Spectral determination of a class of tricyclic graphs

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Abstract

A T-shape tree is a tree with exactly one of its vertices having maximal degree 3. In this paper, we consider a class of tricyclic graphs which is obtained from a T-shape tree by attaching three identical odd cycles C_ks to three vertices of degree 1 of the T-shape tree, respectively, where $k \geq 3$ is odd. It is shown that such graphs are determined by their adjacency spectrum.

Key words: Spectrum; Cospectral graphs; DS-graphs; Tricyclic graphs.

1 Introduction

All graphs considered here are finite and simple. For all notation not given here, we refer the reader to [9]. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. The adjacency matrix A of G is the $n \times n$ matrix with (i, j)-th entry equal to 1 if two vertices v_i and v_j are adjacent and equal to 0 otherwise. Since A is a symmetric matrix, all of its eigenvalues are

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real. The multiset of eigenvalues of A is called the adjacency spectrum of G, denoted by $Spec(G)$, and the largest eigenvalue of G is called the spectral radius of G, denoted by $\rho(G)$. The characteristic polynomial $det(\lambda I - A)$ of G is denoted by $\phi(G;\lambda)$. Two nonisomorphic graphs with the same spectrum are called cospectral. A graph G is said to be determined by its spectrum (or DS, for short) if it does not have a cospectral mate.

E.R. van Dam and W.H. Haemers [2] conjectured that almost all graphs are DS. In the last two decades, many DS graphs have been found including path and its complement [8], the complete graph and the regular complete bipartite graph, cycle and and its complement [7], T-shape tree [14], lollipop graphs and sandglass graphs [16, 20], θ-graphs, dumbbell graphs and ∞-graphs [3–6], the union of complete multipartite graph and some isolated vertices [17], the centipede graph and the graphs with index at most $\sqrt{2+\sqrt{5}}$ [20,21]. One can find that, however, all well-known DS graphs up to now either are small so that they can be proved to be DS by enumeration or have very special properties, and the techniques(e.g., the eigenvalue interlacing technique) involved to prove them to be DS depend heavily on some special properties of the spectrum of these graphs, and can not be applied to general graphs. Moreover, numerous examples of cospectral but non-isomorphic graphs are reported. For example, C.G. Godsil and B.D. Mckay [12, 13] gave some constructions for pairs of cospectral graphs and Schwenk showed that the proportion of trees on n vertices which are characterized by their spectra converges to zero as n increase in [10]. To find more DS graphs is still an interesting but difficult problem.

The degree of a vertex v is denoted by $d(v)$. A graph G is said to be $(r, r+1)$-almost regular if $V(G)$ can be partitioned into two subsets V_1 and V_2 such that $d(v_i) = r$ for $v_i \in V_1$ and $d(v_i) = r+1$ for $v_i \in V_2$. For two graphs G_1 and G_2, $G_1 \cup G_2$ denotes the disjoint union of G_1 and G_2. $G - v$ denotes the graph obtained by removing the vertex v of G and the edges connected with v in G. Let C_n and P_n be respectively the cycle and the path on n vertices. A T-shape tree is a tree, denoted by $T(l_1, l_2, l_3)$ (where $l_1 \geq l_2 \geq l_3 \geq 1$), which exactly has one vertex v of degree 3 such that $T(l_1, l_2, l_3) - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$.

Many trees [2, 7, 11, 14, 19] have been found they are determined by their spectrum. However, few graphs with cycles are been investigated. Including unicyclic graphs (e.g., cycles and lollipop graphs) and bicyclic graphs (e.g., sandglass graphs, θ-graphs and dumbbell graphs), if a graph has more cycles as its subgraphs, then it is more difficult to be determined by its spectrum. Especially, it is more difficult to calculate and compare the characteristic polynomials of these graphs. The technique in [14] used in calculating and comparing of the characteristic polynomials of graphs is suitable in trees. But not necessarily fits for the graphs with cycles, especially, for tricyclic graphs. More concisely, we use another method to calculate and compare the characteristic polynomials of a class of tricyclic graphs shown as follow, which not only have the similar properties of the cospectral graphs of T-shape tree and dumbbell graphs, but also have those of almost regular graphs.

The class of tricyclic graphs is called Γ-graphs, which obtained from a T-shape tree by attaching three identical odd cycles C_ks to three vertices of degree 1 of the T-shape tree, respectively, where $k \geq 3$ is odd. A Γ-graph is denoted by $\Gamma_k(a_1, a_2, a_3)$, where a_i is the vertex number of path P_{a_i} $(i = 1, 2, 3)$ appeared in $T(a_1, a_2, a_3) - v$, C^i $(i = 1, 2, 3)$ is equal to the cycle C_k and v is the vertex of degree 3 of the subgraph $T(a_1, a_2, a_3)$ of Γ-graph. The Γ-graph is shown in Fig 1.

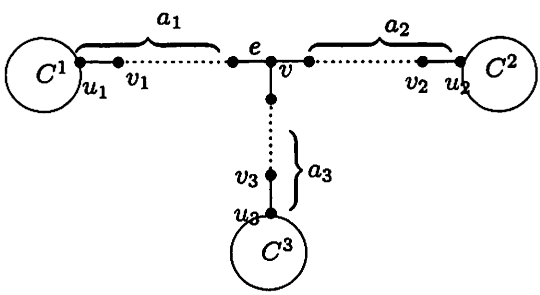

Fig 1. Graph $\Gamma_k(a_1, a_2, a_3)$, where $C^1 = C^2 = C^3 = C_k$

In this paper, we mainly show that Γ-graphs are determined by their adjacency spectrum.

2 Preliminaries

Several useful results on the spectrum are shown as follows, which will play an important role in the proof of main result.

Lemma 1 [2]. Let G and H be two graphs such that $\phi(G; \lambda) = \phi(H; \lambda)$. Then

(i) $n(G) = n(H)$ and $m(G) = m(H)$;

(ii) G is bipartite if and only if H is bipartite;

(iii) G is k-regular if and only if H is k-regular;

(iv) G is k-regular with girth g if and only if H is k-regular with girth

g;

(v) G and H have the same number of closed walks of any fixed length.

Lemma 2 [3]. Let two graphs G and H be cospectral. Then both the length and the number of shortest odd cycles in G and H are the same.

Lemma 3 [3]. Let G be a $(r, r+1)$-almost regular graph without cycle C_4 as its subgraph. If H is a graph such that $Spec(H) = Spec(G)$, then

(i) H contains no cycle C_4 as its subgraph;

(ii) H is a $(r, r+1)$-almost regular graph with the same degree sequence as G.

Lemma 4 [9]. Let uv be an edge of a graph G and let $C(uv)$ denote the collection of cycles containing uv. Then the characteristic polynomial of G satisfies

$$\phi(G; \lambda) = \phi(G - uv; \lambda) - \phi(G - u - v; \lambda) - 2 \sum_{C \in C(uv)} \phi(G - V(C); \lambda)$$

An internal path of G is a walk $v_0 v_1 \cdots v_k$ $(k \geq 1)$ in G such that the vertices v_1, v_2, $\cdots$, v_k are distinct (v_0, v_k do not need to distinct), $d(v_0) \geq 2$, $d(v_k) \geq 2$ and $d(v_i) = 2$ for $0 < i < k$.

Lemma 5 [9]. Let G be a connected graph that is not isomorphic to one of $\{C_n, W_n\}$, and G_{uv} be the graph obtained from G by subdividing the edge uv of G, where W_n can be obtained by attaching two pendent vertices to each of endpoints of path P_{n-4}, respectively. If uv lies on an internal path of G, then $\rho(G_{uv}) < \rho(G)$.

