Distance labelings and (total-)neighbor-distinguishing colorings of the edge-multiplicity-paths-replacements *

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abstract: As a promotion of the channel assignment problem, an $L(1,1,1)$-*labeling* of a graph G is an assignment of nonnegative integers to $V(G)$ such that the difference between labels of adjacent vertices is at least 1, and the difference between labels of vertices that are distance two and three apart is at least 1. About 10 years ago, many mathematicians considered colorings (proper, general, total or from lists) such that vertices (all or adjacent) are distinguished either by sets or multisets or sums. In this paper, we will study $L(1,1,1)$-labeling-number and $L(1,1)$-edge-labeling-number of the edge-path-replacement. From this, we will consider the total-neighbor-distinguishing coloring and the neighbor-distinguishing coloring of the edge-multiplicity-paths-replacements, give a reference for the conjectures: $tndi_{\sum}(G) \leq \Delta+3$, $ndi_{\sum}(G) \leq \Delta+2$ and $tndi_S(G) \leq \Delta+3$ for the edge-multiplicity-paths-replacements $G(rP_k)$ with $k \geq 3$ and $r \geq 1$.
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iplicity-paths-replacements.

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1 Introduction

As a promotion of the channel assignment problem, an $L(1,1,1)$-*labeling* of a graph G is an assignment of nonnegative integers to $V(G)$ such that the difference between labels of adjacent vertices is at least 1, and the difference between labels of vertices that are distance two and three apart is at least 1. The span of an $L(1,1,1)$-labeling of a graph G is the difference between the maximum and minimum integers used by it. The $L(1,1,1)$-labeling-number of G is the minimum span over all $L(1,1,1)$-labelings of G, denoted by $\lambda_{1,1,1}(G)$.

This problem is first known as the $L(j,k)$-labeling problem of a graph G. In 1992, Griggs and Yeh introduced this labeling with $j = 2$ and $k = 1$ in [12]. This notion has been studied many times and gives many challenging problems. The reader is referred to the surveys [5,11,33]. In particular, in [32], Whittlesty, Georges and Mauro studied the $L(2,1)$-labeling of the *subdivision graph* of G, which is the graph obtained from G by replacing each edge by a path with order 3. This labeling is called (2,1)-total labeling of G, which was introduced by Havet and Yu in 2002 [13,14] and generalized to the $(d,1)$-total labeling of a graph G. If $d = 1$, then the (1,1)-total labeling is the traditional total coloring. The total coloring has been intensively studied in [2–4,16,30,31].

About 10 years ago, a new trend originated in the topic of graph colorings. Many mathematicians considered colorings (proper, general, total or from lists) such that vertices (all or adjacent) are distinguished either by sets or multisets or sums. Karoński,Luczak and Thomason in [22] considered general colorings of edges, and they conjectured that three colors are enough to distinguish adjacent vertices by sums. This conjecture is almost proved—Kalkowski showed that five colors are enough [21]. A similar conjecture by Przybyło and Woźniak states that two colors are enough by general total coloring [27].

Recently, some other authors in [9] considered a proper coloring of edges distinguishing adjacent vertices by sums. As usually, we are interested in the smallest number of colors in a neighbor-distinguishing coloring of G. Suppose that $c : E \longrightarrow \{1, 2, \cdots, k\}$ is a proper edge coloring of G. For a vertex v, let $f(v)$ denote the total sum of colors of the edges incident to v. If the function f distinguishes adjacent vertices of G, we say the coloring c is a neighbor-distinguishing coloring. The smallest such k is called the neighbor-distinguishing index by Sum, and denoted by $ndi_{\sum}(G)$. Evidently, when searching for the neighbor-distinguishing index it is sufficient to restrict our attention to connected graphs. Observe also, that $G = K_2$ does not have any neighbor-distinguishing coloring. So, we shall consider only connected graphs with at least three vertices. This invariant has been introduced by Zhang et al. [34]. It is easy to see that $ndi(C_5) = 5$ and in [34] it is conjectured that $ndi_{\sum}(G) \leq \Delta(G) + 2$ for any connected graph $G \neq C5$ on $n \geq 3$ vertices.

