A characterization of the degree sequence of the graph with cyclomatic number k *

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Abstract. Let $(d_1, d_2, \ldots, d_n)$ be a sequence of positive integers with $n-1 \geq d_1 \geq d_2 \geq \cdots \geq d_n$. We give a characterization of $(d_1, d_2, \ldots, d_n)$ that is the degree sequence of the graph with cyclomatic number k. This simplifies the characterization of Erdős-Gallai.
Keywords. Graph, degree sequence, cyclomatic number.

1. Introduction

In this paper, the graphs considered are finite, undirected and simple. Let $G = (V, E)$ be a graph and $V = \{v_1, v_2, \ldots, v_n\}$, the degree sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ of G is denoted by $d(G)$. The sequence $d = (d_1, d_2, \ldots, d_n)$ of positive integers is said to be *graphic* if there exists a simple graph G such that $d(G) = d$, and G is called a *realization* of d. It is obvious that if $(d_1, d_2, \ldots, d_n)$ is graphic, then $d_i \leq n-1$ for each i and $\sum\limits_{i=1}^{n} d_i$ is even. However, these two conditions together do not ensure that a sequence will be graphic. A well-known characterization of $(d_1, d_2, \ldots, d_n)$ that is graphic is the following

Theorem 1.1 (Erdős and Gallai [3]) Let $(d_1, d_2, \ldots, d_n)$ be a sequence of positive integers with $n-1 \geq d_1 \geq d_2 \geq \cdots \geq d_n$. Then $(d_1, d_2, \ldots, d_n)$ is graphic if and only if $\sum\limits_{i=1}^{n} d_i$ is even and $\sum\limits_{i=1}^{s} d_i \leq s(s-1) + \sum\limits_{i=s+1}^{n} min\{s, d_i\}$ for each s with $1 \leq s \leq n$.

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Let $G = (V, E)$ be a connected graph. We say that $|E| - |V| + 1$ is the *cyclomatic number* of G, denoted by $c(G)$. If $c(G)$ =1, 2 and 3, then G is said to be a *unicyclic* graph, *bicyclic* graph and *tricyclic* graph, respectively. Zhou and Cai [1] gave a characterization of $(d_1, d_2, \dots, d_n)$ that is the degree sequence of unicyclic graph, bicyclic graph and tricyclic graph, respectively.

Theorem 1.2 (Zhou and Cai [1]) Let $d = (d_1, d_2, \dots, d_n)$ be a sequence of positive integers with $n - 1 \geq d_1 \geq d_2 \geq \dots \geq d_n$.

(1) There exists a unicyclic graph G such that $d(G) = d$ if and only if $n \geq 3$, $\sum_{i=1}^{n} d_i = 2n$ and $d_3 \geq 2$.

(2) There exists a bicyclic graph G such that $d(G) = d$ if and only if $n \geq 4$, $\sum_{i=1}^{n} d_i = 2n + 2$ and $d_4 \geq 2$.

(3) There exists a tricyclic graph G such that $d(G) = d$ if and only if $n \geq 4$ and d satisfies one of the following conditions: (i) $n = 4$ and $\sum_{i=1}^{n} d_i = 2n + 4$; (ii) $n \geq 5$, $\sum_{i=1}^{n} d_i = 2n + 4$, $d_5 = 1$ and $d_4 \geq 3$; (iii) $n \geq 5$, $\sum_{i=1}^{n} d_i = 2n + 4$ and $d_5 \geq 2$.

In this paper, we give a characterization of $(d_1, d_2, \dots, d_n)$ that is the degree sequence of the graph with cyclomatic number k as follows. This simplifies the characterization of Erdős-Gallai.

Theorem 1.3 Let $d = (d_1, d_2, \dots, d_n)$ be a sequence of positive integers with $n - 1 \geq d_1 \geq d_2 \geq \dots \geq d_n$. Then there exists a graph G with cyclomatic number k such that $d(G) = d$ if and only if $n \geq \lceil \frac{3+\sqrt{1+8k}}{2} \rceil$, $\sum_{i=1}^{n} d_i = 2n + 2k - 2$, $d_{\lceil \frac{3+\sqrt{1+8k}}{2} \rceil} \geq 2$ and $\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$ for $t = 2, \dots, \lceil \frac{3+\sqrt{1+8k}}{2} \rceil - 1$.

2. Proof of Theorem 1.3

We also need a lemma as follows.

Lemma 2.1 (Edmonds [2]) If $d = (d_1, d_2, \dots, d_n)$ is a graphic sequence with $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$ and $\sum_{i=1}^{n} d_i \geq 2n - 2$, then d has a connected realization.