Lemma 6 [9]. Let H be a proper subgraph of a connected graph G, then

$$\lambda_1(H) < \lambda_1(G).$$

Lemma 7 [9]. For a graph G of n vertices with $v \in V(G)$, let $H = G - v$, then

$$\lambda_1(G) \geq \lambda_1(H) \geq \lambda_2(G) \geq \lambda_2(H) \geq \cdots \geq \lambda_{n-1}(H) \geq \lambda_n(G).$$

The following five kinds of graphs as shown in Fig 2 will be used in the proof of the main result in section 3.

The lollipop graph, denoted by $L_{p,q}$, is obtained by appending a cycle C_p to a pendent vertex of a path P_{q+1}, where $p \geq 3, q \geq 1$.

The dumbbell graph, denoted by $D_{a,b,c}$, is obtained by appending a cycle C_c to the pendent vertex of a lollipop graph $L_{a,b}$, where $a \geq 3, b \geq 1, c \geq 3$.

The θ-graph is a graph consisting of three paths, with their end vertices in common. Obviously, a θ-graph is a bicyclic graph which can be also regarded as a graph consisting of two cycles, say C_x and C_y, sharing a common path. For convenience, we denote a θ-graph by $\theta_{x,y}$, where $x \geq 3, y \geq 3$.

The so-called U-graph is denoted by $U_k(x, y)$, which is obtained by appending an odd cycle C_k to the pendent vertices of the lollipop graphs $L_{k,x}$ and $L_{k,y}$, respectively, where $x \geq 1, y \geq 1$.

The graph $D_{b_1,b_2,k} \cup D_{k,b_3,k}$ is the disjoint union of two dumbbells $D_{b_1,b_2,k}$ and $D_{k,b_3,k}$, where $b_1 \geq 3, b_2 \geq 1$ and $b_3 \geq 1$.

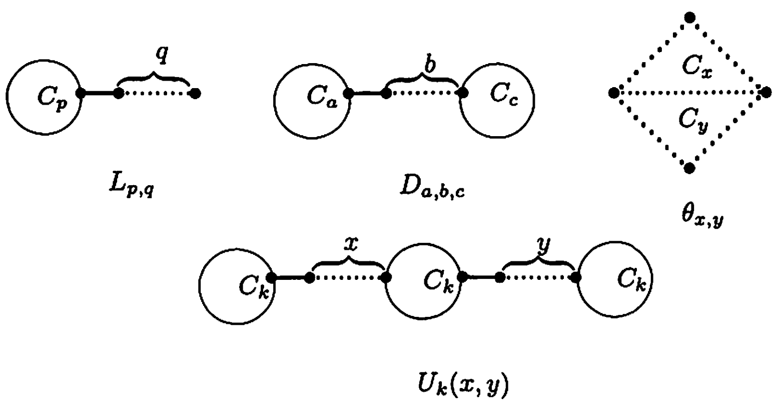

Fig 2. Graphs: $L_{p,q}, D_{a,b,c}, \theta_{x,y}$ and $U_k(x, y)$

3 Main results

First, we characterize the structures of cospectral graphs of Γ-graphs.

Lemma 8 [18]. If $G = T(l_1, l_2, l_3)$, then $\rho(G) = 2$ if and only if $(l_1, l_2, l_3) \in \{(5, 2, 1), (3, 3, 1), (2, 2, 2)\}$.

For convenience, denote $\phi(P_r; \lambda)$ by $p_r = p_r(\lambda)$, by convection, let $p_0 = 1, p_{-1} = 0$ and $p_{-2} = -1$.

Lemma 9. For a Γ-graph $\Gamma_k(a_1, a_2, a_3)$, $2 \in Spec(\Gamma_k(a_1, a_2, a_3))$ if and only if $(a_1, a_2, a_3) \in \{(7, 4, 3), (5, 5, 3), (4, 4, 4)\}$, where $a_1 \geq a_2 \geq a_3 \geq 1$. Moreover, for these graphs, 2 is the fourth largest eigenvalue and has multiplicity 1.

Proof. For simplicity, denote $\Gamma_k(a_1,a_2,a_3)$ by G. Let u_iv_i(i=1,2,3) (see Fig 1.) be the edge of G satisfying one end vertex u_i of u_iv_i of degree 3 on cycle C^i and the other not on C^i. Then, using Lemma 4 for each edge u_iv_i(i=1,2,3), we can get

$$\begin{aligned}\phi(G;\lambda) &= \phi(C^1;\lambda)\phi(G-V(C^1);\lambda)-\phi(C^2;\lambda)p_{k-1}\phi(G-V(C^1)\cup\\ &\quad V(C^2)-v_1;\lambda)+\phi(C^3;\lambda)p_{k-1}^2\phi(G-V(C^1)\cup V(C^2)\cup\\ &\quad V(C^3)-v_1-v_2;\lambda)-p_{k-1}^3\phi(G-V(C^1)\cup V(C^2)\cup V(C^3)\\ &\quad -v_1-v_2-v_3;\lambda)\end{aligned}$$

Obviously, $G-V(C^1)\cup V(C^2)\cup V(C^3)-v_1-v_2-v_3$ is a T-shape tree. Since $\rho(C_k)=2$ and $\rho(P_{k-1})<2$, then, by Lemma 8 and the characteristic polynomial of G, it is easy to see that $2\in Spec(G)$ if and only if $(a_1-2,a_2-2,a_3-2)\in\{(5,2,1),(3,3,1),(2,2,2)\}$. That is, $2\in Spec(G)$ if and only if $(a_1,a_2,a_3)\in\{(7,4,3),(5,5,3),(4,4,4)\}$.

Now we prove that 2 is the fourth largest eigenvalue of G and has multiplicity 1. From Fig 1, we see that v is the vertex of G of degree 3 not on cycles. By Lemma 7, we have

$$\begin{aligned}&\lambda_1(G)\geq\lambda_1(G-v)\geq\lambda_2(G)\geq\lambda_2(G-v)\geq\lambda_3(G)\\ &\geq\lambda_3(G-v)\geq\lambda_4(G)\geq\lambda_4(G-v)\geq\lambda_5(G)\end{aligned}\qquad(1)$$

and

$$\begin{aligned}&\lambda_5(G)\leq\lambda_4(G-v)\leq\lambda_3(G-v-u_1)\leq\lambda_2(G-v-u_1-u_2)\leq\\ &\lambda_1(G-v-u_1-u_2-u_3)=\lambda_1(3P_{k-1}\cup P_{a_1-1}\cup P_{a_2-1}\cup P_{a_3-1})\end{aligned}\qquad(2)$$

Since $\rho(P_n)<2$, then by (2), $\lambda_5(G)<2$. Since $G-v$ is composed of three lollipop graphs and the lollipop graph has a cycle as its subgraph, by Lemma 6, we have $\lambda_3(G-v)>2$. Thus, by (1), $\lambda_4(G)=2$. □

Lemma 10.

(i) If a connected graph G is cospectral with a Γ-graph, then G may be a Γ-graph or a U-graph;

(ii) If a disconnected graph G is cospectral with a Γ-graph, then G may be one of the following graphs: $D_{b_1,b_2,k}\cup D_{k,b_3,k}$, $\theta_{k,l}\cup D_{k,b,k}$, $\theta_{k,l}\cup\theta_{k,k}$, $D_{k,b,l}\cup\theta_{k,k}$, $\Gamma_k(a_1,a_2,a_3)\cup C_z$, $D_{b_1,b_2,k}\cup D_{k,b_3,k}\cup C_z$, $U_k(x,y)\cup C_z$, $\theta_{k,l}\cup D_{k,b,k}\cup C_z$, $\theta_{k,l}\cup\theta_{k,k}\cup C_z$, $D_{k,b,l}\cup\theta_{k,k}\cup C_z$, where $z>k$, $b_1>k$ and $l>k$ otherwise the length of the shortest cycle is z, b_1 or l instead of k.

