COMBINATORIAL IDENTITIES FOR ℓ-REGULAR OVERPARTITIONS

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ABSTRACT. We explore new combinatorial properties of overpartitions which are natural generalizations of integer partitions. Building on recent work we state general combinatorial identities between standard partition, overpartition and ℓ-regular partition, functions. We provide both generating function and bijective proofs. We also prove the congruences for certain overpartition functions combinatorially.

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1. INTRODUCTION

An overpartition of a positive integer n is a partition of n, where the first occurrence of each part-size may be overlined. Overpartitions generalize ordinary partitions. We denote the number of overpartitions of n by $\overline{p}(n)$, with $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ enumerates the following overpartitions:

$$(3), (\overline{3}), (2,1), (\overline{2},1), (2,\overline{1}), (\overline{2},\overline{1}), (1,1,1), (\overline{1},1,1).$$

The three overpartitions with no overlined parts are the ordinary partitions of 3. Given a positive integer ℓ a partition λ is called ℓ-regular if no part of λ is a multiple of ℓ.

We will consider combinatorial identities which connect certain restricted enumeration functions of ordinary partitions, strict overpartitions and ℓ-regular overpartitions. We also highlight few congruence properties of restricted overpartition functions. Related investigations have previously appeared in [7] whereby the authors found an identity between ℓ-regular overpartitions and a class of overpartitions. George Andrews [2] considered the enumeration of *singular overpartitions* which correspond to ℓ-regular overpartitions in which the parts satisfy prescribed congruences. Subsequently, Chen, Hirschhorn and Sellers [5] developed the arithmetic properties of these singular overpartition functions.

In a recent work Munagi and Sellers [14] proved new identities between sets of restricted partitions and certain overpartitions in which the overlined parts belong to specified residue classes. Shen [15] recently proved a finite set of congruences satisfied by ℓ-regular overpartitions of terms in arithmetic progressions of the forms $an+b$, , $a > b > 0$, where $a \in \{4, 9, 12\}$. Further relevant work on the congruence properties for $\overline{p}(n)$ may be found in the following papers among others: Hirschhorn and Sellers [9, 10, 11], Chen, Sang and Shi [6], Fortin, Jacob and Mathieu [8], Kim [12] and Mahlburg [13].

We will use the notation $R_\ell(n)$ to denote the number of ℓ-regular partitions of n, and $\overline{R_\ell}(n)$ for the number of ℓ-regular overpartitions of n. Recall the standard generating functions:

$$\sum_{n=0}^{\infty} \overline{p}(n) = \prod_{n\geq 1} \frac{1+q^n}{1-q^n}.$$

$$\sum_{n=0}^{\infty} \overline{R_\ell}(n) = \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1+q^n)}{(1-q^n)(1+q^{\ell n})}.$$

In section 2, we prove a general theorem (Main Theorem) which connects $\overline{R_\ell}(n)$ with five other restricted partition functions. In section 3, we give an identity for color partitions which extends the results of the main theorem. The final section is devoted to new combinatorial proofs of fundamental congruence properties of two restricted overpartition functions.

2. A general partition theorem

This section is devoted to the statement and proof of a sequence of related partition identities connecting $\overline{R_\ell}(n)$ with different classes of restricted partitions and overpartitions.

We first establish a simple identity between overpartitions and ordinary partitions. The bijective proof of the latter highlights our approach in assigning a partition λ to an image under a map. Generally we define a sub-function, say τ, to act on individual parts of λ and then assign λ to the union of the images of the parts under τ.

Proposition 2.1. *The number of overpartitions of n equals the number of partitions of $2n$ in which odd parts occur with even multiplicity.*

Proof. For a generating function proof let $E(n)$ denote the number of partitions of n in which odd parts occur with even multiplicity. Then

$$\sum_{n=0}^{\infty} E(2n)q^n = \prod_{n=1}^{\infty}(1+q^{2n}+q^{4n}+\cdots)(1+q^{2(2n-1)}+q^{4(2n-1)}+\cdots)$$
$$= \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})(1-q^{2(2n-1)})}.$$

On replacing q^2 by q the right-hand side becomes

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^n)(1-q^{2n-1})} = \prod_{n=1}^{\infty}\frac{(1+q^n)}{(1-q^n)} = \sum_{n=0}^{\infty}\overline{p}(n)q^n.$$

The bijective proof is more insightful. In the sequel square brackets are used to indicate corresponding enumerated sets, and exponents indicate multiplicities of parts.

Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \dots) \in E[2n]$, $c_1 > c_2 > \cdots$, $u_j \geq 1\,\forall j$, and define the map $f : E[2n] \to \overline{p}[n]$ by $\lambda \mapsto f(\lambda) = \cup_{c\in\lambda} f_c(c^k)$ (multiset union), where

$$f_c(c^k) = \begin{cases} \frac{c}{2} & \text{if } k=1,\\ \overline{c} & \text{if } k=2,\\ \frac{c}{2}, \overline{c} & \text{if } k=3,\\ c^{k/2} & \text{if } k=4; \end{cases}$$

and if $4 < k \equiv r \pmod 4$, $1 \leq r \leq 4$, the image is a sequence of parts:

$$f_c(c^k) = f_c(c^r), c^{\frac{k-r}{2}}.$$

The inverse map $f^{-1} : \overline{p}[n] \to E[2n]$ is analogously given by:

$$f_c^{-1}(\overline{c}) = c^2;$$

$$f_c^{-1}(c^k) = \begin{cases} 2c & \text{if } k=1,\\ c^4 & \text{if } k=2; \end{cases}$$

and if $2 < k \equiv r \pmod 2$, $1 \leq r \leq 2$, then

$$f_c^{-1}(c^k) = f_c^{-1}(c^r), c^{2(k-r)}.$$

∎

These bijections are illustrated in Table 1 when $n = 3$; the lists under respective sets correspond one-to-one under the bijection.

Remark 2.2. *The action of the map f on a part of a partition λ is to halve the part (if it is even) or halve its multiplicity (up to possible overlining). The inverse of f reverses these operations. Thus f preserves ℓ-regularity provided that ℓ is odd, that is, if λ is ℓ-regular, then so is $f(\lambda)$, and conversely.*

$E[6]$	$\xrightarrow{f}$	$\overline{p}[3]$
(6)	$\to$	(3)
(4,2)	$\to$	(2,1)
(4,1,1)	$\to$	$(2,\overline{1})$
(3,3)	$\to$	$(\overline{3})$
(2,2,2)	$\to$	$(\overline{2},1)$
(2,2,1,1)	$\to$	$(\overline{2},\overline{1})$
(2,1,1,1,1)	$\to$	(1,1,1)
(1,1,1,1,1,1)	$\to$	$(\overline{1},1,1)$

TABLE 1. The bijections of Proposition 2.1 for $n = 4$.

We now state our main result.

Theorem 2.3. *Main Theorem. Let ℓ and n be positive integers with $\ell, n > 1$.*
Let $B_\ell(n)$ denote the number of partitions of n in which odd parts occur with multiplicity $2, 4, \ldots,$ or $2(\ell-1)$ and even parts appear at most $\ell - 1$ times.
Let $Q_\ell(n)$ denote the number of ℓ^2-regular partitions of n in which parts not divisible by ℓ appear 0 or ℓ times. Then

$$B_\ell(2n) = Q_\ell(\ell n) = \overline{R_\ell}(n); \tag{1}$$

Let $\ell \equiv 1 \pmod 2$ and let $G_\ell(n)$ denote the number of ℓ-regular partitions of n in which odd parts occur with even multiplicities. Then

$$G_\ell(2n) = \overline{R_\ell}(n); \tag{2}$$

Let $\ell \equiv 0 \pmod 2$ and let $H_\ell(n)$ denote the number of 2ℓ-regular partitions of n in which odd parts occur with even multiplicities and each part $\equiv \ell \pmod{2\ell}$ appears at most once. Then

$$H_\ell(2n) = \overline{R_\ell}(n). \tag{3}$$

We present both a generating function proof and a bijective proof of the main theorem.

