Implicit degree sum condition for hamiltonian cycles

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Abstract: The hamiltonian problem is a classical problem in graph theory. Most of the research on hamiltonian problem is looking for sufficient conditions for a graph to be hamiltonian. For a vertex v of a graph G, Zhu, Li and Deng introduced the concept of implicit degree $id(v)$, according to the degrees of its neighbors and the vertices at distance 2 with v in G. In this paper, we will prove that: Let G be a 2-connected graph on $n \geq 3$ vertices. If the maximum value of the implicit degree sums of 2 vertices in S is more than or equal to n for each independent set S with $\kappa(G)+1$ vertices, then G is hamiltonian.

Keywords: Implicit degree sum; Independent set; Hamiltonian cycle

1 Introduction

In this paper, we consider only finite, undirected and simple graphs. Notation and terminology not defined here can be found in [2]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and H be a subgraph of G. For a vertex $u \in V(G)$, let $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and $d_H(u) = |N_H(u)|$. If $G = H$, we always use $N(u)$ and $d(u)$ in place of $N_G(u)$ and $d_G(u)$ respectively. Let $N_2(u) = \{v \in V(G) : d(u,v) = 2\}$, where $d(u,v)$ denotes the distance between vertices u and v in G.

Let $\alpha(G)$ and $\kappa(G)$ denote the independence number and the connectivity of G, respectively. For a nonempty set $S \subset V(G)$, let $\Delta_k(S) = \max\{\sum_{x\in X} d(x) : X \text{ is a subset of } S \text{ with } k \text{ vertices}\}$.

A cycle containing all vertices of G is called a hamiltonian cycle. A graph G is called hamiltonian if it contains a hamiltonian cycle. The hamiltonian problem is an important problem in graph theory. Various sufficient conditions for a graph to be hamiltonian have been given in terms of degree conditions. The following two sufficient conditions are essential.

Theorem 1 ([7]). *Let G be a graph on $n \geq 3$ vertices. If $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x and y, then G is hamiltonian.*

Theorem 2 ([5]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $\max\{d(x), d(y)\} \geq n/2$ for every pair of vertices x and y at distance 2, then G is hamiltonian.*

In 1972, by considering the relationship between the independence number and the connectivity of a graph, Chvátal and Erdös gave a sufficient condition for a 2-connected graph to be hamiltonian.

Theorem 3 ([3]). *Let G be a 2-connected graph with $\alpha(G) \leq \kappa(G)$. Then G is hamiltonian.*

Recently, Yamashita improved Ore's Theorem (Theorem 1) as follows.

Theorem 4 ([8]). *Let G be a 2-connected graph of on $n \geq 3$ vertices. If $\Delta_2(S) \geq n$ for every independent set S of order $\kappa(G) + 1$, then G is hamiltonian.*

In order to generalize and improve the classic results of hamiltonian problem, Zhu, Li and Deng [9] gave the concept of implicit degree of a vertex.

Definition 1 ([9]). *Let v be a vertex of a graph G and $k = d(v) - 1$. Set $M_2 = \max\{d(u) : u \in N_2(v)\}$ and $m_2 = \min\{d(u) : u \in N_2(v)\}$. Suppose $d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_k \leq d_{k+1} \leq \ldots$ is the degree sequence of vertices in $N(v) \cup N_2(v)$. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, define*

$$d^*(v) = \begin{cases} m_2, & \text{if} \quad d_k < m_2; \\ d_{k+1}, & \text{if} \quad d_{k+1} > M_2; \\ d_k, & \text{otherwise}, \end{cases}$$

then the implicit degree *of v is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then $id(v) = d(v)$.*

Clearly, $id(v) \geq d(v)$ for each vertex v from the definition of implicit degree. For $S \subseteq V(G)$ with $S \neq \emptyset$, let $i\Delta_k(G) = \max\{\sum_{x \in X} id(x) : X$ is a subset of S with k vertices$\}$. The authors [9] used implicit degree sum instead of degree sum in Ore's theorem (Theorem 1), and got a sufficient condition for a graph to be hamiltonian.