Proof. If graph G is cospectral with a Γ-graph, then by Lemma 1, they have the same vertex number and edge number. And by Lemma 3, the cospectral graph without C_4 as its subgraph has four vertices of degree 3 and other vertices of degree 2. From Lemma 2, it has three identical odd cycles C_ks as its subgraph. So the possible cospectral graph is a Γ-graph or a U-graph when G is connected. Thus, (i) holds.

Now, we prove (ii). If a disconnected graph G is cospectral with a Γ-graph and G has cycles as its components, then, by Lemma 2, Lemma 3 and Lemma 9, the possible cospectral graphs are $\Gamma_k(a_1, a_2, a_3) \cup C_z$, $D_{b_1,b_2,k} \cup D_{k,b_3,k} \cup C_z$, $U_k(x, y) \cup C_z$, $\theta_{k,l} \cup D_{k,b,k} \cup C_z$, $\theta_{k,l} \cup \theta_{k,k} \cup C_z$ and $D_{k,b,l} \cup \theta_{k,k} \cup C_z$. Moreover, $z > k$ and $b_1 > k$ by Lemma 2 and Lemma 3.

If a disconnected graph G is cospectral with a Γ-graph without cycles as its components, then, by Lemma 2 and Lemma 3, the possible cospectral graphs are $D_{b_1,b_2,k} \cup D_{k,b_3,k}$, $\theta_{k,l} \cup D_{k,b,k}$, $\theta_{k,l} \cup \theta_{k,k}$ and $D_{k,b,l} \cup \theta_{k,k}$, where $l > k$ and $b_1 > k$. □

Next, we show that no two Γ-graphs are cospectral. We compute the characteristic polynomial of the Γ-graph as follow. First, use Lemma 4 with edge e , where e is denoted in Fig 1. Then use Lemma 4 with edge u_iv_i $(i = 1, 2, 3)$, repeatedly. So we get the characteristic polynomial of $\Gamma_k(a_1, a_2, a_3)$ in terms of the characteristic polynomial of of paths.

$$\begin{aligned}
&\phi(\Gamma_k(a_1, a_2, a_3); \lambda) = \\
&(p_{a_1+k-1} - p_{a_1-1}p_{k-2} - 2p_{a_1-1})[(p_{a_3+k} - p_{a_3}p_{k-2} - 2p_{a_3}) \\
&(p_{a_2+k-1} - p_{a_2-1}p_{k-2} - 2p_{a_2-1}) - (p_{a_3+k-1} - p_{a_3-1}p_{k-2} - 2p_{a_3-1}) \\
&(p_{a_2+k-2} - p_{a_2-2}p_{k-2} - 2p_{a_2-2})] - (p_{a_1+k-2} - p_{a_1-2}p_{k-2} - 2p_{a_1-2}) \\
&(p_{a_3+k-1} - p_{a_3-1}p_{k-2} - 2p_{a_3-1})(p_{a_2+k-1} - p_{a_2-1}p_{k-2} - 2p_{a_2-1})
\end{aligned} \tag{3}$$

By Lemma 4, we have $p_r = \lambda p_{r-1} - p_{r-2}$. Solving this recurrence equation, we find that for $r \geq -2$,

$$p_r = \frac{x^{2r+2} - 1}{x^{r+2} - x^r}, \tag{4}$$

where x satisfies $x^2 - \lambda x + 1 = 0$. If we substitute (4) in (3), then we obtain

$$\frac{(x^2 - 1)^6}{x^{2-3k-s}(x^k - 1)^3}\phi(\Gamma_k(a_1, a_2, a_3); \lambda) - f(x) = P(a_1, a_2, a_3; x) \tag{5}$$

where $s = a_1 + a_2 + a_3$,

$$\begin{aligned} f(x) = \quad & -1 + 8x^2 - 24x^4 + 32x^6 - 16x^8 + 3x^k - 3x^{2k} + x^{3k} - 18x^{2+k} \\ & +36x^{4+k} - 24x^{6+k} + 12x^{2+2k} - 12x^{4+2k} - 2x^{2+3k} + 2x^{6+2s} \\ & -x^{8+2s} + 12x^{4+k+2s} - 12x^{6+k+2s} + 3x^{8+k+2s} + 24x^{2+2k+2s} \\ & -36x^{4+2k+2s} + 18x^{6+2k+2s} - 3x^{8+2k+2s} + 16x^{3k+2s} \\ & -32x^{2+3k+2s} + 24x^{4+3k+2s} - 8x^{6+3k+2s} + x^{8+3k+2s} \end{aligned} \tag{6}$$

and $P(a_1, a_2, a_3; x)$ is shown in the appendix in section 4. For each part of it, the terms are sorted in not decreasing order on their powers.

Lemma 11. No two nonisomorphic Γ-graphs are cospectral.

Proof. We assume that $\Gamma_k(a_1, a_2, a_3)$ and $\Gamma_{k'}(a'_1, a'_2, a'_3)$ are cospectral, where $a_1 \geq a_2 \geq a_3$ and $a'_1 \geq a'_2 \geq a'_3$. Then they have the same vertex number and edge number by Lemma 1. By Lemma 2, $k = k'$. Since $\phi(\Gamma_k(a_1, a_2, a_3); \lambda) = \phi(\Gamma_{k'}(a'_1, a'_2, a'_3); \lambda)$, then by (5), we have

$$P(a_1, a_2, a_3; x) = P(a'_1, a'_2, a'_3; x) \tag{7}$$

From $P(a_1, a_2, a_3; x)$, the smallest powers of x in $P(a_1, a_2, a_3; x)$ and $P(a'_1, a'_2, a'_3; x)$ are $4 + 2a_3$ and $4 + 2a'_3$, respectively. By (7), $a_3 = a'_3$. By (17) from $P(a_1, a_2, a_3; x)$ shown in the appendix in section 4, we define function

$$\begin{aligned} \psi(y) \quad = \quad & x^{4+2y} - 4x^{6+2y} + 4x^{8+2y} + 2x^{2+k+2y} - 11x^{4+k+2y} \\ & +16x^{6+k+2y} - 4x^{8+k+2y} - 4x^{2+2k+2y} + 11x^{4+2k+2y} \\ & -4x^{6+2k+2y} + 2x^{2+3k+2y} - x^{4+3k+2y} \end{aligned}$$

Since $a_3 = a'_3$, $\psi(a_3) = \psi(a'_3)$. The smallest powers of x in $P(a_1, a_2, a_3; x) - \psi(a_3)$ and $P(a'_1, a'_2, a'_3; x) - \psi(a'_3)$ are $4+2a_2$ and $4+2a'_2$, respectively. Thus, $a_2 = a'_2$. Since $a_1 + a_2 + a_3 = a'_1 + a'_2 + a'_3$, then $a_1 = a'_1$. □

Lemma 12. Let $n_G(L_{k,1})$ be the number of subgraph $L_{k,1}$ in graph G. If G is cospectral with $\Gamma_k(a_1, a_2, a_3)$, then $n_G(L_{k,1}) = n_{\Gamma_k(a_1,a_2,a_3)}(L_{k,1})$.