2.1. **A generating function proof of Theorem 2.3.** The generating functions for $B_\ell(2n)$ and $Q_\ell(\ell n)$ are given by

$$\sum_{n=0}^{\infty} B_\ell(2n)q^n = \prod_{n=1}^{\infty}(1 + q^{1\cdot 2n} + \cdots + q^{(\ell-1)\cdot 2n})(1 + q^{2(2n-1)} + \cdots + q^{2(\ell-1)(2n+1)})$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{2\ell n})}{(1-q^{2n})} \frac{(1-q^{2\ell(2n-1)})}{(1-q^{2(2n-1)})}. \tag{4}$$

(5) $$\sum_{n=0}^{\infty} Q_\ell(\ell n)q^{\ell n} = \prod_{n=1}^{\infty} \frac{(1-q^{\ell^2 n})(1+q^{\ell n})}{(1-q^{\ell n})(1+q^{\ell^2 n})} = \prod_{n=1}^{\infty} \frac{(1-q^{2\ell n})(1-q^{\ell^2 n})^2}{(1-q^{\ell n})^2(1-q^{2\ell^2 n})}.$$

On the other hand,

$$\sum_{n=0}^{\infty} \overline{R_\ell}(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1+q^n)}{(1-q^n)(1+q^{\ell n})}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1+q^n)}{(1-q^n)(1+q^{\ell n})} \times \frac{(1-q^n)(1-q^{\ell n})}{(1-q^n)(1-q^{\ell n})}$$

(6) $$= \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{\ell n})^2}{(1-q^n)^2(1-q^{2\ell n})}.$$

Replacing q by q^ℓ yields (cf. (5))

$$\prod_{n=1}^{\infty} \frac{(1-q^{2\ell n})(1-q^{\ell^2 n})^2}{(1-q^{\ell n})^2(1-q^{2\ell^2 n})} = \sum_{n=0}^{\infty} Q_\ell(\ell n)q^{\ell n}.$$

To complete the proof of (1) we note that (6) implies

$$\sum_{n=0}^{\infty} \overline{R_\ell}(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{\ell(2n-1)})(1-q^{\ell n})}{(1-q^n)(1-q^{2n-1})}.$$

Replacing q by q^2 gives (cf. (4))

$$\prod_{n=1}^{\infty} \frac{(1-q^{2\ell(2n-1)})(1-q^{2\ell n})}{(1-q^{2n})(1-q^{2(2n-1)})} = \sum_{n=0}^{\infty} B_\ell(2n)q^n.$$

In order to prove (2) we assume that ℓ is odd and consider the generating function

$$\sum_{n=0}^{\infty} G_\ell(2n)q^n$$

$$= \prod_{n=1}^{\infty} \frac{(1+q^{2n}+q^{4n}+\cdots)(1+q^{2(2n-1)}+q^{4(2n-1)}+\cdots)}{(1+q^{\ell(2n)}+q^{2\ell(2n)}+\cdots)(1+q^{\ell\cdot 2(2n-1)}+q^{2\ell\cdot 2(2n-1)}+\cdots)}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{2\ell n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})(1-q^{2(2n-1)})}$$

$$= \sum_{n=0}^{\infty} B_\ell(2n)q^n.$$

Thus (2) is established.

Lastly for (3) we assume that ℓ is even and consider the generating function

$$\sum_{n=0}^{\infty} H_\ell(2n) = \prod_{n=0}^{\infty} \frac{1-q^{\ell n}}{1-q^{2n}} \times \frac{1+q^{\ell n}}{1+q^{2\ell n}} \times \frac{1}{1-q^{2(2n-1)}}.$$

Indeed a partition enumerated by $H_\ell(2n)$ is 2ℓ-regular and contains at most one distinct copy of each part $\equiv \ell \pmod{2\ell}$. This is enumerated by the function

$$\frac{(1-q^\ell)(1-q^{2\ell})\cdots}{(1-q^2)(1-q^4)\cdots}(1+q^\ell)(1+q^{\ell+2\ell})(1+q^{\ell+4\ell})\cdots$$

$$= \prod_{n=0}^{\infty} \frac{1-q^{\ell n}}{1-q^{2n}} \times \frac{1+q^{\ell n}}{1+q^{2\ell n}}.$$