Theorem 5 ([9]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $id(u) + id(v) \geq n$ for each pair of nonadjacent vertices u and v in G, then G is hamiltonian.*

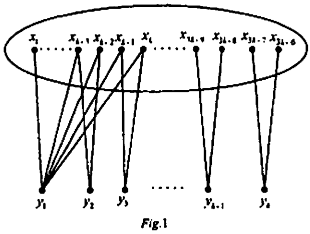

*Fig.*1

From Theorem 5, we can easily deduce Fan's Theorem (Theorem 2). Moreover, the authors in [9] gave an example to illustrate Theorem 5 is stronger than Fan's theorem. Motivated by the results of Theorem 4 and Theorem 5, we study implicit degrees and the hamiltonicity of graphs and obtain the following main result.

Theorem 6. *Let G be a 2-connected graph on $n \geq 3$ vertices. If $i\Delta_2(S) \geq n$ for every independent set S of order $\kappa(G)+1$, then G is hamiltonian.*

We postpone the proof of Theorem 6 in next section. Here we give two remarks. One shows that Theorem 6 is much stronger than Theorem 4 and Theorem 5; the other shows that the condition " $i\Delta_2(S) \geq n$" could not be weaken to " $i\Delta_2(S) \geq n-1$".

Let $G_1, \ldots, G_k$ be k vertex disjoint graphs. The union of $G_1, \ldots, G_k$, denoted by $G_1 \cup \ldots \cup G_k$, is the graph with vertex set $\cup_{i=1}^{k} V(G_i)$ and edge set $\cup_{i=1}^{k} E(G_i)$. We use kQ instead of $G_1 \cup \ldots \cup G_k$ if each G_i is isomorphic to Q. The join of $G_1, \ldots, G_k$, denoted by $G_1 \vee \ldots \vee G_k$, is the graph obtained from $\cup_{i=1}^{k} G_i$ by joining each vertex of G_i to each vertex of G_j for $i \neq j$.

Remark 1. *The graph in Fig.1 shows that Theorem 6 is much stronger than Theorem 4 and Theorem 5. Let $k \geq 4$, $V(K_{3k-6}) = \{x_1, x_2, \ldots, x_{3k-6}\}$, $V(kK_1) = \{y_1, y_2, \ldots, y_k\}$. We construct a graph G, with vertex set $V(G) = V(K_{3k-6} \cup kK_1)$, and edge set $E(G) = E(K_{3k-6}) \cup \{y_1 x_i : i = 1, 2, \ldots, k\} \cup \{y_i x_{2i+k-7}, y_i x_{2i+k-6} : i = 2, 3, \ldots, k\}$. It is easy to check that G is a 2-connected hamiltonian graph on $n = 4k-6$ vertices. Choose $S = \{y_1, y_2, y_3\}$, then $\Delta_2(S) = k+2 < n$ and G does not satisfy the condition in Theorem 4. Since $id(y_1) = 3k-6$, $id(y_2) = id(y_3) = k$, $id(y_j) = 3k-6$ for $j = 4, 5, \ldots, k$, $id(x_2) = 2k+1$ and $id(x_l) \geq d(x_l) \geq 3k-6$ for $l = 1, 2, \ldots, 3k-6$. Since y_2 and y_3 are two nonadjacent vertices of G and $id(y_2) + id(y_3) = 2k < n$, G does not satisfy the condition in Theorem 5. But it is easy to check that G satisfies the condition in Theorem 6.*

Remark 2. *Let $G = K_k \vee (k+1)K_1$. Clearly, G is a k-connected non-hamiltonian graph of order $n = 2k+1$. Let $S = V((k+1)K_1)$. It is* *easy to check that* $i\Delta_2(S) = 2k = n-1$, *This implies that the condition "* $i\Delta_2(S) \geq n$*" in Theorem 6 is best possible.*

2 Proof of Theorem 6

A path P connecting x and y is called an H-path if $V(P) \cap V(H) = \{x, y\}$ and $E(P) \cap E(H) = \emptyset$. For a cycle C in G with a given orientation and a vertex x in C, x^+ and x^- denote the successor and the predecessor of x in C, respectively. For any $I \subseteq V(C)$, let $I^- = \{x : x^+ \in I\}$ and $I^+ = \{x : x^- \in I\}$. For two vertices x and y in C, we define xCy to be the path of C from x to y. $y\bar{C}x$ denotes the path from y to x in the reversed direction of C. A similar notation is used for paths.