Conjecture 1.1 *[34] For any connected graph $G \neq C_5$ on $n \geq 3$ vertices, $ndi_{\sum}(G) \leq \Delta(G) + 2$.*

The conjecture has been confirmed by Balister et al. [1] for bipartite graphs and for graphs G with $\Delta(G) = 3$. Edwards et al. [8] have shown even that $ndi_{\sum}(G) \leq \Delta(G) + 1$ if G is bipartite, planar, and of maximum degree $\Delta(G) \geq 12$.

Suppose that $c : V \cup E \longrightarrow \{1, 2, \cdots, k\}$ is a proper total coloring of G. For a vertex v, let $f(v)$ denote the total sum of colors of the edges incident to v and the color of v. If the function f distinguishes adjacent vertices of G, we say the coloring c is a total-neighbor-distinguishing coloring. The smallest such k is called the total-neighbor-distinguishing index by Sum, and denoted by $tndi_{\sum}(G)$.

[28] considered the total-neighbor-distinguishing coloring, and conjectured that $\Delta+3$ colors suffice to distinguish adjacent vertices in any simple graph. In [28] they shown that this holds for complete graphs, cycles, bipartite graphs, cubic graphs and graphs with maximum degree at most three.

Conjecture 1.2 *[28] For every graph $G = (V, E)$, the total-neighbor-distinguishing index by sums $tndi_{\sum}(G)$ satisfies the inequality $tndi_{\sum}(G) \leq \Delta + 3$.*

Zhang, Chen, Li, Yao, Lu and Wang in [35] investigated a proper total coloring of G, but to every vertex v they assigned a set $S(v)$ of colors of the edges incident to v and the color of v. By χ_a'' or by χ_{at}'' they denoted the smallest number k of colors so that there exists a proper total coloring with k colors that distinguishes adjacent vertices by sets (i.e., $S(u)$ is different from $S(v)$ for every pair of adjacent vertices u, v). We propose to denote this index by $tndi_S(G)$-total-neighbor-distinguishing index. They considered the cases of cliques, paths, cycles, fans,wheels, stars, complete graphs, complete bipartite graphs and trees.They showed (giving exact bounds for $tndi_S$) that $\Delta + 3$ colors are enough in these cases and formulated Conjecture 1.3 as follows. Next, Chen in [7] proved this conjecture for bipartite graphs and for graphs with maximum degree at most three. Hulgan in [15] gave a really short proofs of his results.

Conjecture 1.3 *[35] For every graph $G = (V, E)$, the total-neighbor-distinguishing-index by sets $tndi_S(G)$ satisfies the inequality $tndi_S(G) \leq \Delta + 3$.*

It is easy to observe, that if two vertices are distinguished by sums then they are also distinguished by sets, but not necessarily conversely. Thus $tndi_S(G) \leq tndi_{\sum}(G)$.

Like the classical edge coloring of a graph G, it is natural to investigate the edge version of $L(j, k)$-labeling. An $L(j, k)$-edge-labeling of a graph G is an assignment of nonnegative integers to the edge sets $E(G)$ such that the difference between labels of adjacent edges is at least j, and the difference between labels of edges that are distance two apart is at least k. The minimum span of an $L(j,k)$-labeling of G is denoted by $\lambda'_{j,k}(G)(\lambda'_{1,1}(G)=\lambda'_1(G))$.

The edge version of distance two labeling was first investigated by Georges and Mauro [10]. Several classes of graphs were studied by Georges and Mauro. Among the results, the authors determined the $L(2,1)$-edge-labeling numbers of Δ-regular trees for $\Delta \geq 2$ and n-dimensional cubes for small n. References [6, 10, 25] are the papers that we can find on the $L(j,k)$-edge-labeling problem.

For $r \geq 1$, *the edge-multiplicity-paths-replacement* $G(rP_k)$ *of a graph* G is a graph obtained by replacing each edge uv with r vertex-disjoint paths P_k^i: $ux_{uv}^{i_1}x_{uv}^{i_2}\cdots x_{uv}^{i_{k-2}}v$, where $i=1,2,\cdots r$. Note that the vertices of G are called as the nodes of $G(rP_k)$. It is easily seen that $G(rP_{2k-1})$ is the subdivision graph of $G(rP_k)$, and for $r\Delta \geq 2$, the maximum degree of $G(rP_k)$ is $r\Delta$ where Δ is the maximum degree of G. We can consider $G(rP_k)$ with $r=1$ as the edge-path-replacement of a graph G.

