ferrers diagram can be decomposed into: its Durfee square; a row of f boxes directly below the Durfee square; and the remaining boxes placed in such a way that the shape below row $f + 1$ has the transpose shape to the right of the Durfee square.

Note that the degree sequence of a threshold graph uniquely defines the graph itself and it is completely determined once the conjugate degrees d_i^* are given for $i \in \{1, \cdots, f\}$. This is easily seen from the Ferrers diagram. There the part strictly below the diagonal boxes is the transpose of the part on and above the diagonal.

Also, there are some equivalent ways to characterize threshold graphs. Another way of obtaining a threshold graph is through an iterative process which starts with an isolated vertex by adding a new vertex that is either connected to no other vertex (an isolated vertex) or connected to every other vertex (a cone vertex). The sequence of operations is called the building sequence of a threshold graph. Therefore, we may represent the building sequence of a threshold graph on n vertices using a binary sequence $b = (b_1, b_2, \cdots, b_n)$. Here b_i is 0 if vertex v_i was added as an isolated vertex, and b_i is 1 if v_i was added as a cone vertex. In our representation b_1 is always 0, and b_n is always 1 if G is connected. Figure 1 illustrates a threshold graph and its Ferrers diagram. Its degree sequence (reordered) is $d = (6,6,4,3,3,2,2)$, and its building sequence is $b = (0,0,1,0,0,1,1)$.

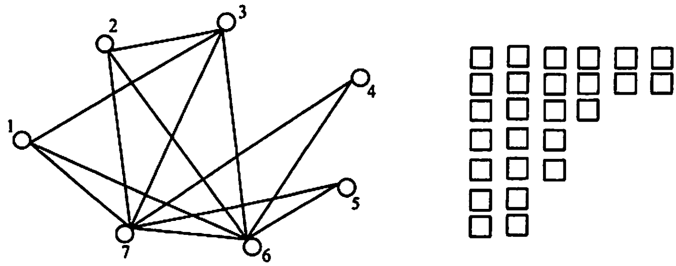

Figure 1: A threshold graph for $(n, m, f) = (7,13,3)$ and its Ferrers diagram.

Now, we explain the construction of the Ferrers diagram of the candidate graphs for extremal laplacian estrada index, see [5].

We call *Type I* the connected threshold graph with n vertices, m edges and trace f constructed in such a way that its conjugate degree sequence d^* is lexicographically maximal. In the algorithmic construction of such a sequence it suffices to describe the placement of the m boxes below the diagonal, because for threshold graphs the other m boxes have to be placed on and above the diagonal in the corresponding transposed positions. In order to obtain a sequence with trace f, below the diagonal the first row up to row $f+1$ have to be filled with boxes then the remaining $m - f(f+1)/2$ boxes are placed in column-wise order, i.e., in the sequence $(f+2,1)$, $(f+3,1), \cdots, (n,1)$, $(f+2,2), (f+3,2), \cdots$. In fact, Figure 1 illustrates a *Type I* threshold graph.

We call *Type II* the connected threshold graph with n vertices, m edges and trace f constructed in such a way that its conjugate degree sequence d^* is lexicographically minimal. Since we only consider connected graphs we have $d_1 = n-1$. Thus a procedure to construct such a graph is, after filling the first row, to fill the positions on and above the diagonal in column-wise order without exceeding row index f, i.e., the sequence reads $(2,2)$, $(2,3)$, $(3,3)$, $(2,4), \cdots, (f,f)$, $(2,f+1), \cdots, (f,f+1)$, $(2,f+2), \cdots$ until m boxes have been placed (the corresponding m boxes below the diagonal need to be placed in row-wise order without exceeding column f). Figure 2 illustrates a *Type II* connected threshold graph, its degree sequence (reordered) is $d = (6,5,4,4,4,2,1)$, and its building sequence is $b = (0,1,1,0,1,0,1)$.

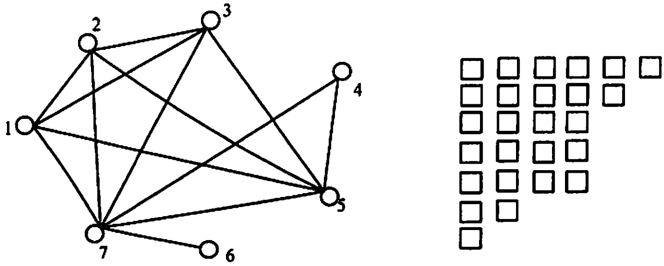

Figure 2: A *Type II* connected threshold graph for $(n, m, f) = (7, 13, 4)$ and its Ferrers diagram.

