Edge Even Graceful Labeling of Some Path and Cycle Related Graphs

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Abstract

Graceful labeling of graphs is used in radar codes. In this work, we introduce a new version of gracefulness we call it edge even graceful labeling of graphs. A necessary and sufficient condition for edge even graceful labeling of path graph, P_n , cycle graph, C_n , and star graph $K_{1,n}$ .We also prove some necessary and sufficient conditions for some path and cycle related graphs, namely, Friendship, Wheel, Double wheel, and Fan graphs.

Keywords: Graph labeling; Graceful labeling; Edge even graceful labeling.
Mathematics Subject Classification: 05C78.

1. Introduction

The field of Graph Theory plays an important role in various areas of pure and applied sciences. In Graph theory a main problem is graph labeling. Graph labeling is the assignment of some integers for vertices, edges or both of the graphs respectively with certain conditions. Graph labeling is used in many applications like coding theory, radar, astronomy, circuit design, missile guidance, communication network addressing, X - ray crystallography and data base management. See [3,4,5] for some details and concerning applications.

A graceful labeling of a graph with m edges is a labeling of its vertices with some subset of the integers between 0 and m inclusive, such that no two vertices share a label, and such that each edge is uniquely identified by the positive, or absolute difference between its endpoints [9]. A graph which admits a graceful labeling is called a graceful graph.

The name "graceful labeling" is due to Solomon W. Golomb [8]; this labeling was originally given the name β -labeling by A. Rosa in a paper on graph labeling [1].

In this work, we mean by a graph, $G=(V(G),E(G))$ a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$.

Definition1.1 A function f is called graceful labeling of a graph G if $f:V(G)\rightarrow\{0,1,2,...,q=|E(G)|\}$ is an injective function and the induced function $f^*:E(G)\rightarrow\{1,2,...,q\}$ defined as $f^*(e=uv)=|f(u)-f(v)|$ is bijective. A graph which admits a graceful labeling is called a graceful graph.

The concept of odd graceful labeling was introduced by Gnanajothi [6] as follows:

Definition1.2. A function f is called odd graceful labeling of a graph G if $f:V(G)\rightarrow\{0,1,2,...,2q-1\}$ is an injective function and the induced function $f^*:E(G)\rightarrow\{1,3,...,2q-1\}$ defined as $f^*(e=uv)=|f(u)-f(v)|$ is bijective. A graph which admits an odd graceful labeling is called an odd graceful graph.

The concept of edge graceful labeling, introduced by Sheng-Ping [7], uses modular arithmetic in the additive group $(Z_p,\oplus_p)$, where $p=|V(G)|$ as follows:

Definition1.3. A function f is called edge graceful labeling of a graph G if $f:E(G)\rightarrow\{1,2,...,q\}$ is a bijection and the induced function $f^*:V(G)\rightarrow Z_p$ defined as $f^*(x)=\sum_{xy\in E(G)} f(xy)(\bmod p)$, is an injective function. Clearly, the sum is taken over all edges $xy\in E(G)$ having x is an endpoint. A graph which admits edge graceful labeling is called an edge graceful graph.

The concept of edge odd graceful labeling was introduced by Solairaju [2] as follows:

Definition1.4. A function f is called edge odd graceful labeling of a graph G if $f:E(G)\rightarrow\{1,3,...,2q-1\}$ is a bijection and the induced function $f^*:V(G)\rightarrow\{0,1,2,...,2q-1\}$ defined as $f^*(x)=\sum_{xy\in E(G)} f(xy)(\bmod 2q)$, is an injective function. A graph which admits edge odd graceful labeling is called an edge odd graceful graph.

Here, we introduce edge even graceful labeling as follows:

Definition1.5. A function f is called edge even graceful labeling of a graph G if $f:E(G)\rightarrow\{2,4,...,2q\}$ is a bijection and the induced function $f^*:V(G)\rightarrow\{0,2,...,2q\}$ defined as $f^*(x)=\sum_{xy\in E(G)} f(xy)(\bmod 2k)$, is an injective function, where $k=\max(p,q)$. The graph which admits an edge even graceful labeling is called an edge even graceful graph.

Illustration: The graph shown in Fig.(1)(a) is an edge even graceful graph, while the graph shown in Fig.(1)(b), is not edge even graceful graph because for any bijective function $f: E(G) \to \{2,4,\ldots,12\}$, there is no injective induced function satisfying the requirements of definition 1.5.

(a) Sierpinski triangle is edge even graceful.

(b) Wheel graph W_3 is not edge even graceful.

Fig. (1)

2. Paths , Cycles and Star Graphs

Lemma 2.1 Let $H = 2Z_{2n} = \{0,2,4,\ldots,2n-2\}$.

i. The only element in H having its own inverse is 0 or n.

ii. $\sum_{x \in H} x = \sum_{i=0}^{n}(2i) \equiv \begin{cases} n, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases} \pmod{2n}$.

Proof:

i. $x \equiv -x \pmod{2n} \Leftrightarrow x + x \equiv 0 \pmod{2n} \Leftrightarrow x \equiv 0$ or $x \equiv n \pmod{2n}$.

ii. $\sum_{i=1}^{n} 2i = 2+\ldots+2n \equiv 2+\ldots+(2n-2) \equiv [2+(2n-2)]+\ldots+n+\ldots+[4+(2n-4)] \pmod{2n}$

This sum is clearly 0 when n is odd as the number is even and it is n when n is even as the number is odd. □

Theorem 2.2

i. The path graph on n vertices, P_n, is edge even graceful if and only if n is odd.

ii. The cycle graph on n vertices, C_n, is edge even graceful if and only if n is odd.

Proof:

i. Suppose $H = \{\pi(2),\ldots,\pi(2n-2)\}$,where

$\pi: \{2,\ldots,2n-2\} \to \{2,\ldots,2n-2\}$ is some permutation (bijective function)

i.e., π is some labeling of edges of P_n.