Proof of Theorem 1.3. For the necessity, let $G = (V, E)$ be a graph with cyclomatic number k and $d(G) = d$. Then $|E| = n - 1 + k$ and $\sum_{i=1}^{n} d_i = 2|E| = 2n + 2k - 2$. By $n - 1 + k = |E| \leq \frac{n(n-1)}{2}$, we have that $n^2 - 3n + 2 - 2k \geq 0$. We can get that $n \geq \frac{3+\sqrt{1+8k}}{2}$, thus $n \geq \lceil \frac{3+\sqrt{1+8k}}{2} \rceil$.

By Theorem 1.1, it is obvious that $\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$ for $t = 2, \ldots, \lceil \frac{3+\sqrt{1+8k}}{2} \rceil - 1$.

If $d_{\lceil \frac{3+\sqrt{1+8k}}{2} \rceil} = 1$, we let $t = \lceil \frac{3+\sqrt{1+8k}}{2} \rceil - 1$, then $\sum_{i=1}^{t} d_i = 2n + 2k - 2 - (n - \lceil \frac{3+\sqrt{1+8k}}{2} \rceil + 1) = n + 2k + \lceil \frac{3+\sqrt{1+8k}}{2} \rceil - 3$. On the other hand, $t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\} = (\lceil \frac{3+\sqrt{1+8k}}{2} \rceil - 1)(\lceil \frac{3+\sqrt{1+8k}}{2} \rceil - 2) + (n - \lceil \frac{3+\sqrt{1+8k}}{2} \rceil + 1) = n + (\lceil \frac{3+\sqrt{1+8k}}{2} \rceil)^2 - 4\lceil \frac{3+\sqrt{1+8k}}{2} \rceil + 3$.

Since $x^2 - 5x$ is a monotone increasing function when $x \ge \frac{5}{2}$ and $\frac{3+\sqrt{1+8k}}{2} + 1 > \lceil \frac{3+\sqrt{1+8k}}{2} \rceil \ge \frac{5}{2}$, we have that $(\lceil \frac{3+\sqrt{1+8k}}{2} \rceil)^2 - 5\lceil \frac{3+\sqrt{1+8k}}{2} \rceil < (\frac{3+\sqrt{1+8k}}{2} + 1)^2 - 5(\frac{3+\sqrt{1+8k}}{2} + 1) = 2k - 6$. This implies that $\sum_{i=1}^{t} d_i = n + 2k + \lceil \frac{3+\sqrt{1+8k}}{2} \rceil - 3 = n + 2k - 6 + \lceil \frac{3+\sqrt{1+8k}}{2} \rceil + 3 > n + (\lceil \frac{3+\sqrt{1+8k}}{2} \rceil)^2 - 4\lceil \frac{3+\sqrt{1+8k}}{2} \rceil + 3 = t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$, a contradiction.

For the sufficiency, we assume that $n \ge \lceil \frac{3+\sqrt{1+8k}}{2} \rceil$, $\sum_{i=1}^{n} d_i = 2n + 2k - 2$, $d_{\lceil \frac{3+\sqrt{1+8k}}{2} \rceil} \ge 2$ and $\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$ for $t = 2, \ldots, \lceil \frac{3+\sqrt{1+8k}}{2} \rceil - 1$. By Lemma 2.1, we only need to prove that d is a graphic sequence. By Theorem 1.1, it is enough to check that $\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$ for $t = 1$ and $\lceil \frac{3+\sqrt{1+8k}}{2} \rceil \le t \le n$. If $t = 1$, then $d_1 \le n - 1 = t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$. Assume that $\lceil \frac{3+\sqrt{1+8k}}{2} \rceil \le t \le n$. Then $\sum_{i=1}^{t} d_i \le 2n + 2k - 2 - (n - t) = n + 2k + t - 2$. On the other hand, $t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\} \ge t^2 - t + (n - t) = n + t^2 - 2t$.

Since $x^2 - 3x$ is a monotone increasing function when $x \ge \frac{3}{2}$ and $t \ge \lceil \frac{3+\sqrt{1+8k}}{2} \rceil \ge \frac{3+\sqrt{1+8k}}{2} \ge \frac{3}{2}$, we have that $t^2 - 3t \ge (\frac{3+\sqrt{1+8k}}{2})^2 - 3(\frac{3+\sqrt{1+8k}}{2}) = 2k - 2$. This implies that $\sum_{i=1}^{t} d_i \le n + 2k + t - 2 \le n + t^2 - 2t \le t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}$. This completes the proof of Theorem 1.3. □

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