Proof. Obviously, $n_{\Gamma_k(a_1,a_2,a_3)}(L_{k,1}) = 3$. There are two types of $(k+2)$-closed walks in $\Gamma_k(a_1, a_2, a_3)$. The first type is the closed walks around the cycle C_k where one edge is used three times, whose number precisely is $6k(k+2)$. The second type, related with $L_{k,1}$, is the closed walks around C_k that go one step up and down the edge uv, where uv is the edge not contained in C_k such that $d(u) = 3$, $u \in V(C_k)$ and $v \notin V(C_k)$, whose number precisely is $6(k+2)$. Therefore, the total number n_1 of $(k+2)$-closed walks in $\Gamma_k(a_1, a_2, a_3)$ is

$$n_1 = 6k(k+2) + 6(k+2).$$

By Lemma 2, G and $\Gamma_k(a_1, a_2, a_3)$ have the same number of C_k. So G has the same number of the first type of closed walks with $\Gamma_k(a_1, a_2, a_3)$, which is $6k(k+2)$. From Lemma 10-(ii), C_k is not the cycle component of G if G is disconnected and G has cycles as its components. Therefore, the number of the subgraph $L_{k,1}$ in G is at least 3. From Lemma 1, G and $\Gamma_k(a_1, a_2, a_3)$ have the same number of closed walks of given length $k+2$. Thus, the number of the second type of closed walks of G is $6(k+2)$ and the number of subgraph $L_{k,1}$ in graph G is 3. So, $n_G(L_{k,1}) = n_{\Gamma_k(a_1,a_2,a_3)}(L_{k,1})$. □

We easily find that the number of subgraph $L_{k,1}$ in each graph shown in Lemma 13 is more than 3. Thus, by Lemma 12, we have

Lemma 13. Let G be a cospectral graph to $\Gamma_k(a_1, a_2, a_3)$.

(i)If G is disconnected, then $G \not\cong U_k(x, y)$.

(ii)If G is connected, then G could not be any graph from $\{\theta_{k,l} \cup D_{k,b,k},\ \theta_{k,l} \cup \theta_{k,k},\ D_{k,b,l} \cup \theta_{k,k},\ U_k(x,y) \cup C_z,\ \theta_{k,l} \cup D_{k,b,k} \cup C_z,\ \theta_{k,l} \cup \theta_{k,k} \cup C_z,\ D_{k,b,l} \cup \theta_{k,k} \cup C_z\}$, where $z > k$ and $l > k$ otherwise the length of the shortest cycle is z or l instead of k.

From Lemma 9, if a disconnected graph G cospectral with a Γ-graph and G has only one cycle as its component, then the Γ-graph must be chosen from $\{\Gamma_k(4,4,4), \Gamma_k(5,5,3), \Gamma_k(7,4,3)\}$. Therefore, we need to prove the following two lemmas.

Lemma 14. There does not exist a graph $\Gamma_k(a_1, a_2, a_3) \cup C_z$ cospectral with any graph from $\{\Gamma_k(4,4,4), \Gamma_k(5,5,3), \Gamma_k(7,4,3)\}$, where $z \geq 3$.

Proof. We distinguish the following three cases.

Case 1. Suppose that $\Gamma_k(a_1, a_2, a_3) \cup C_z$ is cospectral with $\Gamma_k(4,4,4)$. Then $a_1 + a_2 + a_3 + z = 12$ and $z \neq 3, 4$ by Lemma 1, Lemma 2 and Lemma 3. Thus, $5 \leq z < 10$. If $z = 5$, then $a_1 + a_2 + a_3 = 7$. Since C_k is the minimal odd cycle, then $k = 3$ by Lemma 2.

Case 1.1. When $(a_1, a_2, a_3) = (5, 1, 1)$, by direct calculation,
$\rho(\Gamma_3(5,1,1) \cup C_5) \approx 2.437$ and $\rho(\Gamma_3(4,4,4)) \approx 2.269$.

Case 1.2. $(a_1, a_2, a_3) \in \{(4,2,1), (3,3,1)\}$. Since $\Gamma_3(4,4,4)$ can be obtained by subdividing the edges on the internal path of $\Gamma_3(4,2,1)$ or $\Gamma_3(3,3,1)$, repeatedly, then by Lemma 5, we have

$\rho(\Gamma_3(4,4,4)) < \rho(\Gamma_3(4,2,1))$ or $\rho(\Gamma_3(4,4,4)) < \rho(\Gamma_3(3,3,1))$.

Thus, $\rho(\Gamma_3(4,4,4)) < \rho(\Gamma_3(4,2,1) \cup C_5)$ and $\rho(\Gamma_3(4,4,4)) < \rho(\Gamma_3(3,3,1) \cup C_5)$.

If $5 < z < 10$, with the same method as case 1.2, we have $\rho(\Gamma_k(4,4,4)) < \rho(\Gamma_k(a_1, a_2, a_3) \cup C_z)$. Thus, the suppose does not follow by the determination of cospectral graphs.

Case 2. Suppose that $\Gamma_k(a_1, a_2, a_3) \cup C_z$ is cospectral with $\Gamma_k(5,5,3)$. Then $a_1 + a_2 + a_3 + z = 13$ and $5 \leq z < 11$. If $z = 5$, then $a_1 + a_2 + a_3 = 8$.

By Lemma 2, $k = 3$.

Case 2.1. When $(a_1, a_2, a_3) = (6, 1, 1)$, by direct calculation, $\rho(\Gamma_3(6, 1, 1) \cup C_5) \approx 2.437$ and $\rho(\Gamma_3(5, 5, 3)) \approx 2.269$.

Case 2.2. $(a_1, a_2, a_3) \in \{(5, 2, 1), (4, 3, 1), (3, 3, 2)\}$. Since $\Gamma_3(5, 5, 3)$ can be obtained by subdividing the edges on the internal path of $\Gamma_3(a_1, a_2, a_3)$ repeatedly, then by Lemma 5, we have $\rho(\Gamma_3(5, 5, 3)) < \rho(\Gamma_3(a_1, a_2, a_3))$. So $\rho(\Gamma_3(5, 5, 3)) < \rho(\Gamma_3(a_1, a_2, a_3) \cup C_5)$ and the suppose does not hold.

Case 3. Suppose that $\Gamma_k(a_1, a_2, a_3) \cup C_z$ is cospectral with $\Gamma_k(7, 4, 3)$. Then $a_1 + a_2 + a_3 + z = 14$ and $5 \leq z < 12$. Using the same method as case 1.2, we have $\rho(\Gamma_k(7, 4, 3)) < \rho(\Gamma_k(a_1, a_2, a_3) \cup C_z)$. So the suppose does not follow. □

Lemma 15. There does not exist a graph $D_{b_1,b_2,k} \cup D_{k,b_3,k} \cup C_z$ cospectral with any graph from $\{\Gamma_k(4, 4, 4), \Gamma_k(5, 5, 3), \Gamma_k(7, 4, 3)\}$, where $z \geq 3$.

Proof. We discuss the following three cases.

Case 1. Suppose that $D_{b_1,b_2,k} \cup D_{k,b_3,k} \cup C_z$ is cospectral with $\Gamma_k(4, 4, 4)$. Then $b_1 + b_2 + b_3 + z = 12$ and $z \neq 3, 4$ by Lemma 1 , Lemma 2 and Lemma 3. Since $b_1 \geq 5, b_2 \geq 1$ and $b_3 \geq 1$, then $(b_1, b_2, b_3, z) = (5, 1, 1, 5)$. Since C_k is the minimal odd cycle, then $k = 3$ from Lemma 2. Thus, by direct calculation, $\rho(D_{5,1,3} \cup D_{3,1,3} \cup C_5) \approx 2.414$ and $\rho(\Gamma_3(4, 4, 4)) \approx 2.269$, which is a contradiction with the suppose.