If P_n is edge even graceful, then the labels of its vertices satisfy

$\pi(2) + [\pi(2)+\pi(4)] + [\pi(4)+\pi(6)] + \ldots + [\pi(2n-4)+\pi(2n-2)] + \pi(2n-2)$

$\equiv 0+2+\ldots+(2n-2) \pmod{2n}$ and hence $2[2+4+\ldots+(2n-2)] \equiv 0+\ldots+(2n-2) \pmod{2n}$, which is satisfied by Lemma 2.1 only when n is odd, but this congruence yields a contradiction, $2n \equiv n(\bmod 2n)$, when n is even. The rest of the proof is to show a labeling which satisfies definition 1.5 when n is odd. Simply, label the edges in order by $2,4,\ldots,2n-2$, which yields the labels $2,6,\ldots,0;\ 4,8,\ldots,2n-2$ on vertices respectively.

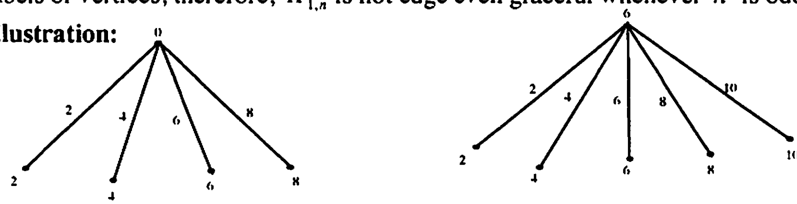
ii. Similar argument works in the case of cycle graph C_n. □

Theorem 2.3 The star graph, $K_{1,n}$ is edge even graceful if and only if n is even.

Proof: If n is even, then labeling the edges by $2,4,\ldots,2n$, yields the same labels on the pendent vertices, and the label on the central vertex will be $2+4+\ldots+2n \ (\bmod 2n+2)$, which by Lemma 2.1, is congruent to $2+4+\ldots+2n+(2n+2) \equiv 0(\bmod 2n+2)$ as $n+1$ is odd, which proves that $K_{1,n}$ is edge even graceful in this case.

If n is odd, the only element in $H=\{2,4,\ldots,2n+2\}$ which is its own inverse is $n+1$ which is even and hence, $\sum_{x\in H} x = 2+4+\ldots+2n \equiv n+1\ (\bmod 2n+2)$ by Lemma 2.1 and hence this sum is $n+1 \in H$ as n is odd. We get some repetition of labels of vertices, therefore, $K_{1,n}$ is not edge even graceful whenever n is odd. □

Illustration:



(a) $K_{1,4}$ is edge even graceful. (b) $K_{1,5}$ is not edge even graceful.

Fig. (2)

3. Some paths and cycles related graphs

The friendship graph Fr_n is a planar undirected graph with $2n+1$ vertices and $3n$ edges constructed by joining n copies of the cycle graph C_3 with a common vertex.

Theorem 3.1 The Friendship graph is an edge even graceful graph.

Proof: There are three cases:

Case (1): $n \equiv 0(\bmod 3)$. Let the friendship graph be as in Fig.(3) and the triangles in it are $T_1, T_2, \ldots, T_n$. Name the center by v_0 and name the other two vertices of T_i by v_{2i-1}, v_{2i}, $i=1,2,\ldots,n$. Label the edges incident to the center (hub) by $4,6,8,\ldots,4n+2$ and then label the outer edges, beginning from the base of triangle T_1 to the triangle T_{n-1} by $4n+4, 4n+6, 4n+8, \ldots, 6n$, then label the base of T_n by 2.

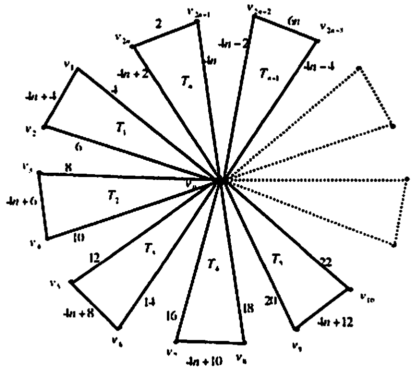


Fig. (3)

Hence we have the labels of v_i, $i = 1,2,\ldots,n$ as follows:

$4n+2, 4n+4;\ 4n+8, 4n+10;\ 4n+14, 4n+16;\ 4n+20, 4n+22;\ldots$

By rearranging this sequences, the pattern is clear. Now the label assigned to v_0 is given by $f^*(v_0) \equiv 4+6+\ldots+4n+(4n+2) \equiv \frac{2n}{2}[4+(4n+2)] \pmod{6n}$

Thus $f^*(v_0) = n(4n+6)$, since $n \equiv 0 \pmod 3$, thus $n = 3m$, we have

$f^*(v_0) = 3m(4n+6) = 12mn + 6n \equiv 0 \pmod{6n}$.

Therefore, the friendship graph admits an edge even graceful labeling for this case.

Case (2): $n \equiv 1 \pmod 3$. Let the friendship graph be as in Fig. (4).

First, label the outer edges of $T_1, T_2, \ldots, T_n$ by $2,4,\ldots,2n$. Second, label the inner edges (incident to v_0) by $2n+2,\ 2n+4;\ 2n+6,\ 2n+8;\ \ldots;\ 6n-2,\ 6n$

The odd rank and the even rank sequence are two arithmetic subsequences each with base increment 4.

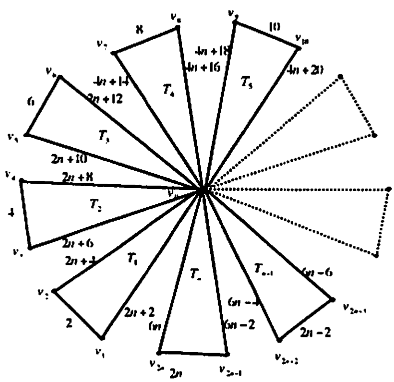


Fig. (4)

The induced labels of vertices are as follows: the center label is
$f^*(v_0) \equiv (2n+2)+(2n+4)+\ldots+(6n-2)+6n \equiv n(8n+2) \equiv 4n \pmod{6n}$.

For the rest of vertices in T_m, $m = \frac{2}{3}(n-1)+1$ its two vertices have labels 0,2 and T_{m+1} has vertices with labels 12,14 and so on. So we only miss 4, 10, 16,... which is arithmetic sequence with base 6 and hence the labels of the two vertices of the last triangle T_{m+n} are $6n-6, 6n-4$.

