Some Tree-book Ramsey Numbers

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Abstract: For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. In this paper, we study a large class of tree $\mathbb{T}$ as studied by Cockayne in [3], including paths, trees which have a vertex of degree one adjacent to vertex of degree two, as special cases. We evaluate some $R(T'_m, B_m)$, where $T'_n \in \mathbb{T}$ and B_m is a book of order $m+2$. Besides, some bounds for $R(T'_n, B_m)$ are obtained.

Key words: Ramsey number, Tree, Book

1 Introduction

All graphs considered in this paper are finite simple graphs without loops. Let $G = (V(G), E(G))$ be a graph. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *minimum degree* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $d_G(v)$ be the degree of vertex v in G. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. We use P_n and mK_n to denote a path of order

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n and m vertex disjoint complete graphs K_n, respectively. A *tree* with order n, T_n, is an undirected graph in which any two vertices are connected by exactly one path. A *star* of order m is a complete bipartite $K_{1,m-1}$. A *book* B_m is m triangles sharing one edge, that is, $B_m = K_2 + \overline{K_m} = K_1 + K_{1,m}$. The length of the longest path of G is denoted by $p(G)$. Let $\lceil x \rceil$ denote the greatest integer not larger than x.

In[9], Rousseau and Sheehan studied the Ramsey numbers involving trees versus other graphs. They showed that the Ramsey numbers of tree versus books are given by:

$$\max\{q_1, q_2\} + 1 \leq R(T_n, B_m) \leq m + 2n - 2.$$

where $q_1 = (k+2)(n-1)$, $q_2 = m - 1 + 2\lceil \frac{m-1}{k+1} \rceil$ and $k = \lceil \frac{m-1}{n-1} \rceil$.

Specifically, for some m, n, they derived that

Theorem 1.1. *(1)$R(T_n, B_m) = m + 2n - 2$ when $m \equiv 1(mod(n-1))$; (2)$m + 2n - 3 \leq R(T_n, B_m) \leq m + 2n - 2$ when $m \equiv 2(mod(n-1))$; (3)$m + 2n - 4 \leq R(T_n, B_m) \leq m + 2n - 2$ when $m \equiv 0(mod(n-1))$.*

Other results about Tree-book Ramsey are stated as follows.

Theorem 1.2. *[5] $R(B_m, T_n) = 2n - 1$ for $n \geq 3m - 3$.*

In this paper, we evaluate Ramsey numbers concerning a large class of tree, $\mathbb{T}$, defined in next section, which includes paths as special cases. The main result are stated as follows: let T'_n be a tree in $\mathbb{T}$, then

- $R(T'_n, B_m) = m + 2n - 3$ for $m \equiv 2(mod(n-1))$;
- $R(T'_n, B_m) = m + 2n - 4$ for $m \equiv 0(mod(n-1))$;
- $R(T'_n, B_m) = m + 2n - 4$ for $m \equiv n - 2(mod(n-1))$;
- $R(T'_n, K_{1,m}) = m + n - 2$ for $m \equiv n - 2(mod(n-1))$ and $m > n - 2$.

In addition, some bounds about Tree-book Ramsey numbers are provided in this paper.

For Ramsey numbers concerning trees or books, many results have been obtained[5, 6, 8, 9]. For a survey, we refer readers to see [7].

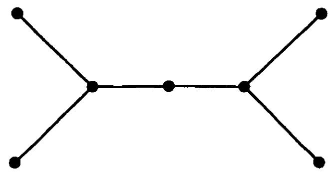

Figure 1: A tree not in $\mathbb{T}$

2 The Class of Tree $\mathbb{T}$

The class of tree studied here is firstly defined by Cockayne[3], where he studied the Tree-star Ramsey numbers. In this section, for the sake of descriptive integrality, we here restate the description as appeared in [3].