Case 2. Suppose that $D_{b_1,b_2,k} \cup D_{k,b_3,k} \cup C_z$ is cospectral with $\Gamma_k(5, 5, 3)$. Then $b_1 + b_2 + b_3 + z = 13$ and $z \neq 3, 4$. By $b_1 \geq 5, b_2 \geq 1$ and $b_3 \geq 1$, at least one of $\{b_1, z\}$ is 5. Since C_k is the minimal odd cycle, then $k = 3$ from Lemma 2. Thus, by direct calculation, we have
Table 1 as follow.

b_1	b_2	b_3	z	$\rho(D_{b_1,b_2,3} \cup D_{3,b_3,3} \cup C_z)$
5	2	1	5	2.414
5	1	2	5	2.359
5	1	1	6	2.414
6	1	1	5	2.414

Table 1.

But $\rho(\Gamma_3(5, 5, 3)) \approx 2.269$. Thus, from Table 1, the suppose does not follow.

Case 3. Assume that $D_{b_1,b_2,k} \cup D_{k,b_3,k} \cup C_z$ is cospectral with $\Gamma_k(7, 4, 3)$. Then $b_1 + b_2 + b_3 + z = 14$. By $b_1 \geq 5, b_2 \geq 1$, $b_3 \geq 1$, and $z \neq 3, 4$, we have either at least one of $\{b_1, z\}$ is 5 or $(b_1, z) = (6, 6)$. First, we discuss the former. Since C_k is the minimal odd cycle, then $k = 3$ from Lemma 2. Therefore, by direct calculation, we have Table 2 as follow.

b_1	b_2	b_3	z	$\rho(D_{b_1,b_2,3} \cup D_{3,b_3,3} \cup C_z)$
5	3	1	5	2.414
5	1	3	5	2.359
5	2	2	5	2.343
5	2	1	6	2.414
5	1	2	6	2.359
6	2	1	5	2.414
6	1	2	5	2.348
7	1	1	5	2.414
5	1	1	7	2.414

Table 2.

However, $\rho(\Gamma_3(7,4,3)) \approx 2.272$. So all radiuses shown in Table 2 are larger than 2.272. Next, we discuss the case of $(b_1, z) = (6,6)$ in order to show that no disconnected graph $D_{6,b_2,k} \cup D_{k,b_3,k} \cup C_6$ is cospectral with $\Gamma_k(7,4,3)$. When $k = 3$, $\rho(D_{6,1,3} \cup D_{3,1,3} \cup C_6) \approx 2.414$. When k is an odd integer more than 3, by Lemma 9, $\Gamma_k(7,4,3)$ has three eigenvalues more than 2 and one eigenvalue 2. But by removing the vertices of degree 3 of $D_{6,1,k} \cup D_{k,1,k} \cup C_6$, and by Lemma 6 and Lemma 7, $D_{6,1,k} \cup D_{k,1,k} \cup C_6$ exactly has two eigenvalues more than 2 and one eigenvalue 2. Therefore, the assertion does not hold. □

Finally, we mainly show that Γ-graph and the graph $D_{b_1,b_2,k} \cup D_{k,b_3,k}$ are not cospectral. Using Lemma 4, we first compute the characteristic polynomial of the graph $D_{b_1,b_2,k} \cup D_{k,b_3,k}$. Then we have

$$\begin{aligned}\phi(D_{b_1,b_2,k} \cup D_{k,b_3,k};\lambda) = \; & [(p_{b_1+b_2+k-1} - p_{b_1+b_2-1}p_{k-2} - 2p_{b_1+b_2-1}) \\ & -p_{b_1-2}(p_{b_2+k-1} - p_{b_2-1}p_{k-2} - 2p_{b_2-1}) \\ & -2(p_{b_2+k-1} - p_{b_2-1}p_{k-2} - 2p_{b_2-1})] \cdot \\ & [(p_{b_3+2k-1} - p_{b_3+k-1}p_{k-2} - 2p_{b_3+k-1}) \\ & -p_{k-2}(p_{b_3+k-1} - p_{b_3-1}p_{k-2} - 2p_{b_3-1}) \\ & -2(p_{b_3+k-1} - p_{b_3-1}p_{k-2} - 2p_{b_3-1})] \qquad (8)\end{aligned}$$

If we substitute (4) in (8), then we obtain

$$\frac{(x^2-1)^6}{x^{2-3k-s}(x^k-1)^3}\phi(D_{b_1,b_2,k} \cup D_{k,b_3,k};\lambda) - f(x) = Q(b_1,b_2,b_3;x) \qquad (9)$$

where $s = b_1 + b_2 + b_3$ and $f(x)$ is (6). $Q(b_1,b_2,b_3;x)$ is given in section 4. For each part of it, the terms are sorted in not decreasing order on their powers.

Lemma 16. There is no Γ-graph cospectral with the graph $D_{b_1,b_2,k} \cup D_{k,b_3,k}$.

Proof. Suppose that $G = \Gamma_k(a_1, a_2, a_3)$ is cospectral with $G' = D_{b_1,b_2,k} \cup D_{k,b_3,k}$, where $a_1 \geq a_2 \geq a_3$. Since G and G' have the same vertex number, we have

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 \tag{10}$$

By (5) and (9), we obtain

$$P(a_1, a_2, a_3; x) = Q(b_1, b_2, b_3; x) \tag{11}$$

Let f be the smallest power of x in $P(a_1, a_2, a_3; x)$(also in $Q(b_1, b_2, b_3; x)$ by (11)). By $P(a_1, a_2, a_3; x)$, $f = 4 + 2a_3$. From $Q(b_1, b_2, b_3; x)$, $f = b_1$ or $4 + 2b_2$ or $4 + 2b_3$. Since the coefficient of x^f in $P(a_1, a_2, a_3; x)$ is 1, by (11), then the coefficient of x^f in $Q(a_1, a_2, a_3; x)$ should be 1.

If $b_1 \leq min\{4 + 2b_2, 4 + 2b_3\}$, then $f = b_1$. But the coefficient of x^f in $Q(a_1, a_2, a_3; x)$ is at least 2, which is a contraction. We distinguish the following two cases when $b_1 > min\{4 + 2b_2, 4 + 2b_3\}$.

Case 1. $b_2 > b_3$. Thus, $4 + 2b_2 > 4 + 2b_3$, $f = 4 + 2b_3$ in $Q(b_1, b_2, b_3; x)$ and $4 + 2a_3 = 4 + 2b_3$(i.e., $a_3 = b_3$). Then $a_1 + a_2 = b_1 + b_2$ by (10). Let

$$\begin{aligned} P_1(a_1, a_2, a_3; x) = \; & P(a_1, a_2, a_3; x) - (x^{4+2a_3} - 4x^{6+2a_3} + 4x^{8+2a_3} \\ & -4x^{3k+2a_1+2a_2} + 4x^{2+3k+2a_1+2a_2} - x^{4+3k+2a_1+2a_2}) \\ Q_1(b_1, b_2, b_3; x) = \; & Q(b_1, b_2, b_3; x) - (x^{4+2b_3} - 4x^{6+2b_3} + 4x^{8+2b_3} \\ & -4x^{3k+2b_1+2b_2} + 4x^{2+3k+2b_1+2b_2} - x^{4+3k+2b_1+2b_2}) \end{aligned}$$

Obviously, $P_1(a_1, a_2, a_3; x) = Q_1(b_1, b_2, b_3; x)$. Denote by h_1 and h'_1 the smallest powers of $P_1(a_1, a_2, a_3; x)$ and $Q_1(b_1, b_2, b_3; x)$, respectively. So, $h_1 = h'_1$. By $P_1(a_1, a_2, a_3; x)$, h_1 is the smallest power of $A_1 = x^{4+2a_1} + x^{4+2a_2} + 2x^{2+k+2a_3}$. From $Q_1(b_1, b_2, b_3; x)$, h'_1 is the smallest power of $A'_1 = 2x^{b_1} + x^{4+2b_2} + 4x^{2+k+2b_3}$.