Case (3): $n \equiv 2 \pmod 3$. Let the friendship graph be as in Fig.(5).

First, label the outer edges of $T_1, T_2, \ldots, T_n$ by $2, 4, \ldots, 2n$.

Second, label the inner edges by $2n+2, 2n+4$; $2n+6, 2n+8$; ...; $6n-2, 6n$.

The odd rank and the even rank sequence are two arithmetic subsequence each with base 4.

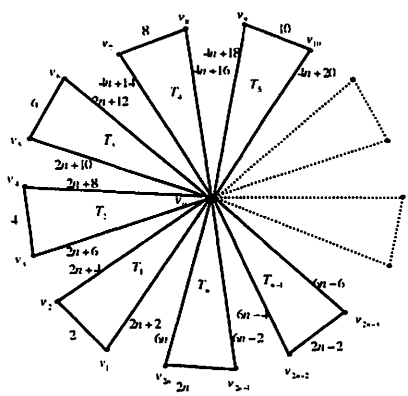
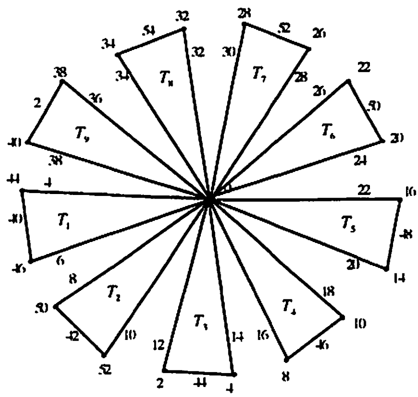


Fig. (5)

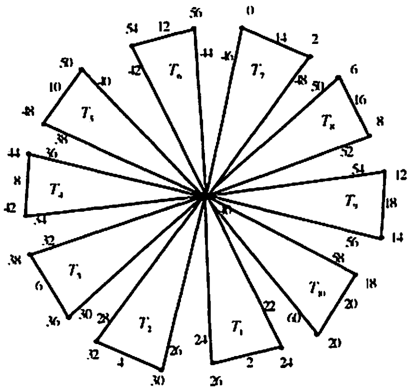
The induced labels of vertices are as follows: the center label is
$f^*(v_0) \equiv (2n+2)+(2n+4)+\ldots+(6n-2)+6n \equiv n(8n+2) \equiv 0 \pmod{6n}$.

For the rest of vertices in T_m, $m = \frac{2}{3}(n-2)+2$ its two vertices have labels 2,4 and T_{m+1} has vertices with labels 8,10 and so on. So we only miss 6,12,18,... which forms an arithmetic sequence with base 6 and hence the labels of the two vertices of the last triangle T_{m+n-1} are $6n-4, 6n-2$. □

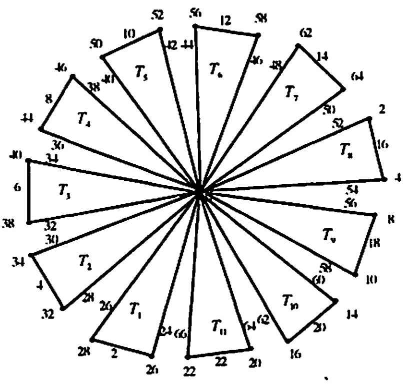
Illustration: The edge even graceful labeling of the friendship graphs Fr_9, Fr_{10} and Fr_{11} are shown in Fig.(6).



(a) Fr_9



(b) Fr_{10}



(c) Fr_{11}

Fig. (6)

Theorem 3.2 The wheel graph $W_n = K_1 + C_n$; $n \geq 3$ is an edge even graceful graph.
Proof: Using standard notation $p = |V(G)| = n+1$, $q = |V(G)| = 2n$, $k = \max(p,q) = 2n$, there are four cases:

Case (1): $n \equiv 0 (\bmod 4)$. Let the wheel graph $G_0 = W_n$, $n \equiv 0 (\bmod 4)$ as in Fig.(7) and the central vertex be v_0.

Label the interior edges: $v_0v_1, v_0v_2, ..., v_0v_n$ by $2, 4, ..., 2n$. Label the cycle edges: $v_1v_n, v_nv_{n-1}, ..., v_2v_1$ by $2n+2, 2n+4, ..., 4n$. These assignments simply means that the edge labeling function $f: E \rightarrow \{2,4,...,4n\}$ is given by $f(v_0v_i) = 2i$.

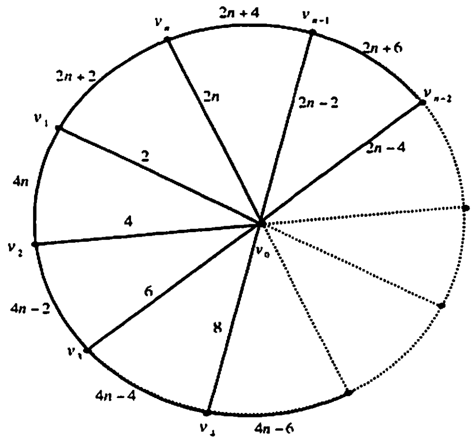


Fig. (7)

The corresponding labels of vertices will be :

$f^*(v_2) \equiv 2 \pmod{4n}$, $f^*(v_1) \equiv 2n+4 \pmod{4n}$, $f^*(v_n) \equiv 2n+6 \pmod{4n}$, $f^*(v_{n-1}) \equiv 2n+8 \pmod{4n}$,......, $f^*(v_3) \equiv 0 \pmod{4n}$, and

$f^*(v_0) \equiv 2+4+...+2n \equiv n(n+1) \equiv n^2+n \pmod{4n}$, which is an even number different from $2n+4, 2n+6,...,4n$ .This proves that G_0 is an edge even graceful graph.

Case (2): $n \equiv 1 \pmod 4$. Let the wheel graph $G_1 = W_n$, $n \equiv 1 \pmod 4$ as in Fig.(8).