Let T be a spanning tree of G which has n vertices, a permutation π of $V(G)$ is *T-preserving* if and only if for each $[u, v] \in E(T), [\pi(u), \pi(v)] \in E(G)$. Let u be a vertex of G and U be the set of all images of u under T-preserving permutation of $V(G)$. If the assumption that each $u \in U$ has degree $n-1$ in G implies that G is the complete graph K_n, (T, u) is defined as a *complete pair*. See an example in [3].

A tree T is in the class $\mathbb{T}$ if there exists some vertex v with $d_T(v) = 1$ and its incident edge $[v, w]$ such that $(T - \{v\}, w)$ is a complete pair. Cockayne has showed that all trees which have a vertex of degree one adjacent to vertex of degree two are in $\mathbb{T}$. Therefore, $\mathbb{T}$ contains all paths P_n. It is worth noting that many trees without this property are also in $\mathbb{T}$. For non-star trees with $n \leq 7$ vertices, the only tree not in $\mathbb{T}$ is that shown in Figure 1. In addition, at most five trees among the 23 8-vertex trees are not in $\mathbb{T}$.

3 Preliminary

In this section, we will give some results obtained in previous research, which will be used in our proof.

Theorem 3.1. *([1]). For* $m \geq n > 1$,
(1)$R(T_n, K_{1,m}) = m + n - 1$ *for* $m \equiv 1(mod(n-1))$;
(2)$R(T_n, K_{1,m}) \leq m + n - 1$;

Theorem 3.2. *([3]). Let T'_n be a tree in $\mathbb{T}$, then $R(T'_n, K_{1,m}) = m+n-2$ if one of the following four conditions holds*
(1)$m \equiv 0, 2(mod(n-1))$;
(2)$m \not\equiv 1(mod(n-1))$ and $m \geq (n-3)^2$;
(3)$m \not\equiv 1(mod(n-1))$ and $m \equiv 1(mod(n-2))$;
(4)$m \equiv n-2(mod(n-1))$ and $m > n-2$.

Lemma 3.1. *Let G be a graph with $|G| \geq R(T_n, K_{1,m}) + 1$. If there is a vertex $v \in V(G)$ such that $|N[v]| \leq |G| - R(T_n, K_{1,m})$ and G contains no T_n, then $\overline{G}$ contains a B_m.*

Proof. Let $G' = G - N[v]$, then $|G'| \geq R(T_n, K_{1,m})$. Since G contains no T_n, then G' contains no T_n. This implies that $\overline{G'}$ contains a $K_{1,m}$. Hence $\overline{G}$ contains a $B_m = v + K_{1,m}$. ▮

4 The Ramsey Numbers

Let T'_n be a tree in $\mathbb{T}$. In this section, we will evaluate the Ramsey numbers of $R(T'_n, B_m)$ by separately studying those values according to the relationship between m and n. We will first give the values of Ramsey numbers when $m \leq n$.

Lemma 4.1. *$R(T'_n, B_m) \leq m + 2n - 3$ for $m \not\equiv 1(mod(n-1))$.*

Proof Let G be a graph with $|G| = m + 2n - 3$. If G contains no T'_n and there is a vertex $v \in V(G)$ such that $|N[v]| \leq n - 1$, we have $|G - N[v]| \geq m + n - 2 \geq R(T'_n, K_{1,m})$. Hence $\overline{G - N[v]}$ contains a $K_{1,m}$. Therefore, $\overline{G}$ contains a $B_m = v + K_{1,m}$. Thus, for any vertex v, $|N[v]| \geq n$, which shows that $\delta(G) \geq n - 1$. So G contains every tree with m vertices. This completes the lemma. ▮

Theorem 4.1. *If $m \geq n$, then*
(1) $R(T'_n, B_m) = m + 2n - 3$ for $m \equiv 2(mod(n-1))$;
(2) $R(T'_n, B_m) = m + 2n - 4$ for $m \equiv 0(mod(n-1))$;
(3) $R(T'_n, B_m) = m + 2n - 4$ for $m \equiv n - 2(mod(n-1))$.