Since odd parts occur with even multiplicities, we have the contribution

$$\prod_{n=0}^{\infty}(1+q^{2(2n-1)}+q^{4(2n-1)}+q^{6(2n-1)}+\cdots) = \prod_{n=0}^{\infty} \frac{1}{1-q^{2(2n-1)}}.$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} H_\ell(2n) &= \prod_{n=0}^{\infty} \frac{1-q^{2\ell n}}{(1-q^{2n})(1+q^{2\ell n})(1-q^{2(2n-1)})} \cdot \frac{1-q^{2\ell n}}{1-q^{2\ell n}} \\
&= \prod_{n=0}^{\infty} \frac{1-q^{2\ell n}}{(1-q^{2n})(1-q^{2(2n-1)})} \cdot \frac{1-q^{2\ell n}}{1-q^{4\ell n}} \\
&= \prod_{n=0}^{\infty} \frac{1-q^{2\ell n}}{(1-q^{2n})(1-q^{2(2n-1)})} \cdot (1-q^{2\ell(2n-1)}) \\
&= \prod_{n=0}^{\infty} \frac{(1-q^{2\ell n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})(1-q^{2(2n-1)})} \\
&= \sum_{n=0}^{\infty} B_\ell(2n) q^n.
\end{aligned}$$

This completes the generating function proof of Theorem 2.3. ∎

2.2. **A combinatorial proof of Theorem 2.3.** We provide combinatorial proofs of the three parts of the theorem in the following order.

First we establish the the bijection $Q_\ell[\ell n] \iff \overline{R_\ell}[n]$. Then we prove the remaining parts according to the schemes

$B_\ell[2n] \iff G_\ell[2n] \iff \overline{R_\ell}[n]$ and $B_\ell[2n] \iff H_\ell[2n] \iff \overline{R_\ell}[n]$, corresponding to odd and even ℓ respectively.

Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \cdots) \in Q_\ell[\ell n]$. Define the map $Q_\ell[\ell n] \to \overline{R_\ell}[n]$ by $\lambda \mapsto \cup_{c\in\lambda} w_c(c)$, where

$$w_c : c^{u_j} \mapsto \begin{cases} \left(\frac{c}{\ell}\right)^{u_j} & \text{if } \ell | c, \\ \overline{c} & \text{if } u_j = \ell. \end{cases}$$

In other words, each multiple of ℓ is divided by ℓ and each non-multiple (which occurs exactly ℓ times) is replaced by one overlined copy. The inverse map is

$$w_c^{-1} : x \mapsto \begin{cases} c^\ell & \text{if } x = \overline{c}, \\ k\ell & \text{if } x = c. \end{cases}$$

This proves the bijection $Q_\ell[\ell n] \iff \overline{R_\ell}[n]$.

Next we define a new bijection θ to compose with f:

$$B_\ell[2n] \xrightarrow{\theta} G_\ell[2n] \xrightarrow{f} \overline{R_\ell}[n].$$

If $\lambda = (c_1 \geq c_2 \geq \cdots) \in B_\ell[2n]$, then each $c_i = c$ can be expressed uniquely in the form $c = \ell^r m$ with $r \geq 0$ such that $\ell \nmid m$. Define $\theta : B_\ell[2n] \to G_\ell[2n]$ by setting $\theta(\lambda) = \cup_{c\in\lambda}\theta_c(c)$, with

$$\theta_c(c) = \theta_c(\ell^r m) = m^{\ell^r}.$$

It may be verified that θ is invertible. Note that θ is similar to the classical bijection of Glaisher between odd and strict ordinary partitions, see [3, 4]. To insure that the image is not divisible by ℓ, each part c is mapped to x copies of c/x, where x is the highest power of ℓ dividing c.

The fact that f is the required bijection between $G_\ell[2n]$ and $\overline{R}_\ell[n]$ follows from the proof of Proposition 2.1 and Remark 2.2.

The second part of the proof, $B_\ell[2n] \iff H_\ell[2n] \iff \overline{R_\ell}[n]$, also relies on the composition of two bijections ϕ and f:

$$B_\ell[2n] \xrightarrow{\phi} H_\ell[2n] \xrightarrow{f} \overline{R_\ell}[n].$$

We define $\phi : B_\ell[2n] \to H_\ell[2n]$ by $\phi(\lambda) = \cup_{c\in\lambda}\phi_c(c)$, where ϕ_c is explained below.

Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \cdots) \in B_\ell[2n]$ and consider $c^{u_j} \in \lambda$. Then one can write $c = \ell^r m$ where $0 \leq r \leq 1$ and $\ell \nmid m$. If m is odd and u_j is odd, then there are two cases:

$$\phi_c : c^{u_j} \mapsto \begin{cases} c & \text{if } u_j = 1, \\ c, m^{\ell^r(u_j-1)} & \text{if } u_j > 1. \end{cases}$$

Note that when $u_j > 1$, ϕ_c fixes one copy of c but assigns the other copies to m^{ℓ^r} apiece.

For all other cases apply the following transformation to each $c^k \in \lambda$:

$$\phi_c : c^k \mapsto m^{\ell^r k}.$$

To complete the proof, we give the inverse of ϕ. Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \cdots) \in H_\ell[2n]$. In order to assign each $c^u \in \lambda$, we first obtain the ℓ-adic expansion of u: $u = m_0 + m_1\ell + \cdots + m_r\ell^r$, $m_i \in \{0, \ldots, \ell-1\}$. Thus each $c^u \in \lambda$ is equivalent to $c^u = c^{m_1\ell}, c^{m_2\ell^2}, \ldots, c^{m_r\ell^r}$. Then if $\lambda_i = c^{m_i\ell^i}$, we have $\phi^{-1}(\lambda) = \cup_{\lambda_i \in \lambda}\phi_{\lambda_i}^{-1}(\lambda_i)$ with

$$\phi_{\lambda_i}^{-1} : \lambda_i = c^{m_i\ell^i} \mapsto \begin{cases} c^{\ell^i}, (\ell^i c)^{m_i-1} & \text{if } c \equiv m_i \equiv 1 \pmod 2 \text{ and } 0 \le i \le 1; \\ (\ell^i c)^{m_i} & \text{otherwise.} \end{cases}$$

■

An illustration of the bijection $B_\ell[2n] \iff H_\ell[2n] \iff \overline{R_\ell}[n]$ is given for some of the partitions when $\ell = 4$ and $n = 25$ in Table 2.

$B_4(50)$	$\xrightarrow{\phi}$	$H_4(50)$	$\xrightarrow{f}$	$\overline{R}_4(25)$
$(48, 2)$	$\to$	$(3^{16}, 2)$	$\to$	$(3^8, 1)$
$(32, 12, 4, 2)$	$\to$	$(12, 4, 2^{17})$	$\to$	$(6, 2^9, 1)$
$(24^2, 2)$	$\to$	$(6^8, 2)$	$\to$	$(6^4, 1)$
$(24, 16, 8, 1^2)$	$\to$	$(6^4, 2^4, 1^{18})$	$\to$	$(6^2, 2^2, 1^8, \bar{1})$
$(16^3, 2)$	$\to$	$(2, 1^{48})$	$\to$	(1^{25})
$(16^2, 12, 4, 2)$	$\to$	$(12, 4, 2, 1^{32})$	$\to$	$(6, 2, 1^{17})$
$(16, 12^2, 8, 1^2)$	$\to$	$(12, 3^4, 2^4, 1^{18})$	$\to$	$(6, 3^2, 2^2, 1^8, \bar{1})$
$(12^3, 8, 6)$	$\to$	$(12, 6, 3^8, 2^4)$	$\to$	$(6, 3^5, 2^2)$
$(8^3, 7^2, 4^3)$	$\to$	$(7^2, 4, 2^{12}, 1^8)$	$\to$	$(\bar{7}, 2^7, 1^4)$
$(8, 4^3, 3^6, 2^3, 1^6)$	$\to$	$(4, 3^6, 2^7, 1^{14})$	$\to$	$(3^2, \bar{3}, 2^3, \bar{2}, 1^7, \bar{1})$

TABLE 2. The bijections of Theorem 2.3 for $n = 25$, $\ell = 4$.

3. A partial identity for colored partitions

We state a partition identity involving 2-color partitions.

Proposition 3.1. *Let $T_4(n)$ denote the number of partitions of n in which even parts are of two kinds and distinct, and odd parts occur with multiplicity 4. Then*

$$\overline{R_4}(n) = T_4(2n). \tag{7}$$

We remark that one part-size with two different colors are treated as distinct parts in Proposition 3.1. It is a special case ($\ell = 4$) of the following generalization to every even integer $\ell > 0$.