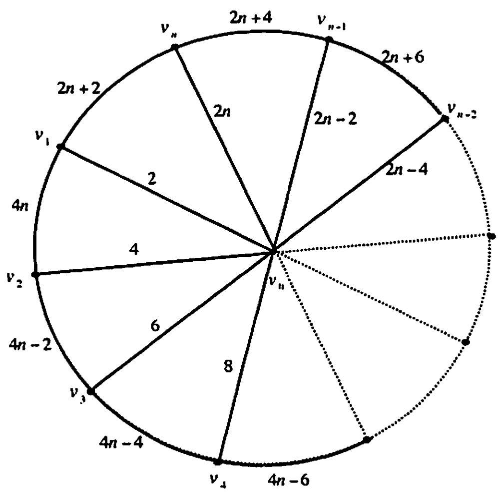


Fig. (8)

Label the edges and vertices follow in the same pattern in case (1) which one exception.

$$f^*(v_0) \equiv [2+4+\ldots+(2n-2)]+2n \equiv \frac{n-1}{2}[2+(2n-2)]+2n \pmod{4n}$$

$$\equiv (n-1)n+2n \equiv 4m\,n+2n \equiv 0+2n \equiv 2n \pmod{4n};\ (n=4m+1)$$

Clearly, $f^*(v_0)$ is different from all the labels of the vertices

$f^*(v_1)=2n+1, f^*(v_n)=2n+6, f^*(v_{n-1})=2n+8, \ldots, f^*(v_3)=4n\equiv 0, f^*(v_2)=2.$

This proves that G_1 is an edge even graceful graph.

Case (3): $n\equiv 2 \pmod 4$. Let the wheel graph $G_2=W_n$, $n\equiv 2 \pmod 4$ as in Fig. (9).

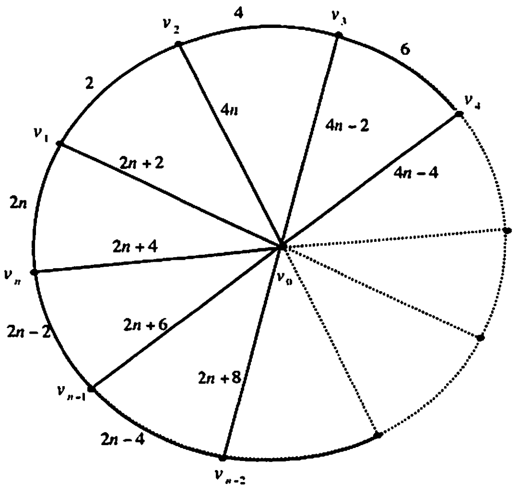


Fig. (9)

Label the outer edges as follows:

$f(v_1v_2)=2, f(v_2v_3)=4, f(v_3v_4)=6, \ldots f(v_nv_1)=2n.$

Then move anticlockwise to label the inner edges by:

$f(v_0v_1)=2n+2, f(v_0v_n)=2n+4, \ldots, f(v_0v_2)=4n$.

The corresponding labels of vertices will be

$f^*(v_1)\equiv 4, f^*(v_2)\equiv 6, f^*(v_3)\equiv 8, \ldots, f^*(v_n)\equiv 2n+2 \pmod{4n}$.

The center vertex v_0 has the label

$$f^*(v_0)\equiv(2n+2)+(2n+4)+\ldots+(4n)\equiv\frac{n-1}{2}[(2n+2)+(4n-2)]\equiv 3n(n-1)\equiv 3n \pmod{4n}.$$

Clearly, $f^*(v_0)$ is different from all the labels of the vertices. This proves that G_2 is an edge even graceful graph.

Case (4): $n\equiv 3 \pmod 4$. Let the wheel graph $G_3=W_n$, $n\equiv 3 \pmod 4$ as in Fig.(10).

Label the edge v_1v_2 by 2 .Then move clockwise to label

$f(v_2v_3)=2n+4, f(v_3v_4)=2n+6, \ldots, f(v_{n-1}v_n)=4n-2, f(v_nv_1)=4n$

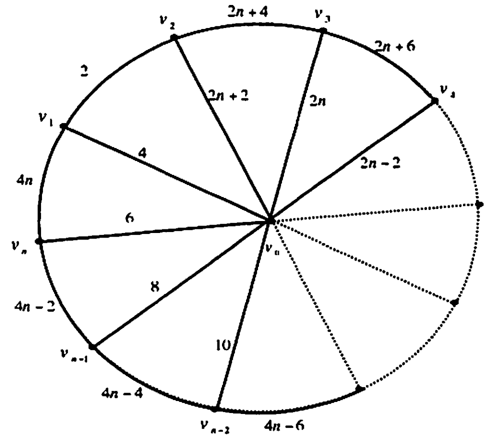


Fig. (10)

Again go outside, anticlockwise to label the inner edges as follows

$f(v_0v_1)=4, f(v_0v_n)=6, f(v_0v_{n-1})=8, \ldots, f(v_0v_3)=2n, f(v_0v_2)=2n+2$.

The corresponding labels of vertices will be

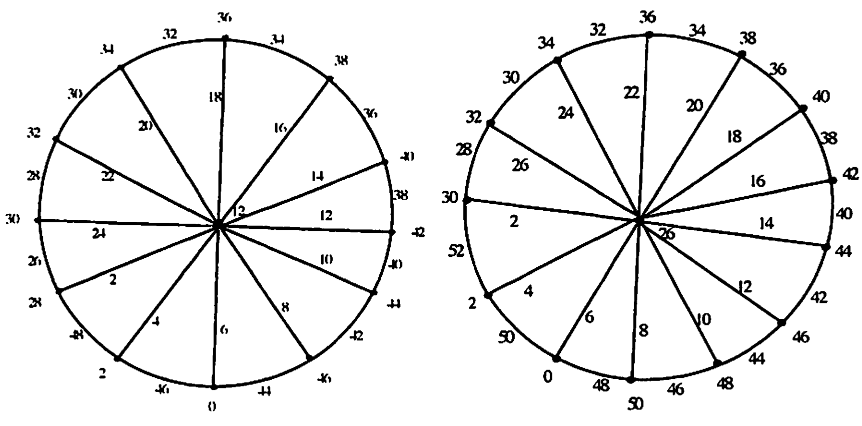
$f^*(v_1)=4n+2+4\equiv 4n+6, f^*(v_n)\equiv 4n-2+4n+6\equiv 4n+4, f^*(v_{n-1})\equiv 4n-4+4n-2+8\equiv 4n+2,$