3 Connected threshold graphs with maximal Laplacian Estrada index

Note that for a connected graph G on n vertices and m edges, the feasible range of traces f is determined by the constraints

$$f(f-1) \le 2(m-n+1) \text{ and } f(f+1) + 2(n-1-f)f \ge 2m.$$

Thus,

$$\underline{f}(n,m) = \lceil n - \frac{1}{2} - \sqrt{n^2 - n + \frac{1}{4} - 2m} \rceil \le f \le \lfloor \frac{1}{2} + \sqrt{2(m-n) + \frac{9}{4}} \rfloor = \overline{f}(n,m).$$

We will only write $\underline{f}$ and $\overline{f}$ if the arguments n and m are clear from the context.

Also note that for a threshold graph G, its conjugate degree sequence d^* gives the spectrum of the Laplacian matrix L, that is, $0 = \lambda_1 = d^*_n \le \lambda_2 = d^*_{n-1} \le \ldots \le \lambda_n = d^*_1$, see [17].

Now, we are ready to consider connected threshold graphs with fixed n, m and f, and determine which graph has maximal Laplacian Estrada index.

Lemma 1. *Among all connected threshold graphs with n vertices, m edges and degree sequence with trace f, the Type I graph has maximal Laplacian Estrada index.*

Proof. From the Ferrers diagram of *Type I* graph G, one can write its Laplacian spectrum as

$$\{\underbrace{n, \cdots, n}_{k}, d^*_{k+1}, \underbrace{f+1, \cdots, f+1}_{f-k-1}, \underbrace{k+1, \cdots, k+1}_{d^*_{k+1}-f-1}, \underbrace{k, \cdots, k}_{n-d^*_{k+1}}, 0\},$$

where $k = \lfloor \frac{m-f(f+1)/2}{n-1-f} \rfloor$, $(\leq f)$, if $k < f$, $d^*_{k+1} = f+1+m-f(f+1)/2-k(n-1-f)$, and it suffices to consider this case, because the threshold graph is determined uniquely if $k = f$, so the theorem holds. Now consider all connected threshold graphs with n vertices, m edges and degree sequence of trace f that have maximal Laplacian Estrada index. Assume, for contradiction, that G is not in this set. From the set we pick $\hat{G}$ with degree sequence $\hat{d}$ so that the largest index with $\hat{d}^*_i > d^*_i$ is minimal and $\hat{d}^*_i - d^*_i$ is minimal as well. Let $\bar{l}$ be the corresponding index. Because d^* is lexicographically maximal, there must be an index $1 < \hat{l} < \bar{l}$ with $d^*_{\hat{l}} > \hat{d}^*_{\hat{l}}$. Now we consider the conjugate degree sequence $\tilde{d}^*$ of a threshold graph defined by

$$\tilde{d}^*_i = \begin{cases} \hat{d}^*_i, & i \in \{1, \cdots, n\} \setminus \{\hat{l}, \bar{l}\}, \\ \hat{d}^*_i + 1, & i = \hat{l}, \\ \hat{d}^*_i - 1, & i = \bar{l}. \end{cases}$$

The corresponding graph $\tilde{G}$ is again a connected threshold graph on n vertices, m edges with degree sequence of trace f. It remains to show that $LEE(\tilde{G}) > LEE(\hat{G})$, then by the choice of $\hat{G}$ this yields the desired contradiction. In fact, we have

$$\begin{aligned} LEE(\tilde{G}) - LEE(\hat{G}) &= (e^{\hat{d}^*_{\hat{l}}+1} + e^{\hat{d}^*_{\bar{l}}-1}) - (e^{\hat{d}^*_{\hat{l}}} + e^{\hat{d}^*_{\bar{l}}}) \\ &= (e-1)(e^{\hat{d}^*_{\hat{l}}} - e^{\hat{d}^*_{\bar{l}}-1}) \\ &> 0 \end{aligned}$$

The last inequality holds because $\hat{d}^*_{\hat{l}} \geq \hat{d}^*_{\bar{l}} - 1$.