Case 1.1. $2+k+2a_3 < 4+2a_2$. Then $h_1 = 2+k+2a_3$. Since the coefficient of x^{h_1} in $P_1(a_1, a_2, a_3; x)$ is 2 and $P_1(a_1, a_2, a_3; x) = Q_1(b_1, b_2, b_3; x)$, then the coefficient of $x^{h'_1}$ in $Q_1(b_1, b_2, b_3; x)$ is 2 and $h'_1 = b_1$. Then $b_1 < 2+k+2b_3$, which is a contradiction with $b_1 = 2+k+2a_3 = 2+k+2b_3$.

Case 1.2. $4+2a_2 \leq 2+k+2a_3$. Then $h_1 = 4+2a_2$. Since the coefficient of x^{h_1} in $P_1(a_1, a_2, a_3; x)$ is at least 1, by $P_1(a_1, a_2, a_3; x) = Q_1(b_1, b_2, b_3; x)$, then the coefficient of $x^{h'_1}$ in $Q_1(b_1, b_2, b_3; x)$ should be at least 1.

Subcase 1.2.1. If the coefficient of x^{h_1} in $P_1(a_1, a_2, a_3; x)$ is 2, then $h_1 = 4+2a_1 = 4+2a_2$(i.e., $a_1 = a_2$). Since $P_1(a_1, a_2, a_3; x) = Q_1(b_1, b_2, b_3; x)$ and the coefficient of $x^{h'_1}$ in $Q_1(b_1, b_2, b_3; x)$ is 2, then $h'_1 = b_1 = 4 + 2a_1$. Then by $a_1 = a_2$ and $a_1 + a_2 = b_1 + b_2$, we have $4 + b_2 = 0$, which is impossible.

Subcase 1.2.2. The coefficient of x^{h_1} in $P_1(a_1, a_2, a_3; x)$ is 3. Since the coefficients of x^{4+2a_2} and x^{2+k+2a_3} in A_1 are 1 and 2, respectively, then $h_1 = 4 + 2a_2 = 2 + k + 2a_3$, which is a contraction because k is odd.

Subcase 1.2.3. If the coefficient of x^{h_1} in $P_1(a_1, a_2, a_3; x)$ is 4, then $h_1 = 4 + 2a_1 = 4 + 2a_2 = 2 + k + 2a_3$ from A_1, which is a contraction because k is odd.

Subcase 1.2.4. If the coefficient of x^{h_1} in $P_1(a_1, a_2, a_3; x)$ is 1, then by A_1 and A'_1, $4 + 2a_2 = 4 + 2b_2$(i.e., $a_2 = b_2$). From (10), $a_1 = b_1$.

Let

$$P_2(a_1, a_2, a_3; x) = P_1(a_1, a_2, a_3; x) - f_1(x)$$

where $f_1(x)$ is composed of (13) and (16) from $P(a_1, a_2, a_3; x)$.

Let

$$Q_2(b_1, b_2, b_3; x) = Q_1(b_1, b_2, b_3; x) - g_1(x)$$

where $g_1(x)$ is composed of (23) from $Q(b_1, b_2, b_3; x)$ and

$$\begin{aligned}\{&x^{4+2b_1+2b_3} - 2x^{6+2b_1+2b_3} + 4x^{2+k+2b_1+2b_3} - 11x^{4+k+2b_1+2b_3}\\ &+4x^{6+k+2b_1+2b_3} + 4x^{2k+2b_1+2b_3} - 16x^{2+2k+2b_1+2b_3} + 11x^{4+2k+2b_1+2b_3}\\ &-2x^{6+2k+2b_1+2b_3} - 4x^{3k+2b_1+2b_3} + 4x^{2+3k+2b_1+2b_3} - x^{4+3k+2b_1+2b_3}\}\end{aligned}$$

chosen from (20) in $Q(b_1, b_2, b_3; x)$.

Obviously, $f_1(x) = g_1(x)$ and $P_2(a_1, a_2, a_3; x) = Q_2(b_1, b_2, b_3; x)$. Denote by h_2 and h'_2 the smallest powers of $P_2(a_1, a_2, a_3; x)$ and $Q_2(b_1, b_2, b_3; x)$, respectively. So, $h_2 = h'_2$. By $P_2(a_1, a_2, a_3; x)$, h_2 is the smallest power of $A_2 = x^{4+2a_1} + x^{4+2a_2+2a_3} + 2x^{2+k+2a_3}$. From $Q_2(b_1, b_2, b_3; x)$, h'_2 is the smallest power of $A'_2 = 2x^{b_1} - x^{8+2b_2+2b_3} + 4x^{2+k+2b_3}$. By A_2, A'_2 and $a_i = b_i (i = 1, 2, 3)$, obviously, the coefficient of x^{h_2} must not be equal to the coefficient of $x^{h'_2}$, which is a contraction with $P_2(a_1, a_2, a_3; x) = Q_2(b_1, b_2, b_3; x)$.

Case 2. $b_2 \leq b_3$. If $b_2 = b_3$, then $4 + 2b_2 = 4 + 2b_3$ and the coefficient of x^f in $Q(a_1, a_2, a_3; x)$ is at least 2, which is a contraction. Thus, we just need to discuss the case of $b_2 < b_3$. Since $b_2 < b_3$, by (11), $P(a_1, a_2, a_3; x)$ and $Q(b_1, b_2, b_3; x)$, then $4 + 2a_3 = 4 + 2b_2$(i.e., $a_3 = b_2$). Therefore, by (10), $a_1 + a_2 = b_1 + b_3$.

Let

$$P_3(a_1, a_2, a_3; x) = P(a_1, a_2, a_3; x) - f_2(x)$$

where $f_2(x)$ is composed of (12) and (17) from $P(a_1, a_2, a_3; x)$.

Let

$$Q_3(b_1, b_2, b_3; x) = Q(b_1, b_2, b_3; x) - g_2(x)$$

where $g_2(x)$ is composed of (23) from $Q(b_1, b_2, b_3; x)$ and

$$\{x^{4+2b_1+2b_3} - 2x^{6+2b_1+2b_3} + 4x^{2+k+2b_1+2b_3} - 11x^{4+k+2b_1+2b_3}$$
$$+4x^{6+k+2b_1+2b_3} + 4x^{2k+2b_1+2b_3} - 16x^{2+2k+2b_1+2b_3} + 11x^{4+2k+2b_1+2b_3}$$
$$-2x^{6+2k+2b_1+2b_3} - 4x^{3k+2b_1+2b_3} + 4x^{2+3k+2b_1+2b_3} - x^{4+3k+2b_1+2b_3}\}$$

chosen from (20) in $Q(b_1, b_2, b_3; x)$.