$f^*(v_{n-2})\equiv(4n-6)+(4n-4)+10\equiv 0\equiv 4n \; ,\ldots, f^*(v_2)\equiv 2+2n+4+2n+2\equiv 4n+8 \pmod{4n}.$

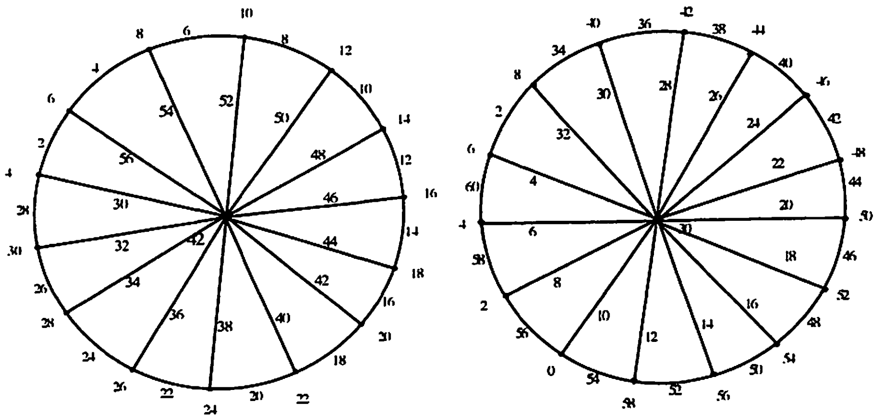
The central vertex v_0 has the label $f^*(v_0)\equiv n(n+3)\equiv 2n \pmod{4n}$.

Clearly, $f^*(v_0)$ is different from all the labels of the vertices. This proves that G_3 is an edge even graceful graph. □

Illustration: The edge even graceful labeling of the wheel graphs W_{12}, W_{13}, W_{14} and W_{15} are shown in Fig.(11).



(a) W_{12} (b) W_{13}



(c) W_{14} (d) W_{15}

Fig. (11)

Theorem 3.3 The Fan graph $F_n = K_1 + P_n$; $n \ge 2$ is an edge even graceful graph.

Proof: Using standard notations $p = |V(G)| = n+1$, $q = |V(G)| = 2n-1$, $k = \max(p,q) = 2n-1$, there are four cases:

Case (1): $n \equiv 0 \pmod 4$, $n \ge 4$.

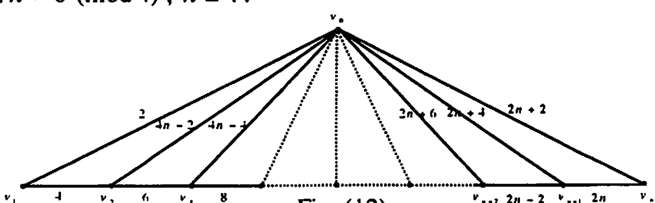


Fig. (12)

When $n > 8$: Let v_0 be the central vertex and $v_1, v_2, \ldots, v_n$ be the vertices of P_n as in Fig.(12). Let $f : E \to \{2, 4, \ldots, 2q = 4n-2\}$ be defined by:

$f(v_{r-1}v_r) = 2r$, $1 \le r \le n$, and

$f(v_0v_1) = 2, f(v_0v_n) = 2n+2, f(v_0v_{n-1}) = 2n+4, \ldots, f(v_0v_2) = 4n-2.$

Therefore, labels of vertices are:

$f^*(v_1) = 6 \bmod (4n-2), f^*(v_2) = 10 \bmod (4n-2), \ldots, f^*(v_{n-1}) = 2n+4 \bmod (4n-2),$

$f^*(v_n) = (2n+(2n+2)) \bmod (4n-2) = ((4n-2)+4) \bmod (4n-2) = 4 \bmod (4n-2),$

and the label of the central vertex is

$f^*(v_0) \equiv 2 + [(2n+2)+(2n+4)+\ldots+(4n-4)+(4n-2)] \equiv 3n(n-1)+2(\bmod(4n-2))$

Since $n \equiv 0 \pmod 4 \Rightarrow n = 4m \Rightarrow p = 4m+1, q = k = 8m-1,$

$f^*(v_0) \equiv 3(4m-1).4m+2 \equiv 3(16m^2-4m)+2 \equiv 3(2m-4m)+2 \equiv -6m+2 \equiv 10m \equiv (\bmod 16m-2)$

$\equiv 10.\frac{n}{4} \equiv \frac{5}{2}n \pmod{4n-2}.$

Notice that F_4 and F_8 are even edge graceful graphs but they don't follow this rule. See Fig. (13).

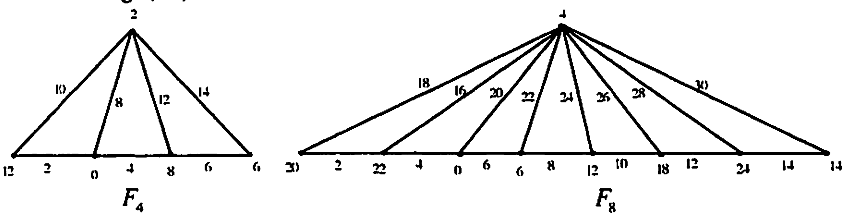


Fig. (13)

Case (2): $n \equiv 1 (\mod 4)$

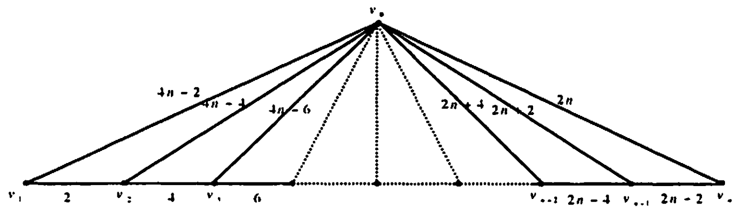


Fig. (14)

Let $f : E \rightarrow \{2,4,...,2q\}$ define edge labels as shown in Fig.(14).