Theorem 3.2. *Let ℓ be an even positive integer and let $T_\ell(2n)$ denote the number of partitions of $2n$ in which odd parts occur with multiplicity ℓ and* *even parts are of two kinds such that even parts of one kind are distinct and each even part of the other kind appears at most $\frac{\ell-2}{2}$ times. Then*

$$\overline{R_\ell}(n) = T_\ell(2n). \tag{8}$$

Remark 3.3. *If we combine Theorem 3.2 with the compatible functions defined the main theorem (Theorem 2.3) we obtain the following five-way identity for every even integer $\ell > 0$:*

$$B_\ell(2n) = Q_\ell(\ell n) = H_\ell(2n) = T_\ell(2n) = \overline{R_\ell}(n).$$

Proof of Theorem 3.2. Since $\overline{R_\ell}(n) = B_\ell(2n)$ from Theorem 2.3 it will suffice to prove $B_\ell(2n) = T_\ell(2n)$. The generating function for $T_\ell(2n)$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} T_\ell(2n)q^n &= \prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n}+\cdots+q^{(\frac{\ell-2}{2})2n})(1+q^{\ell(2n-1)}) \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^{2n})}\frac{(1-q^{\ell n})}{(1-q^{2n})}\frac{(1-q^{2\ell(2n-1)})}{(1-q^{\ell(2n-1)})}.
\end{aligned}$$

From Equation (4) we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} B_\ell(2n)q^n &= \prod_{n=1}^{\infty}\frac{(1-q^{2\ell n})}{(1-q^{2n})}\frac{(1-q^{2\ell(2n-1)})}{(1-q^{2(2n-1)})} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})(1+q^{\ell n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})}\frac{(1-q^{4n})}{(1-q^{2n})} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})(1-q^{4n})(1-q^{2\ell n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})^2(1-q^{\ell n})} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})(1-q^{4n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})^2(1-q^{\ell(2n-1)})} \\
&= \sum_{n=0}^{\infty} T_\ell(2n)q^n.
\end{aligned}$$

We now give a bijection $g : B_\ell[2n] \to T_\ell[2n]$ as usual, according to parities. Let the two kinds or colors in the theorem be distinguished by subscripting with "a" and "b". Thus each even part-size $2r$ has either the form $(2r)_a$ or $(2r)_b$ with $(2r)_a \neq (2r)_b$ such that $(2r)_a$ is distinct while $(2r)_b$ may also occur in the same partition at most $(\ell-2)/2$ times. Since an odd part in $\lambda \in B_\ell[2n]$ has multiplicity $2, 4, \ldots, 2(\ell-1)$, we have

If $c \equiv 1 \pmod 2$, then $g_c : c^k \mapsto \begin{cases} (2c)_b^{\frac{k}{2}} & \text{if } 2 \leq k < \ell; \\ c^\ell & \text{if } k = \ell; \\ (2c)_b^{\frac{k-\ell}{2}}, c^\ell & \text{if } k > \ell. \end{cases}$

If $c \equiv 0 \pmod 2$, then $g_c : c^k \mapsto \begin{cases} c_a & \text{if } k = 1. \\ (2c)_b^{\frac{k-1}{2}}, c_a & \text{if } k > 1 \text{ is odd.} \\ (2c)_b^{\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$

The inverse map is immediately seen to be $g^{-1} : T_\ell[2n] \to B_\ell[2n]$ with

$$g_c^{-1} : \begin{cases} c^\ell \mapsto c^\ell & \text{if } c \equiv 1 \pmod 2; \\ c_a \mapsto c & \text{if } c \equiv 0 \pmod 2; \\ c_b \mapsto (c/2)^2 & \text{if } c \equiv 0 \pmod 2). \end{cases}$$

■

An illustration of the bijection $B_\ell[2n] \iff T_\ell[2n]$ is given for some partitions of $\ell = 6$ and $n = 6$ in Table 3.