□

Theorem 2. *Among all connected threshold graphs on n vertices and m edges, the Type I graph with trace $\underline{f}$ has maximal Laplacian Estrada index.*

Proof. Let G be a connected threshold graph on n vertices and m edges with maximal Laplacian Estrada index, its conjugate degree d^* with trace f. We assume that G is selected so that its trace f is minimal among all optimal connected threshold graphs. By Lemma 1 we may assume d^* is lexicographically maximal. If $f = \underline{f}$ we are done, so assume, for contradiction, that $f > \underline{f}$. Since d^* is lexicographically maximal with $f > \underline{f}$, we have $d^*_f = f+1$, $d^*_{f+1} < f$ and there is a smallest index $1 < \hat{l} < f$ with $d^*_{\hat{l}} < n$. We can define a new conjugate degree sequence $\hat{d}^*$ of a connected threshold graph $\hat{G}$ on n vertices and m edges with trace

$f-1$ via

$$\hat{d}_i^* = \begin{cases} d_i^*, & i \in \{1, \cdots, n\} \setminus \{\hat{l},\ f,\ d_{\hat{l}}^* - f\}, \\ d_i^* + 1, & i = \hat{l}, \\ d_i^* - 2, & i = f, \\ d_i^* + 1, & i = d_{\hat{l}}^* - f. \end{cases}$$

It remains to show that $LEE(\hat{G}) > LEE(G)$, then by the choice of G this yields the desired contradiction. In fact, we have

$$\begin{aligned} LEE(\hat{G}) - LEE(G) &= (e^{d_{\hat{l}}^*+1} + e^{f-1} + e^{\hat{l}}) - (e^{d_{\hat{l}}^*} + e^{f+1} + e^{\hat{l}-1}) \\ &= [e^{d_{\hat{l}}^*} - e^{f-1}(e+1) + e^{\hat{l}-1}](e-1) \\ &> (e^{d_{\hat{l}}^*} - 2e^{f} + e^{\hat{l}-1})(e-1) \\ &> (e^{d_{\hat{l}}^*} - e^{f+1} + e^{\hat{l}-1})(e-1) \\ &> 0 \end{aligned}$$

the last inequality holds because $d_{\hat{l}}^* \geq f+1$.

□

Using the Ferrers diagram, the Laplacian Estrada index of *Type I* connected threshold graph G on n vertices and m edges with trace $\underline{f}$ can be easily calculated

$$\underline{LEE}(n,m) = (\underline{f}-1)e^{n} + e^{d_{\underline{f}}^*} + (d_{\underline{f}}^* - \underline{f} - 1)e^{\underline{f}} + (n - d_{\underline{f}}^*)e^{\underline{f}-1} + 1,$$

where $d_{\underline{f}}^* = \underline{f} + 1 + m - \underline{f}(\underline{f}+1)/2 - (\underline{f}-1)(n-1-\underline{f})$.

For example, for the graph in Figure 1, $\underline{f} = 3$, $d_{\underline{f}}^* = 5$, $\underline{LEE}(7,13) = 2e^7 + e^5 + e^3 + 2e^2 + 1 \approx 2377.5431$, which is the maximal value of Laplacian Estrada index among all connected threshold graphs with 7 vertices and 13 edges.

It is easy to check that the building sequence of *Type I* connected threshold graph G on n vertices and m edges with trace $\underline{f}$ is $b = (0^{n-\underline{f}}, 1^{\underline{f}})$ if $d_{\underline{f}}^* = n$, or $b = (0^{d_{\underline{f}}^* - \underline{f}}, 1, 0^{n-d_{\underline{f}}^*}, 1^{\underline{f}-1})$ if $d_{\underline{f}}^* < n$.

Corollary 3. *Among all connected threshold graphs on n vertices the complete graph K_n has maximal Laplacian Estrada index.*

Proof. Using the notation above, it suffices to prove that $\underline{LEE}(n, m+1) > \underline{LEE}(n,m)$. Then, for given n, $\underline{LEE}(n,m)$ is a increasing function on m, so the maximal is obtained at K_n. In fact we have

$$\underline{LEE}(n, m+1) - \underline{LEE}(n,m) = (e^{d_{\underline{f}}^*+1} + e^{\underline{f}}) - (e^{d_{\underline{f}}^*} + e^{\underline{f}-1}) > 0.$$

□

4 Connected threshold graphs with minimal Laplacian Estrada index

In this section, we consider connected threshold graphs with fixed n and m and determine which graph has minimal Laplacian Estrada index.