$f(v_{r-1}v_r) = 2(r-1),\ 2 \le r \le n,$

$f(v_0 v_n) = 2n, f(v_0 v_{n-1}) = 2n+2, ..., f(v_0 v_3) = 4n-6, f(v_0 v_2) = 4n-4, f(v_0 v_1) = 4n-2,$

and hence labels of vertices are given by

$f^*(v_1) = 4n-2 \equiv 2(\mod 2k),\ f^*(v_2) = (2+4+4n-4) \equiv 4 \ (\mod 2k),$

$f^*(v_3) = (4+6+4n-6) \equiv 6 \ (\mod 2k), ..., f^*(v_r) \equiv 2r(\mod 2k);\ 1 \le r \le n-1.$

$f^*(v_n) = (2(n-1)+2n) \equiv 4n-2 \equiv 0(\mod 2k),$

$$f^*(v_0) = [2n+(2n+2)+(2n+4)+...+(4n-4)+(4n-2)] \equiv \frac{n}{2}(6n-2) \equiv n(3n-1)(\mod 2k),$$

Since $n \equiv 1(\mod 4) \Rightarrow n = 4m+1 \Rightarrow p = 4m+2, q = k = 8m+1$

so, $f^*(v_0) \equiv (4m+1)[3(4m+1)-1] \equiv \frac{7}{2}(n-1)+2 \equiv \frac{1}{2}(7n-3)(\mod 2k).$

Case (3): $n \equiv 2(\mod 4)$

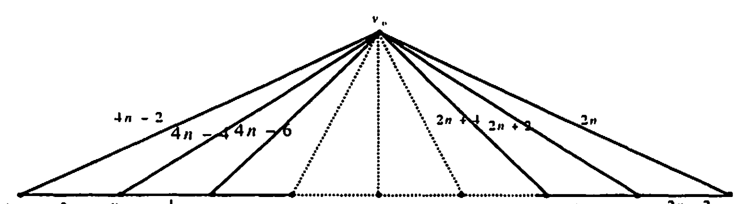


Fig. (15)

Let the labeling function for edges, $f : E \to \{2,4,...,2q\}$ be defined by
$f(v_1v_2) = 2, f(v_2v_3) = 4, ..., f(v_{n-1}v_n) = 2(n-1), f(v_0v_n) = 2n, f(v_0v_{n-1}) = 2n+2, ...$
$f(v_0v_3) = 4n-6, f(v_0v_2) = 4n-4, f(v_0v_1) = 4n-2,$ as shown in Fig.(15), and hence the induced function for vertices will be

$f^*(v_1) = 4n - 2 + 2 \equiv 2 \pmod{2k}, f^*(v_2) = 2 + 4 + 4n - 4 \equiv 4 \pmod{2k}$
$f^*(v_3) = 4 + 6 + 4n - 6 \equiv 6 \pmod{2k}, ..., f^*(v_n) = 4n - 2 \equiv 0 \pmod{2k}.$

and the label of the central vertex v_0 is

$f^*(v_0) = 2n + 2(n+1) + ... + 2(2n-2) + 2(2n-1) \equiv n(3n-1) \pmod{2k}$

Since $n \equiv 2 \pmod 4 \Rightarrow n = 4m+2 \Rightarrow p = 4m+3, q = k = 8m+3$

$$\begin{aligned} f^*(v_0) &= (4m+2)[3(4m+2)-1] = 3(4m+1)^2 - (4m+2) \\ &= 3(16m^2 + 16m + 4) - (4m+2) \equiv 3(-6m - 6 + 4) - (4m+2) \\ &\equiv (-18m - 6) - (4m+2) \equiv -22m - 8 + 32m + 12 \equiv 10m + 4 \pmod{(16m+6)} \\ &\equiv 10\left(\frac{n-2}{4}\right) + 4 \equiv \frac{5}{2}(n-2) + 4 \equiv \frac{1}{2}(5n-2) \bmod (4n-2). \end{aligned}$$

Case (4): $n \equiv 3 \pmod 4$

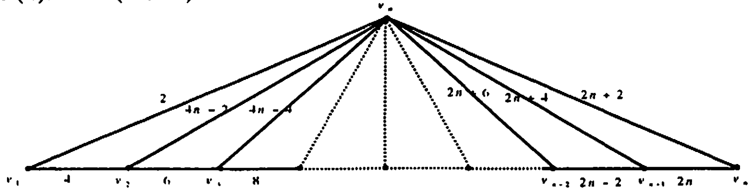


Fig. (16)

When $n > 3$: Let the labeling function $f : E \to \{2,4,...,4n-2\}$ define edge label as in Fig.(16), $f(v_{r-1}v_r) = 2r, \ 1 \le r \le n$,
$f(v_0v_n) = 2n+2, f(v_0v_{n-1}) = 2(n+2), ..., f(v_0v_3) = 4n-4, f(v_0v_2) = 4n-2$.
Hence labels of vertices are given by

$f^*(v_1) \equiv 6 \pmod{2k}, f^*(v_2) \equiv 10 \pmod{2k}, ...,$

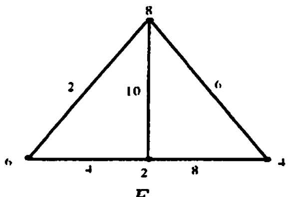
$f^*(v_{n-1}) \equiv 2n + 4 \pmod{2k}, \ f^*(v_n) \equiv 4 \pmod{2k}$

$f^*(v_0) = [(2n+2) + (2n+4) + ... + (4n-2)] + 2 \equiv (n-1)(3n) + 2 (\bmod\ 4n-2)$

Since $n \equiv 3 \pmod 4 \Rightarrow n = 4m+3 \Rightarrow p = 4m+4, q = k = 8m+5$. Therefore,

$$\begin{aligned} f^*(v_0) &\equiv 3[(4m+3-1)(4m+3)] + 2 \equiv 3((4m+3)^2 - (4m+3)) + 2 \\ &\equiv 3[16m^2 + 24m + 9 - 4m - 3] + 2 \equiv 3[-10m + 20m + 6] + 2 \equiv 3[10m + 6] + 2 \\ &\equiv 30m + 20 \equiv 14m + 10 (\bmod\ 16m+10) \equiv 14\left(\frac{n-3}{4}\right) + 10 = \frac{1}{2}(7n-1) \ (\bmod\ 4n-2). \end{aligned}$$

When $n = 3$, F_3 is an edge even graceful graph but it doesn't follow this rule. See Fig.(17).