$B_6[2n]$	$\xrightarrow{g}$	$T_6[2n]$
(12)	$\to$	(12_a)
(6,6)	$\to$	(12_b)
(4,4,4)	$\to$	$(8_b,4_a)$
(8,4)	$\to$	$(8_a,4_a)$
(8,2,2)	$\to$	$(8_a,4_b)$
(4,4,2,2)	$\to$	$(8_b,4_b)$
(8,2,1,1)	$\to$	$(8_a,2_a,2_b)$
(2,2,2,2,2,1,1)	$\to$	$(4_b,4_b,2_a,2_b)$
(2,2,2,2,1,1,1,1)	$\to$	$(4_b,4_b,2_b,2_b)$
(2,2,2,1,1,1,1,1,1)	$\to$	$(4_b,2_a,1,1,1,1,1,1)$

TABLE 3. The bijections of Theorem 3.2 for $n = 6$, $\ell = 6$.

4. Congruences Properties

The number of overpartitions of $n > 0$ is always even. This is because an overpartition is obtained from an ordinary partition $\lambda = (c_1^{u_1}, c_2^{u_2}, \cdots, c_r^{u_r})$ by overlining the first occurrence of each part-size or not. Thus λ alone gives rise to 2^r overpartitions. We strengthen this observation in the following result.

Lemma 4.1. *For all $n \geq 1$,*

$$\overline{p}(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = k^2 \text{ for some integer } k \\ 0 \pmod 4 & \text{otherwise.} \end{cases}$$

Proof. We decompose overpartitions of n into two sets: those containing a unique part-size and those containing two or more different part-sizes. Then we see that the latter set of overpartitions has cardinality 2^r, $r > 1$, that is, a cardinality divisible by 4. On the other hand, partitions with a single part-size arise from divisors of n. Each divisor d of n gives the partition $(d^{n/d})$ which in turn generates 2 overpartitions. Since a square has an odd number of divisors, $\tau(k^2) \equiv 1 \pmod 2$, we deduce that $\overline{p}(k^2) \equiv 2 \pmod 4$. ∎

We can now give a combinatorial proof of the following result which is proved with generating functions by Alanazi-Munagi-Sellers [1].

Theorem 4.2. *For all $n \geq 1$ and an integer $k > 0$,*

$$\overline{R_\ell}(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = k^2 \text{ or } n = \ell k^2, \text{ where } \ell \text{ is not a square;} \\ 0 \pmod 4 & \text{otherwise.} \end{cases}$$

Proof. Let $m(\ell|n)$ be the number of multiples ℓ dividing n. By the proof of Lemma 4.1, it will suffice to find the parity of $\tau(n) - m(\ell|n)$: a divisor d of n generates a single part-size ℓ-regular overpartition provided that ℓ does not divide d. In each case we exclude the divisors enumerated by $m(\ell|n)$ and compare the parity of $\tau(n) - m(\ell|n)$ with the the parity of $\tau(n)$, and conclude that $\overline{R_\ell}(n) \equiv 2 \pmod 4$ or $\overline{R_\ell}(n) \equiv 0 \pmod 4$ if $\tau(n) - m(\ell|n)$ is odd or even respectively.

Consider the first case $n = k^2$ or $n = \ell k^2$ given that ℓ is not a square. If $n = k^2$ and ℓ does not divide n, then $m(\ell|n) = 0$. So $\tau(n) - m(\ell|n)$ is odd. If $n = k^2$ and ℓ divides n, then $m(\ell|n) = \tau(n/\ell)$ which is even. So $\tau(n) - m(\ell|n)$ is still odd. But if $n = \ell k^2$, then $\tau(n)$ is even and $m(\ell|n) = \tau(k^2)$ which is odd. So $\tau(n) - m(\ell|n)$ is odd.

The second case has two parts namely (i) $n = k^2$ with ℓ a square factor of n and (ii) $n \neq k^2$ and $n \neq \ell k^2$. In (i) we find that both $\tau(n)$ and $m(\ell|n)$ are odd; so $\tau(n) - m(\ell|n)$ is even. In (ii) it is clear that both $\tau(n)$ and $m(\ell|n)$ are even. This completes the proof. ∎

Conclusion

The main theorem, Theorem 2.3, as well as Theorem 3.2, contain seemingly incomplete partition identities because some of these are given only for selective parities of ℓ. It will be of interest to obtain extensions of the identities to all integers $\ell > 0$.

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