Obviously, $f_2(x) = g_2(x)$ and $P_3(a_1, a_2, a_3; x) = Q_3(b_1, b_2, b_3; x)$. Let h_3 and h'_3 as the smallest powers of $P_3(a_1, a_2, a_3; x)$ and $Q_3(b_1, b_2, b_3; x)$, respectively. So, $h_3 = h'_3$. By $P_3(a_1, a_2, a_3; x)$ and $a_1 \geq a_2$, then $h_3 = 4+2a_2$ and the coefficient of x^{h_3} is 1. Since $P_3(a_1, a_2, a_3; x) = Q_3(b_1, b_2, b_3; x)$, then the coefficient of $x^{h'_3}$ in $Q_3(a_1, a_2, a_3; x)$ should be 1. From $Q_3(b_1, b_2, b_3; x)$, h'_3 is the smallest power of $2x^{b_1} + x^{4+2b_3}$ and the coefficient of x^{b_1} is at least 2. Thus $h'_3 = 4 + 2b_3$ and $4 + 2a_2 = 4 + 2b_3$(i.e. $a_2 = b_3$). Then $a_1 = b_1$ by (10) and $a_3 = b_2$. Let

$$\begin{aligned} P_4(a_1, a_2, a_3; x) = \; & P_3(a_1, a_2, a_3; x) - (x^{4+2a_2} - 4x^{6+2a_2} + 4x^{8+2a_2} \\ & -4x^{3k+2a_1+2a_3} + 4x^{2+3k+2a_1+2a_3} - x^{4+3k+2a_1+2a_3}) \\ Q_4(b_1, b_2, b_3; x) = \; & Q_3(b_1, b_2, b_3; x) - (x^{4+2b_3} - 4x^{6+2b_3} + 4x^{8+2b_3} \\ & -4x^{3k+2b_1+2b_2} + 4x^{2+3k+2b_1+2b_2} - x^{4+3k+2b_1+2b_2}) \end{aligned}$$

Obviously, $P_4(a_1, a_2, a_3; x) = Q_4(b_1, b_2, b_3; x)$. Let h_4 and h'_4 be the smallest powers of
$P_4(a_1, a_2, a_3; x)$ and $Q_4(b_1, b_2, b_3; x)$, respectively. So, $h_4 = h'_4$.

By $P_4(a_1, a_2, a_3; x)$, h_4 is the smallest power of $A_4 = x^{4+2a_1} + x^{4+2a_2+2a_3} + 2x^{2+k+2a_2}$. From $Q_4(b_1, b_2, b_3; x)$, h'_4 is the smallest power of $A'_4 = 2x^{b_1} + 4x^{2+k+2b_3} - x^{8+2b_2+2b_3}$. Since $a_1 = b_1, a_2 = b_3$ and $a_3 = b_2$, then $h'_4 = min\{a_1, 2 + k + 2a_2, 8 + 2a_2 + 2a_3\}$.

If $2 + k + 2a_2 < min\{4 + 2a_1, 4 + 2a_2 + 2a_3\}$, by $P_4(a_1, a_2, a_3; x)$, then $h_4 = 2 + k + 2a_2$ and the coefficient of of x^{h_4} in $P_4(a_1, a_2, a_3; x)$ is 2. Since $P_4(a_1, a_2, a_3; x) = Q_4(b_1, b_2, b_3; x)$, then the coefficient of $Q_4(b_1, b_2, b_3; x)$ is 2. Since $2 + k + 2a_2 = 2 + k + 2b_3$ and $h_4 = h'_4$, then $h'_4 = 2 + k + 2b_3$. But the coefficient of $x^{h'_4}$ in $Q_4(b_1, b_2, b_3; x)$ is at least 4, which is a contraction.

Otherwise, $2 + k + 2a_2 \geq min\{4 + 2a_1, 4 + 2a_2 + 2a_3\}$.

Case 2.1. Suppose $4 + 2a_1 \leq 4 + 2a_2 + 2a_3$, then $h_4 = 4 + 2a_1$. Since $a_1 < 4 + 2a_1 \leq 4 + 2a_2 + 2a_3 < 8 + 2a_2 + 2a_3$ and $4 + 2a_1 \leq 2 + k + 2a_2$, by $h'_4 = min\{a_1, 2 + k + 2a_2, 8 + 2a_2 + 2a_3\}$, then $h'_4 = a_1 \neq 4 + 2a_1 = h_4$.

Case 2.2. Suppose $4+2a_2+2a_3 < 4+2a_1$, then $h_4 = 4+2a_2+2a_3$. From $P_4(a_1, a_2, a_3; x)$, the coefficient of x^{h_4} is 1. Thus, by $P_4(a_1, a_2, a_3; x) = Q_4(b_1, b_2, b_3; x)$, the coefficient of $x^{h'_4}$ should be 1. Therefore, from A'_4 and $h'_4 = a_1 = 8 + 2a_2 + 2a_3$, we have $h_4 \neq h'_4$, which is a contraction. This completes the proof. ☐

Theorem 1. All Γ-graphs are determined by their adjacent spectrum.

Proof. Let G be a Γ-graph. Suppose G' be a cospectral graph of G. We discuss the following two cases.

Case 1. G' is connected. Then G' is not a Γ-graph by Lemma 11 and $G' \ncong U_k(x,y)$ from Lemma 13-(i).

Case 2. G' is disconnected. Then we have

(i) $G' \notin \{\theta_{k,l} \cup D_{k,b,k},\ \theta_{k,l} \cup \theta_{k,k},\ D_{k,b,l} \cup \theta_{k,k},\ U_k(x,y) \cup C_z,\ \theta_{k,l} \cup D_{k,b,k} \cup C_z,\ \theta_{k,l} \cup \theta_{k,k} \cup C_z,\ D_{k,b,l} \cup \theta_{k,k} \cup C_z\}$ by Lemma 13.

(ii) $G' \ncong \Gamma_k(a_1,a_2,a_3) \cup C_z$ from Lemma 14.

(iii) By Lemma 15, $G' \ncong D_{b_1,b_2,k} \cup D_{k,b_3,k} \cup C_z$.

(iv) From Lemma 16, $G' \ncong D_{b_1,b_2,k} \cup D_{k,b_3,k}$.

Therefore, we have searched all possible cospectral graphs of G, and the assertion does not hold.

This ends the proof. □

4 Appendix

$$P(a_1,a_2,a_3;x) =$$

$$\begin{aligned} &x^{4+2a_1+2a_2} - 2x^{6+2a_1+2a_2} + 4x^{2+k+2a_1+2a_2} - 11x^{4+k+2a_1+2a_2} \\ &+4x^{6+k+2a_1+2a_2} + 4x^{2k+2a_1+2a_2} - 16x^{2+2k+2a_1+2a_2} \\ &+11x^{4+2k+2a_1+2a_2} - 2x^{6+2k+2a_1+2a_2} - 4x^{3k+2a_1+2a_2} \\ &+4x^{2+3k+2a_1+2a_2} - x^{4+3k+2a_1+2a_2} + \end{aligned} \tag{12}$$

$$\begin{aligned} &x^{4+2a_1+2a_3} - 2x^{6+2a_1+2a_3} + 4x^{2+k+2a_1+2a_3} - 11x^{4+k+2a_1+2a_3} \\ &+4x^{6+k+2a_1+2a_3} + 4x^{2k+2a_1+2a_3} - 16x^{2+2k+2a_1+2a_3} \\ &+11x^{4+2k+2a_1+2a_3} - 2x^{6+2k+2a_1+2a_3} - 4x^{3k+2a_1+2a_3} \\ &+4x^{2+3k+2a_1+2a_3} - x^{4+3k+2a_1+2a_3} + \end{aligned} \tag{13}$$

$$x^{4+2a_2+2a_3} - 2x^{6+2a_2+2a_3} + 4x^{2+k+2a_2+2a_3} - 11x^{4+k+2a_2+2a_3}$$
$$+4x^{6+k+2a_2+2a_3} + 4x^{2k+2a_2+2a_3} - 16x^{2+2k+2a_2+2a_3}$$
$$+11x^{4+2k+2a_2+2a_3} - 2x^{6+2k+2a_2+2a_3} - 4x^{3k+2a_2+2a_3}$$
$$+4x^{2+3k+2a_2+2a_3} - x^{4+3k+2a_2+2a_3} + \tag{14}$$