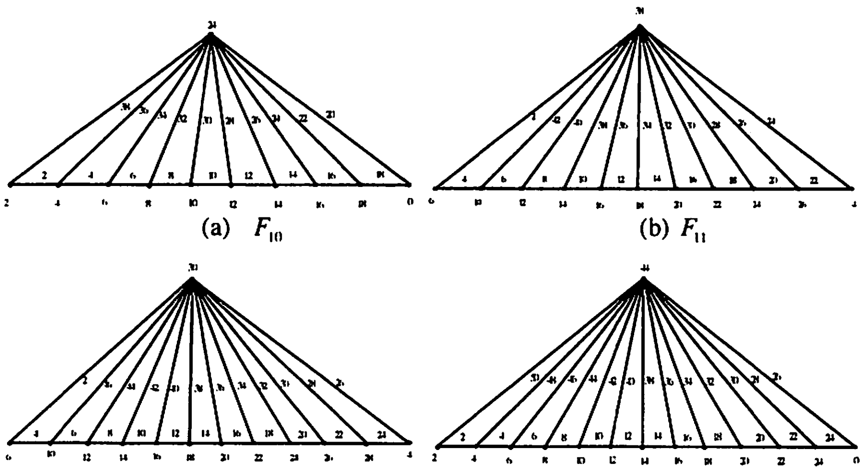


F_3

Fig. (17)

The proof is complete □

Illustration: The edge even graceful labeling of the Fan graphs F_{10}, F_{11}, F_{12} and F_{13} are shown in Fig.(18).



(a) F_{10} (b) F_{11}

(c) F_{12} (d) F_{13}

Fig. (18)

Lemma 3.4

i. If n is an odd integer, then $2n(1+2n) \equiv 6n(\bmod 8n)$.

ii. If n is an even integer, then $2n(1+2n) \equiv 2n(\bmod 8n)$.

Proof:

i. If n is odd $\Rightarrow n = 2m+1, 6n = 12m+6,$ thus

$$2n(1+2n) = 4n^2 + 2n = 4(4m^2+4m+1)+4m+2 = 16m^2+16m+4+4m+2$$
$$= 16m^2+8m+12m+6$$
$$= 8m(2m+1)+12m+6 \equiv 8nm+6n \equiv 6n\ (\bmod 8n).$$

ii. If n is even $\Rightarrow n = 2m$, thus

$$2n(1+2n) = 4m(1+4m) = 16m^2+4m = m(8n)+2n \equiv 0+2n \equiv 2n(\bmod 8n).$$ □

The double wheel graph $W_{n,n}$, consists of two cycles of n vertices connected to a common hub.

Theorem 3.5 The double wheel graph $W_{n,n}$, $n \geq 3$ is an edge even graceful graph.

Proof: Using standard notations,

$p = |V(G)| = 2n+1$, $q = |E(G)| = 4n$, $k = \max(p,q) = 4n$. There are two cases:

Case(1): When n is odd

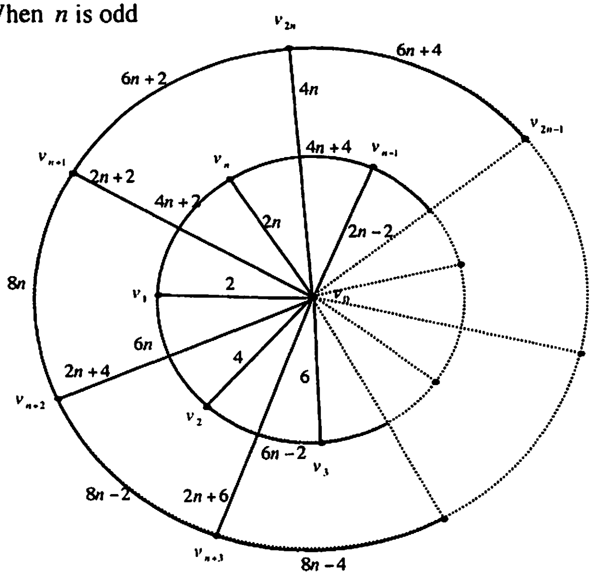


Fig. (19)

We label the $4n$ edges as follows: the inner radial edges by $2,4,\ldots,2n$, the outer radial edges by $2n+2, 2n+4,\ldots,4n$, the inner circle edges by $4n+2, 4n+4,\ldots,6n$ and the outer circle edges by $6n+2, 6n+4,\ldots,8n$, as shown in Fig.(19), and hence the vertices on the outer cycle are labeled as follows:

$f^*(v_1) \equiv 2n+4 \pmod{8n}$, $f^*(v_2) \equiv 4n+2 \pmod{8n}$

$f^*(v_3) \equiv 4n \pmod{8n}, \ldots, f^*(v_{n-1}) \equiv 2n+8 \pmod{8n}, f^*(v_n) \equiv 2n+6 \pmod{8n}$,

Thus, the set of vertices on the inner cycle $v_1, v_2, \ldots, v_n$ are labeled by

$2n+4, 2n+6, 2n+8, \ldots, 4n+2$.

The vertices on the outer cycle are labeled as follows:

$f^*(v_{n+1}) \equiv 4 \pmod{8n}, f^*(v_{n+2}) \equiv 2n+2 \pmod{8n}, f^*(v_{n+3}) \equiv 2n \pmod{8n}, \ldots,$

$f^*(v_{2n-1}) \equiv 8 \pmod{8n}, f^*(v_{2n}) \equiv 6 \pmod{8n}$.

Thus, the set of vertices on the outer cycle $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ are labeled by $4, 6, \ldots, 2n, 2n+2$.