For a path $P = x_1 x_2 \ldots x_p$ of a graph G, let $l_P(x_1) = \max\{i : x_i \in V(P) \text{ and } x_i x_1 \in E(G)\}$ and $l_P(x_p) = \min\{i : x_i \in V(P) \text{ and } x_i x_p \in E(G)\}$. Set $L_P(x_1) = x_{l_P(x_1)}$ and $L_P(x_p) = x_{l_P(x_p)}$. Our proof of Theorem 6 is based on the following lemmas.

Lemma 1 ([1]). *Let G be a 2-connected graph on n vertices and C be a longest cycle of G with length at most $n-1$. If P is a path connecting x and y in G such that $|V(C)| < |V(P)|$, then $d(x) + d(y) < n$.*

Lemma 2 ([4]). *Let G be a 2-connected graph and X be a subset of $V(G)$. If $|X| \leq \kappa(G)$, then G has a cycle that includes every vertex of X.*

Lemma 3 ([6]). *Let G be a 2-connected graph and $P = x_1 x_2 \ldots x_p$ be a path of G. If $x_1 x_p \notin E(G)$, and $d(u) < id(x_1)$ for any $u \in N_{G-V(P)}(x_1) \cup \{x_1\}$, then either*
(1) there exists a vertex $x_j \in N_P(x_1)^-$ such that $d(x_j) \geq id(x_1)$; or
(2) $N_P(x_1)^- = N_P(x_1) \cup \{x_1\} - \{L_P(x_1)\}$, $d(x_j) < id(x_1)$ for any vertex $x_j \in N_P(x_1)^-$ and $id(x_1) = \min\{d(v) : v \in N_2(x_1)\}$.

Proof of Theorem 6. Suppose to the contrary that G is a graph satisfying the condition of Theorem 6 and G is not hamiltonian. Let C be a longest cycle of G and give C a clockwise orientation. Then $|V(C)| < n$.

Let H be a component of $G - V(C)$ and $y_0 \in V(H)$. Set $k = \kappa(G)$. By Lemma 2, $|V(C)| \geq k$. Since the connectivity of G is k, there are k paths $P_1(y_0, y_1), P_2(y_0, y_2), \ldots, P_k(y_0, y_k)$ from y_0 to C having only y_0 in common pairwise and $V(P_i) \cap V(C) = \{y_i\}$ for each $1 \leq i \leq k$. Without loss of generality, we orient P_i from y_0 to y_i. Assume without loss of generality that $y_1, y_2, \ldots, y_k$ occur in this order along C. Let $x_i = y_i^+$ for each $i = 1, 2, \ldots, k$. Let x_0 and z_0 be the predecessor of y_2 on the path P_2 and the predecessor of y_1 on the path P_1, respectively.

Claim 1. $\{x_0, x_1, x_2, \ldots, x_k\}$ is an independent set of G. Similarly, $\{z_0, x_1, x_2, \ldots, x_k\}$ is also an independent set of G.

Proof. If $x_0x_2 \in E(G)$, then $C' = x_0x_2Cy_2x_0$ is a cycle longer than C, a contradiction.

If $x_0x_i \in E(G)$ for $i \neq 2$, then $C' = x_ix_0\bar{P}_2y_0P_iy_i\bar{C}x_i$ is a cycle longer than C, a contradiction.