Lemma 4. *Among all connected threshold graphs with n vertices, m edges and degree sequence with trace f, the Type II graph has minimal Laplacian Estrada index.*

Proof. From the Ferrers diagram of *Type II* graph G, one can write its Laplacian spectrum as

$$\{n,\underbrace{f+h+2,\cdots,f+h+2}_{k},\underbrace{f+h+1,\cdots,f+h+1}_{f-k-1},\underbrace{f,\cdots,f}_{h},k+1,\underbrace{1,\cdots,1}_{n-f-h-2},0\},$$

where $h=\lfloor\frac{m-n+1-f(f-1)/2}{f-1}\rfloor$, $k=m-n+1-f(f-1)/2-(f-1)h,(\leq f-1)$. Now consider all connected threshold graphs with n vertices, m edges and degree sequence of trace f that have minimal Laplacian Estrada index. Assume, for contradiction, that G is not in this set. From the set we pick $\hat{G}$ with degree sequence $\hat{d}$ so that the largest index with $\hat{d}_i^* < d_i^*$ is minimal and $d_i^* - \hat{d}_i^*$ is minimal as well. Let $\bar{l}$ be the corresponding index. Because d^* is lexicographically minimal, there must be an index $1 < \hat{l} < \bar{l}$ with $d_{\hat{l}}^* < \hat{d}_{\hat{l}}^*$. Now we consider the conjugate degree sequence $\tilde{d}^*$ of a threshold graph defined by

$$\tilde{d}_i^* = \begin{cases} \hat{d}_i^*, & i \in \{1,\cdots,n\}\setminus\{\hat{l},\bar{l}\}, \\ \hat{d}_i^* - 1, & i = \hat{l}, \\ \hat{d}_i^* + 1, & i = \bar{l}. \end{cases}$$

The corresponding graph $\tilde{G}$ is again a connected threshold graph on n vertices, m edges with degree sequence of trace f. It remains to show that $LEE(\tilde{G}) < LEE(\hat{G})$, then by the choice of $\hat{G}$ this yields the desired contradiction. In fact, we have

$$\begin{aligned} LEE(\hat{G}) - LEE(\tilde{G}) &= (e^{\hat{d}_{\hat{l}}^*} + e^{\hat{d}_{\bar{l}}^*}) - (e^{\hat{d}_{\hat{l}}^*-1} + e^{\hat{d}_{\bar{l}}^*+1}) \\ &= (e-1)(e^{\hat{d}_{\hat{l}}^*-1} - e^{\hat{d}_{\bar{l}}^*}) \\ &> 0. \end{aligned}$$

The last inequality holds because $\hat{d}_{\hat{l}}^* > d_{\hat{l}}^* \geq d_{\bar{l}}^* > \hat{d}_{\bar{l}}^*$, so $\hat{d}_{\hat{l}}^* - 1 > \hat{d}_{\bar{l}}^*$. □

Theorem 5. *Among all connected threshold graphs on n vertices and m edges the Type II graph with trace $\overline{f}$ has minimal Laplacian Estrada index.*

Proof. Let G be a connected threshold graph on n vertices and m edges with minimal Laplacian Estrada index, its conjugate degree d^* with trace f. We assume G to be selected so that its trace f is maximal among all optimal connected threshold graphs. By Lemma 4 we may assume d^* is lexicographically minimal. If $f = \overline{f}$ we are done, so assume, for contradiction, that $f < \overline{f}$. Using the same notation as in Lemma 4, since d^* is lexicographically minimal with $f < \overline{f}$, we have $d^*_{f+1} = f$, and there is a largest index $1 < \hat{l} < f$ with $d^*_{\hat{l}} > f+h+1$. We can define a new conjugate degree sequence $\hat{d}^*$ of a connected threshold graph $\hat{G}$ on n vertices and m edges with trace $f+1$ via

$$\hat{d}^*_i = \begin{cases} d^*_i, & i \in \{1, \cdots, n\} \setminus \{\hat{l},\ f+1,\ f+h+1\}, \\ d^*_i - 1, & i = \hat{l}, \\ d^*_i + 2, & i = f+1, \\ d^*_i - 1, & i = f+h+1. \end{cases}$$

It remains to show that $LEE(\hat{G}) < LEE(G)$, then by the choice of G this yields the desired contradiction. In fact, we have

$$\begin{aligned} LEE(G) - LEE(\hat{G}) &= (e^{d^*_{\hat{l}}} + e^{d^*_{f+1}} + e^{d^*_{f+h+1}}) - (e^{d^*_{\hat{l}}-1} + e^{d^*_{f+1}+2} + e^{d^*_{f+h+1}-1}) \\ &= (e^{f+h+2} + e^{f} + e^{k+1}) - (e^{f+h+1} + e^{f+2} + e^{k}) \\ &= [e^{f+h+1} - e^{f}(e+1) + e^{k}](e-1) \\ &> (e^{f+h+1} - 2e^{f+1} + e^{k})(e-1) \\ &> (e^{f+h+1} - e^{f+2} + e^{k})(e-1) \\ &> 0. \end{aligned}$$

The last inequality holds because $h \geq 1$.