$$x^{4+2a_1} - 4x^{6+2a_1} + 4x^{8+2a_1} + 2x^{2+k+2a_1} - 11x^{4+k+2a_1}$$
$$+16x^{6+k+2a_1} - 4x^{8+k+2a_1} - 4x^{2+2k+2a_1} + 11x^{4+2k+2a_1}$$
$$-4x^{6+2k+2a_1} + 2x^{2+3k+2a_1} - x^{4+3k+2a_1} + \tag{15}$$

$$x^{4+2a_2} - 4x^{6+2a_2} + 4x^{8+2a_2} + 2x^{2+k+2a_2} - 11x^{4+k+2a_2}$$
$$+16x^{6+k+2a_2} - 4x^{8+k+2a_2} - 4x^{2+2k+2a_2} + 11x^{4+2k+2a_2}$$
$$-4x^{6+2k+2a_2} + 2x^{2+3k+2a_2} - x^{4+3k+2a_2} + \tag{16}$$

$$x^{4+2a_3} - 4x^{6+2a_3} + 4x^{8+2a_3} + 2x^{2+k+2a_3} - 11x^{4+k+2a_3}$$
$$+16x^{6+k+2a_3} - 4x^{8+k+2a_3} - 4x^{2+2k+2a_3} + 11x^{4+2k+2a_3}$$
$$-4x^{6+2k+2a_3} + 2x^{2+3k+2a_3} - x^{4+3k+2a_3} \tag{17}$$

$$Q(b_1, b_2, b_3; x) =$$

$$-2x^{6+b_1+2b_2+2b_3} + 2x^{8+b_1+2b_2+2b_3} - 12x^{4+k+b_1+2b_2+2b_3}$$
$$+18x^{6+k+b_1+2b_2+2b_3} - 6x^{8+k+b_1+2b_2+2b_3} - 24x^{2+2k+b_1+2b_2+2b_3}$$
$$+48x^{4+2k+b_1+2b_2+2b_3} - 30x^{6+2k+b_1+2b_2+2b_3} + 6x^{8+2k+b_1+2b_2+2b_3}$$
$$-16x^{3k+b_1+2b_2+2b_3} + 40x^{2+3k+b_1+2b_2+2b_3} - 36x^{4+3k+b_1+2b_2+2b_3}$$
$$+14x^{6+3k+b_1+2b_2+2b_3} - 2x^{8+3k+b_1+2b_2+2b_3} + \tag{18}$$

$$
\begin{aligned}
&2x^{2+b_1+2b_2} - 10x^{4+b_1+2b_2} + 16x^{6+b_1+2b_2} - 8x^{8+b_1+2b_2} \\
&+4x^{k+b_1+2b_2} - 26x^{2+k+b_1+2b_2} + 54x^{4+k+b_1+2b_2} - 40x^{6+k+b_1+2b_2} \\
&+8x^{8+k+b_1+2b_2} - 8x^{2k+b_1+2b_2} + 30x^{2+2k+b_1+2b_2} - 30x^{4+2k+b_1+2b_2} \\
&+8x^{6+2k+b_1+2b_2} + 4x^{3k+b_1+2b_2} - 6x^{2+3k+b_1+2b_2} + 2x^{4+3k+b_1+2b_2} \\
&-2x^{2+2b_1+2b_2} + 9x^{4+2b_1+2b_2} - 12x^{6+2b_1+2b_2} + 4x^{8+2b_1+2b_2} \\
&-4x^{k+2b_1+2b_2} + 24x^{2+k+2b_1+2b_2} - 43x^{4+k+2b_1+2b_2} + 24x^{6+k+2b_1+2b_2} \\
&-4x^{8+k+2b_1+2b_2} + 8x^{2k+2b_1+2b_2} - 26x^{2+2k+2b_1+2b_2} + 19x^{4+2k+2b_1+2b_2} \\
&-4x^{6+2k+2b_1+2b_2} - 4x^{3k+2b_1+2b_2} + 4x^{2+3k+2b_1+2b_2} - x^{4+3k+2b_1+2b_2}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&-2x^{4+b_1+2b_3} + 6x^{6+b_1+2b_3} - 4x^{8+b_1+2b_3} - 8x^{2+k+b_1+2b_3} \\
&+30x^{4+k+b_1+2b_3} - 30x^{6+k+b_1+2b_3} + 8x^{8+k+b_1+2b_3} - 8x^{2k+b_1+2b_3} \\
&+40x^{2+2k+b_1+2b_3} - 54x^{4+2k+b_1+2b_3} + 26x^{6+2k+b_1+2b_3} - 4x^{8+2k+b_1+2b_3} \\
&+8x^{3k+b_1+2b_3} - 16x^{2+3k+b_1+2b_3} + 10x^{4+3k+b_1+2b_3} - 2x^{6+3k+b_1+2b_3} \\
&+x^{4+2b_1+2b_3} - 2x^{6+2b_1+2b_3} + 4x^{2+k+2b_1+2b_3} - 11x^{4+k+2b_1+2b_3} \\
&+4x^{6+k+2b_1+2b_3} + 4x^{2k+2b_1+2b_3} - 16x^{2+2k+2b_1+2b_3} + 11x^{4+2k+2b_1+2b_3} \\
&-2x^{6+2k+2b_1+2b_3} - 4x^{3k+2b_1+2b_3} + 4x^{2+3k+2b_1+2b_3} - x^{4+3k+2b_1+2b_3}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&-x^{8+2b_2+2b_3} - 6x^{6+k+2b_2+2b_3} + 3x^{8+k+2b_2+2b_3} - 12x^{4+2k+2b_2+2b_3} + \\
&12x^{6+2k+2b_2+2b_3} - 3x^{8+2k+2b_2+2b_3} - 8x^{2+3k+2b_2+2b_3} + 12x^{4+3k+2b_2+2b_3} \\
&-6x^{6+3k+2b_2+2b_3} + x^{8+3k+2b_2+2b_3} +
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&2x^{b_1} - x^{2b_1} - 14x^{2+b_1} + 36x^{4+b_1} - 40x^{6+b_1} + 16x^{8+b_1} \\
&-6x^{k+b_1} + 30x^{2+k+b_1} - 48x^{4+k+b_1} + 24x^{6+k+b_1} + 6x^{2k+b_1} \\
&-18x^{2+2k+b_1} + 12x^{4+2k+b_1} - 2x^{3k+b_1} + 2x^{2+3k+b_1} + 6x^{2+2b_1} \\
&-12x^{4+2b_1} + 8x^{6+2b_1} + 3x^{k+2b_1} - 12x^{2+k+2b_1} + 12x^{4+k+2b_1} \\
&-3x^{2k+2b_1} + 6x^{2+2k+2b_1} + x^{3k+2b_1} +
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&x^{4+2b_2} - 4x^{6+2b_2} + 4x^{8+2b_2} + 2x^{2+k+2b_2} - 11x^{4+k+2b_2} + 16x^{6+k+2b_2} \\
&-4x^{8+k+2b_2} - 4x^{2+2k+2b_2} + 11x^{4+2k+2b_2} - 4x^{6+2k+2b_2} \\
&+2x^{2+3k+2b_2} - x^{4+3k+2b_2} +
\end{aligned} \tag{23}
$$

$$x^{4+2b_3} - 4x^{6+2b_3} + 4x^{8+2b_3} + 4x^{2+k+2b_3} - 19x^{4+k+2b_3} + 26x^{6+k+2b_3} - 8x^{8+k+2b_3} + 4x^{2k+2b_3} - 24x^{2+2k+2b_3} + 43x^{4+2k+2b_3} - 24x^{6+2k+2b_3} + 4x^{8+2k+2b_3} - 4x^{3k+2b_3} + 12x^{2+3k+2b_3} - 9x^{4+3k+2b_3} + 2x^{6+3k+2b_3} \quad (24)$$

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