If $x_ix_j \in E(G)$ for $1 \leq i < j \leq k$, then $C' = x_iCy_j\bar{P}_jy_0P_iy_i\bar{C}x_jx_i$ is a cycle longer than C, a contradiction. Therefore, $\{x_0, x_1, x_2, \ldots, x_k\}$ is an independent set of G. Similarly, $\{z_0, x_1, x_2, \ldots, x_k\}$ is also an independent set of G. □

By Claim 1 and the hypothesis of Theorem 6, there exist at least two vertices in $\{x_0, x_1, x_2, \ldots, x_k\}$ with implicit degree sum more than or equal to n and there exist at least two vertices in $\{z_0, x_1, x_2, \ldots, x_k\}$ with implicit degree sum more than or equal to n.

Case 1. There exist some i and j with $1 \leq i < j \leq k$ such that $id(x_i) + id(x_j) \geq n$.

Set $P = x_iCy_j\bar{P}_jy_0P_iy_i\bar{C}x_j$. Clearly, $|V(P)| > |V(C)|$.

Claim 2. $N_P(x_i)^- \neq N_P(x_i) \cup \{x_i\} - \{L_P(x_i)\}$ and $N_P(x_j)^+ \neq N_P(x_j) \cup \{x_j\} - \{L_P(x_j)\}$.

Proof. By the choices of C, we know $x_iy_i \in E(G), x_iy_0 \notin E(G)$ and $x_jy_j \in E(G), x_jy_0 \notin E(G)$. Since y_0 is before y_i on the path P, $N_P(x_i)^- \neq N_P(x_i) \cup \{x_i\} - \{L_P(x_i)\}$. Since y_j is before y_0 on the path P, $N_P(x_j)^+ \neq N_P(x_j) \cup \{x_j\} - \{L_P(x_j)\}$. □

Claim 3. There exists a path $W(w_1, w_2)$ such that (i) $V(P) \subseteq V(W)$, and (ii) $d(w_1) \geq id(x_i)$ and $d(w_2) \geq id(x_j)$.

Proof. For convenience, set $P = u_1u_2 \ldots u_p$ with $u_1 = x_i$ and $u_p = x_j$. By the choice of C, we have $N_{G-V(P)}(x_i) \cap N_{G-V(P)}(x_j) = \emptyset$.

Case 1.1. There is a vertex $u \in N_{G-V(P)}(x_i) \cup \{x_i\}$ such that $d(u) \geq id(x_i)$.

If there is a vertex $v \in N_{G-V(P)}(x_j) \cup \{x_j\}$ such that $d(v) \geq id(x_j)$, then set

$$W(w_1, w_2) = ux_iPx_jv, \text{ where } w_1 = u \text{ and } w_2 = v.$$

If $d(v) < id(x_j)$ for each vertex $v \in N_{G-V(P)}(x_j) \cup \{x_j\}$, then by Claim 2 and Lemma 3, there exists some vertex $u_l \in N_P(x_j)^+$ such that $d(u_l) \geq id(x_j)$. Set

$W(w_1, w_2) = ux_iPu_{l-1}x_j\bar{P}u_l$, where $w_1 = u$ and $w_2 = u_l$.

Case 1.2. $d(u) < id(x_i)$ for each vertex $u \in N_{G-V(P)}(x_i) \cup \{x_i\}$.

By Claim 2 and Lemma 3, there is some vertex $u_h \in (N_P(x_i))^-$ such that $d(u_h) \geq id(x_i)$. If there is a vertex $v \in N_{G-V(P)}(x_j) \cup \{x_j\}$ such that $d(v) \geq id(x_j)$, then set

$W(w_1, w_2) = u_h\bar{P}x_iu_{h+1}Px_jv$, where $w_1 = u_h$ and $w_2 = v$.

Next, we suppose $d(v) < id(x_j)$ for any vertex $v \in N_{G-V(P)}(x_j) \cup \{x_j\}$. If $h + 1 \leq l_P(x_j)$ (where h is the index of u_h on the path P), then there is some $u_l \in (N_P(x_j))^+$ such that $d(u_l) \geq id(x_j)$ by Claim 2 and Lemma 3. Set

$W(w_1, w_2) = u_h\bar{P}x_iu_{h+1}Pu_{l-1}x_j\bar{P}u_l$, where $w_1 = u_h$ and $w_2 = u_l$.