□

Using the Ferrers diagram, the Laplacian Estrada index of *Type II* connected threshold graph G on n vertices and m edges with trace $\overline{f}$ can be easily calculated

$$\overline{LEE}(n,m) = e^{n} + ke^{\overline{f}+2} + (\overline{f} - k - 1)e^{\overline{f}+1} + e^{k+1} + (n - \overline{f} - 2)e + 1,$$

where $k = m - n + 1 - \overline{f}(\overline{f}-1)/2$, $(\leq \overline{f} - 1)$.

For example, for the graph in Figure 2, $\overline{f} = 4$, $k = 1$, $\overline{LEE}(7, 13) = e^7 + e^6 + 2e^5 + e^2 + e + 1 \approx 1807.9956$, which is the minimal value of Laplacian Estrada index among all connected threshold graphs with 7 vertices and 13 edges.

It is easy to check that the building sequence of *Type II* connected threshold graph G on n vertices and m edges with trace $\overline{f}$ is $b = (0^2, 1^{\overline{f}-1}, 0^{n-\overline{f}-2}, 1)$ if $k = \overline{f} - 1$, or $b = (0, 1^{\overline{f}-k-1}, 0, 1^{k}, 0^{n-\overline{f}-2}, 1)$ if $k < \overline{f} - 1$.

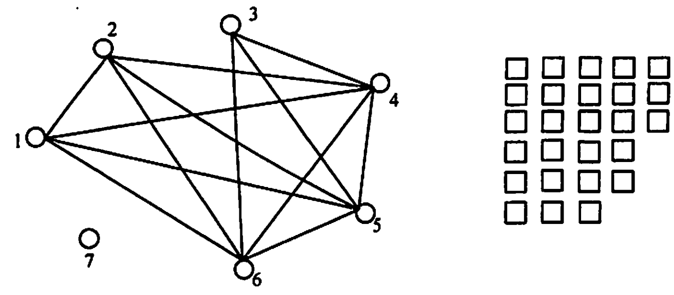

Figure 3: A *Type II* disconnected threshold graph for $(n, m, f) = (7, 13, 4)$ and its Ferrers diagram.

Corollary 6. *Among all connected threshold graphs on n vertices the star graph S_n has minimal Laplacian Estrada index.*

Proof. Using the notation above, it is easy to check that $\overline{LEE}(n, m+1) > \overline{LEE}(n, m)$. Therefore, for given n, $\overline{LEE}(n, m)$ is a increasing function on m, so the minimal is obtained at S_n. □

5 Results on disconnected threshold graphs

In this section, we consider the case of disconnected threshold graphs. Note that, in fact, the analysis for the case of *Type I* graph with lexicographically maximal conjugate degree sequence can be applied without any changes. So Lemma 1 and Theorem 2 still holds for disconnected threshold graphs. In the case of *Type II* graph with lexicographically minimal conjugate degree sequence, without the constraint $d_1^* = n$, its Ferrers diagram should be changed accordingly, and the upper bound on f is now determined from $f(f+1) \leq 2m$, thus the upper bound on f is $\overline{f}_d(n, m) = \lfloor -\frac{1}{2} + \sqrt{2m + \frac{1}{4}} \rfloor$. Therefore, Lemma 4 still holds and Theorem 5 holds also if we replace the argument $\overline{f}$ by $\overline{f}_d(n, m)$. The proofs of these results should be adapted slightly, we omit the details since the lines of arguments are the same.

For example, Figure 3 illustrates a *Type II* disconnected threshold graph, its degree sequence (reordered) is $d = (5, 5, 5, 4, 4, 3, 0)$, and its building sequence is $b = (0, 1, 0, 1, 1, 1, 0)$. In this case we still have $\overline{f}_d(n, m) = 4$, then $\overline{LEE}(7, 13) = 3e^6 + e^5 + e^3 + 2e^0 \approx 1380.7851$, which is the minimal value of Laplacian Estrada index among all threshold graphs with 7 vertices and 13 edges.

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