Proof (1) For each m with $m \equiv 2(mod(n-1))$, let $m = k(n-1) + 2$ and $G = (k+2)K_{n-1}$, so G with $m + 2n - 4$ vertices has no T'_n and $\overline{G}$ has no

B_m, which implies $R(T'_n, B_m) > m + 2n - 4$. According to Lemma 4.1, we have $R(T'_n, B_m) = m + 2n - 3$ for $m \equiv 2(mod(n-1))$.

(2) We now prove the upper bound by induction on m where $m \equiv 0(mod(n-1))$. Suppose the result holds for $m-(n-1)$, i.e. $R(T'_n, B_{m-n+1}) \leq m+n-3$. For m, let G be a graph with order $m+2n-4$. Identical to that of Lemma 4.1, we have $\delta(G) \geq n-2$. By the definition of $\mathbb{T}$, G contains a tree T_{n-1} formed from T'_n by removal of a vertex v of degree one in T'_n and its incident edge vw, and such that (T_{n-1}, w) is a complete pair. We let the vertex set of T_{n-1} be $W = \{v_1, v_2, \cdots, v_{n-1}\}$ and the remaining vertices be $H = \{v_n, \cdots, v_{m+2n-4}\}$. If the image of w under any T_{n-1}-preserving permutation of W is adjacent to any vertex of H, $T'_n \subset G$. Hence there is no such adjacency, and each such image, having degree not smaller than $n-2$ in G, has degree $n-2$ in $G[W]$. Since (T_{n-1}, w) is a complete pair, $G = K_{n-1} \cup G[H]$. In addition, $|H| = m+n-3 \geq R(T'_n, B_{m-n+1})$ implies that $\overline{H}$ contains a book B_{m-n+1}. Hence $\overline{G} = \overline{K_{n-1} \cup G[H]}$ has a book B_m.

Noting that $G = (k-1)K_{n-1} \cup 3K_{n-2}$ with $m+2n-5$ vertices contains no T'_n and $\overline{G}$ contains no B_m, and hence $R(T'_n, B_m) = m + 2n - 4$.

(3) Let $m = k(n-1)+n-2$. The proof of upper bound is identical to the proof in (2). Noting that $G = (k-1)K_{n-1} \cup 4K_{n-2}$ with $m+2n-5$ vertices contains no T'_n for $m \geq n$ and $\overline{G}$ contains no B_m, we have $R(T'_n, B_m) \geq 3m-2$ and hence $R(T'_n, B_m) = m + 2n - 4$. ∎

Theorem 4.2. *If $m \geq n^2 - 7n + 11$ and $m \not\equiv 1(mod(n-1))$, then $m + 2n - 4 \leq R(T'_n, B_m) \leq m + 2n - 3$.*

Proof. If $m \geq n^2 - 7n + 11$, then $m + 2n - 5 \geq (n-2)^2 - (n-2)$. If $m + 2n - 5 = (n-2)^2 - t$ where $0 < t \leq n-2$, then we have $m + 2n - 5 = (n-2)^2 - t = (t-1)(n-2) + (n-2-t)(n-1)$. If $m+2n-5 = (n-2)^2 + t$ where $t = ra + s$ and $0 \leq s < a$, then we have $m + 2n - 5 = (n-2)^2 + t = (r+a-s)(n-2) + s(n-1)$. In both cases, we can find some integers $p, q \geq 0$ such that $m+2n-5 = p(n-2)+q(n-1)$. We can let $G = pK_{n-2} \cup qK_{n-1}$. So G with $m + 2n - 5$ vertices has no T'_n, and $\overline{G}$ contains no B_m since $\forall uv \in E(\overline{G}), |N_{\overline{G}}(u) \cap N_{\overline{G}}(v)| < m$. The proof is completed. ∎

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