If $h + 1 > l_P(x_j)$, set

$$A = \{u_s : u_{s+1} \in N_P(x_j) \text{ and } s < h\},$$
$$B = \{u_s : u_{s-1} \in N_P(x_j) \text{ and } s > h + 1\}, \text{ and}$$
$$C = \{u_s : u_{s+1} \in N_{P_{i,j}}(x_j)\ s \geq h + 1 \text{ and} s \text{ is as small as possible}\}.$$

Clearly, $x_j \in B$ and $|C| = 1$. Then $|A|+|B\backslash\{x_j\}|+|C|+|N_{G-V(P)}(x_j)| = d(x_j)$, $u_{l_P(x_j)-1} \in A \cap N_2(x_j)$. (Since $d(x_j) = d_P(x_j) + d_{G-V(P)}(x_j) = |N_P(x_j)|+|N_{G-V(P)}(x_j)|$ and $|N_P(x_j)| = |N_{V(u_1Pu_h)}(x_j)|+|N_{V(u_{h+1}Pu_p)}(x_j)|$ $= |A| + |B \setminus \{x_j\}| + |C|$.) Since $u_h \notin N(x_j)$, $C \subseteq N_2(x_j)$. By the definition of $id(x_j)$, there is some vertex $u_l \in (A \cup B) - \{x_j\}$ such that $d(u_l) \geq id(x_j)$. When $u_l \in B \setminus \{x_j\}$, set

$W(w_1, w_2) = u_h\bar{P}x_iu_{h+1}Pu_{l-1}x_j\bar{P}u_l$, where $w_1 = u_h$ and $w_2 = u_l$.

When $u_l \in A$, set

$W(w_1, w_2) = u_h\bar{P}u_{l+1}x_j\bar{P}u_{h+1}x_iPu_l$, where $w_1 = u_h$ and $w_2 = u_l$.

Now we complete the proof of Claim 3. □

By Claim 3, there exists a path $W(w_1, w_2)$ such that $|V(W)| \geq |V(P)| > |V(C)|$ and $d(w_1)+d(w_2) \geq id(x_i)+id(x_j) \geq n$. This contradicts Lemma 1.

Case 2. There exists some i with $1 \leq i \leq k$ and $i \neq 2$ such that $id(x_0) + id(x_i) \geq n$.

Assume, without loss of generality, that $id(x_0) + id(x_1) \geq n$ and set $Q = x_0\bar{P}_2y_0P_1y_1\bar{C}x_1$. Clearly, $|V(Q)| > |V(C)|$.

Claim 4. $N_Q(x_0)^- \neq N_Q(x_0) \cup \{x_0\} - \{L_Q(x_0)\}$ and $N_Q(x_1)^+ \neq N_Q(x_1) \cup \{x_1\} - \{L_Q(x_1)\}$.

Proof. By Claim 1, $x_0x_2 \notin E(G)$. Since $x_0y_2 \in E(G)$ and x_2 is the predecessor of y_2 on the path P, $N_Q(x_0)^- \neq N_Q(x_0) \cup \{x_0\} - \{L_Q(x_0)\}$. Since $x_1y_1 \in E(G), x_1x_2 \notin E(G)$ and y_1 is before x_2 on the path Q, $N_Q(x_1)^+ \neq N_Q(x_1) \cup \{x_1\} - \{L_Q(x_1)\}$. □

By similar argument as in Claim 3 to the path Q, we have

Claim 5. There exists a path $W'(w_1', w_2')$ such that (i) $V(Q) \subseteq V(W')$, and (ii) $d(w_1') \geq id(x_0)$ and $d(w_2') \geq id(x_1)$. □

By Claim 5, there exists a path $W'(w_1', w_2')$ such that $|V(W')| \geq |V(Q)| > |V(C)|$ and $d(w_1')+d(w_2') \geq id(x_0)+id(x_1) \geq n$. This contradicts Lemma 1.