Lastly, the label of the central vertex is

$f^*(v_0) = [2+4+\ldots+2n] + [(2n+2)+(2n+4)+\ldots+4n] \equiv 2n(1+2n)(\mathrm{mod}\,8n)$,

by Lemma 3.4 $f^*(v_0) \equiv 6n\,(\mathrm{mod}\,8n)$.

Case (2): When n is even

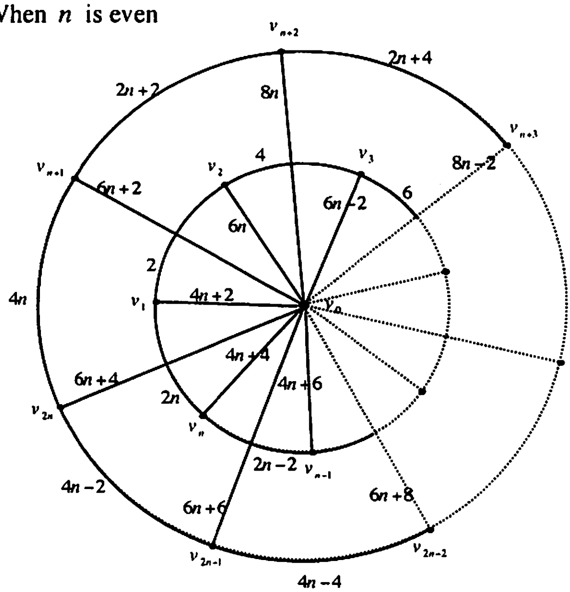


Fig. (20)

We label the $4n$ edges as follows: the inner radial edges by $4n+2, 4n+4, 4n+6, \ldots, 6n$, the outer radial edges by $6n+2, 6n+4, 6n+6, \ldots, 8n$, the inner circle edges by $2, 4, 6, \ldots, 2n$ and the outer circle edges by $2n+2, 2n+4, 2n+6, \ldots, 4n$, as shown in Fig.(20), and hence the vertices on the inner cycle are labeled as follows:

$f^*(v_1) = 6n+4(\mathrm{mod}\,8n), f^*(v_2) = 6n+6(\mathrm{mod}\,8n), f^*(v_3) = 6n+8(\mathrm{mod}\,8n), \ldots,$

$f^*(v_{n-2}) = 8n-2(\mathrm{mod}\,8n), f^*(v_{n-1}) = 0(\mathrm{mod}\,8n), f^*(v_n) = 2(\mathrm{mod}\,8n).$

So the labels of vertices $v_1, v_2, \ldots, v_n$ are as follows

$6n+4, 6n+6, , \ldots, 8n+2, 8n \equiv 0, 8n+2 \equiv 2(\mathrm{mod}\,8n).$

For the vertices on the outer cycle, we have

$f^*(v_{n+1}) \equiv 4n+4(\mathrm{mod}\,8n), f^*(v_{n+2}) \equiv 4n+6(\mathrm{mod}\,8n), f^*(v_{n+3}) \equiv 4n+8(\mathrm{mod}\,8n), \ldots,$

$f^*(v_{2n-1}) \equiv 6n\,(\mathrm{mod}\,8n), f^*(v_{2n}) \equiv 6n+2(\mathrm{mod}\,8n).$

Thus the set of vertices $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ are labeled by

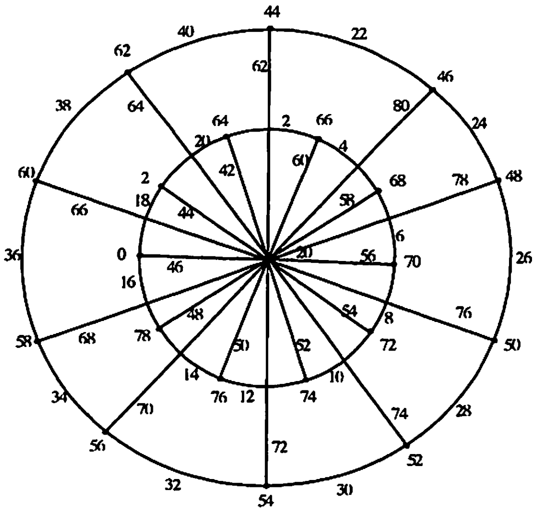
$4n+4, 4n+6, \ldots, 6n, 6n+2.$

Lastly, we calculate $f^*(v_0)$,

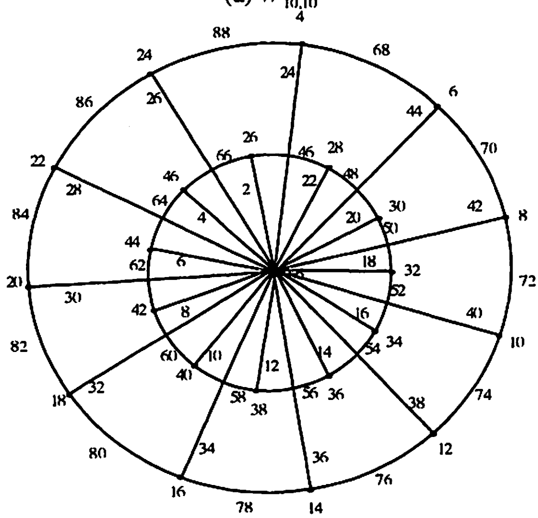
$$f^*(v_0) = (4n+2)+(4n+4)+\ldots+6n+(6n+2)+(6n+4)+\ldots+8n$$
$$= n(4n+2) \equiv 2n \pmod{8n},$$

By Lemma 3.4 the proof is complete. □

Illustration: The edge even graceful labeling of the double wheel graphs $W_{10,10}$ and $W_{11,11}$ are shown in Fig.(21).



(a) $W_{10,10}$



(b): $W_{11,11}$

Fig.(21)

4. Conclusion

Graph labeling is the assignment of some integers for vertices, edges or both of the graphs respectively with certain conditions. Graph labeling is used in many applications like coding theory, radar, astronomy, circuit design, missile guidance and communication network addressing. We introduced a new method for labeling edges and vertices using modular arithmetic. We studied few families of the edge even gracefulness of some families of famous graphs. Edge even graceful labeling is still open for more extensive studies.

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