Case 3. $id(x_0) + id(x_2) \geq n$.

Set $R = x_0y_2\bar{C}x_2$. Clearly, $|V(R)| > |V(C)|$. For convenience, set $R = r_1r_2 \ldots r_s$.

Claim 6. $N_C(x_0) \cap N_C(x_0)^+ = \emptyset$ and $N_R(x_2)^+ \neq N_R(x_2) \cup \{x_2\} - \{L_R(x_2)\}$.

Proof. If $x \in N_C(x_0) \cap N_C(x_0)^+$, then $x_0xCx^-x_0$ is a cycle longer than C, contrary to the choice of C. So $N_C(x_0) \cap N_C(x_0)^+ = \emptyset$. Since $x_2y_2 \in E(G), x_2x_1 \notin E(G)$ and y_2 is before x_1 on the path R, $N_R(x_2)^+ \neq N_R(x_2) \cup \{x_2\} - \{L_R(x_2)\}$. □

Case 3.1. There is a vertex $x \in N_{G-V(R)}(x_0) \cup \{x_0\}$ such that $d(x) \geq id(x_0)$.

If there is a vertex $y \in N_{G-V(R)}(x_2) \cup \{x_2\}$ such that $d(y) \geq id(x_2)$, then $R' = xx_0Rx_2y$ is a path such that $|V(R')| > |V(C)|$ and $d(x)+d(y) \geq id(x_0) + id(x_2) \geq n$. This contradicts Lemma 1.

If $d(y) < id(x_2)$ for each vertex $y \in N_{G-V(R)}(x_2) \cup \{x_2\}$, then by Claim 6 and Lemma 3, there exists some vertex $r_t \in N_R(x_2)^+$ such that $d(r_t) \geq id(x_2)$. Then $R' = xx_0Rr_{t-1}x_2\bar{R}r_t$ is a path such that $|V(R')| > |V(C)|$ and $d(x) + d(r_t) \geq id(x_0) + id(x_2) \geq n$. This contradicts Lemma 1.

Case 3.2. $d(x) < id(x_0)$ for each vertex $x \in N_{G-V(R)}(x_0) \cup \{x_0\}$.

By Claim 6, $N_C(x_0)^+ \subseteq N_2(x_0)$. Then $|N_{G-V(R)}(x_0)| + |N_C(x_0)^+| \geq d_R(x_0) + d_{G-V(R)}(x_0) = d(x_0)$. If $d_C(x_0) \geq 2$, then by the definition of $id(x_0)$, there exists at least one vertex $z \in N_C(x_0)^+ \setminus \{x_2\}$ such that $d(z) \geq id(x_0)$. Then $id(z) + id(x_2) \geq id(x_0) + id(x_2) \geq n$. By similar argument as in Case 1, we can get a contradiction.

Next, suppose $d_C(x_0) = 1$. Then by the definition of $id(x_0)$, $id(x_0) = \min\{d(u) : u \in N_2(x_0)\}$. If there exists a vertex $y \in N_2(x_0) \cap (V(G) - V(R))$, then by similar argument to the path $yy'x_0Rx_2$ with $y' \in N(x_0) \cap N(x)$ as in Case 3.1, we can get a contradiction. So suppose $N_2(x_0) \cap (V(G) - V(R)) = \emptyset$.

By Claim 1 and the hypothesis of Theorem 6, there exist at least two vertices in $\{z_0, x_1, x_2, \ldots, x_k\}$ with implicit degree sum more than or equal to n. If there exists some i with $1 \leq i \leq k$ and $i \neq 1$ such that $id(z_0)+id(x_i) \geq n$, then by similar argument as in Case 2, we can get a contradiction. So suppose $id(z_0) + id(x_1) \geq n$. Then $id(x_0) + id(x_2) + id(z_0) + id(x_1) \geq 2n$. Therefore, $id(z_0) + id(x_2) \geq n$ or $id(x_0) + id(x_1) \geq n$. Thus, by similar argument as in Case 2, we can get a contradiction.

Now the proof of Theorem 6 is completed.

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