In [20, 23, 24], the authors worked on $L(d,1)$-labeling-number of the edge-path-replacement $G(P_k)$ of a graph G. In this paper, we will study $L(1,1,1)$-labeling-number and $L(1,1)$-edge-labeling-number of the edge-path-replacements. From this, we will consider the total-neighbor-distinguishing coloring and the neighbor-distinguishing coloring of the edge-multiplicity-paths-replacements, give a reference for the conjectures: $tndi_{\sum}(G) \leq \Delta+3$, $ndi_{\sum}(G) \leq \Delta+2$ and $tndi_S(G) \leq \Delta+3$ for the edge-multiplicity-paths-replacements $G(rP_k)$ with $k \geq 3$ and $r \geq 1$.

2 L(1,1,1)-labeling of $G(rP_k)$ with $r \geq 1$ and $k \geq 3$

For $r\Delta = 2$, observe, that $G(rP_k)$ is $P_n(P_k)(\cong P_{(k-1)n-k+2})$, $C_n(P_k)(\cong C_{(k-1)n})$, or $P_2(2P_k)(\cong C_{2k-2})$. Then we have the following theorem.

Theorem 2.1 *$\lambda_{1,1,1}(P_n(P_k)) = 3$ and $3 \leq \lambda_{1,1,1}(C_n(P_k)) \leq 4$; for $k \geq 3$, $3 \leq \lambda_{1,1,1}(P_2(2P_k)) \leq 4$.*

We next consider the case $r\Delta \geq 3$ and $k \geq 3$. Note that $\lambda_{1,1,1}(G) \geq \Delta + 1$ except that G is the star.

Theorem 2.2 *Suppose that G is a connected graph with degree Δ. Let $k \geq 9$. Then $\lambda_{1,1,1}(G(P_k)) = r\Delta + 1$ for $r\Delta \geq 3$.*

Proof. It is easy to see that $\lambda_{1,1,1}(G(rP_k)) \geq r\Delta+1$, since there $G(rP_k) \ncong k_{1,r\Delta}$. Thus it suffices to give an L(1,1,1)-labeling of $G(rP_k)$ with span $r\Delta + 1$ for $k \geq 9$.

We first give L(1,1,1)-labelings of $G(rP_k)$ with span $r\Delta + 1$ for $k = 9, 10, 11$ as follows. Label all the nodes of $G(rP_k)$ by $r\Delta + 1$, and label all the adjacent vertices of each node in $[1, r\Delta]$. And label the replacement-path P_9 as follows: $(r\Delta + 1)p0abc0q(r\Delta + 1)$, where $p, q \in [1, r\Delta]$, $p \neq a, b$, $q \neq b, c$, and a, b, c are different to each other and can be in [1,4]; label the replacement-path P_{10} as follows: $(r\Delta + 1)p0abcd0q(r\Delta + 1)$, where $p, q \in [1, r\Delta]$, $p \neq a, b$, $q \neq c, d$, and a, b, c, d are different to each other and can be in [1,4]; label the replacement-path P_{11} as follows: $(r\Delta+1)p0ab(r\Delta+1)cd0q(r\Delta + 1)$, where $p, q \in [1, r\Delta]$, $p \neq a, b$, $q \neq c, d$, and a, b, c, d are different to each other and can be in [1,4]; and label the replacement-path P_{12} as follows: $(r\Delta + 1)p0ab(r\Delta + 1)pcd0q(r\Delta + 1)$, where $p, q \in [1, r\Delta]$, $p \neq a, b, c, d$, $q \neq c, d$, and a, b, c, d can be in [1,4].

Note that the L(1,1,1)-labeling of $G(rP_k)$ with span $r\Delta + 1$ for $k \geq 13$ can be obtained by repeating the labels of P_4 in the replacement-path P_k for $k = 9, 10, 11, 12$. Then $\lambda_{1,1,1}(G(rP_k)) = r\Delta + 1$ for $k \geq 9$. ∎

Theorem 2.3 *Suppose that G is a connected graph with degree Δ. let $7 \leq k \leq 8$. Then $r\Delta + 1 \leq \lambda_{1,1,1}(G(rP_k)) \leq r\Delta + 2$ for $r\Delta \geq 3$.*

Proof. It is easy to see that $\lambda_{1,1,1}(G(rP_k)) \geq r\Delta + 1$. Thus it suffices to give an L(1,1,1)-labeling of $G(rP_k)$ with span $r\Delta + 2$. Label all the nodes of $G(rP_k)$ by $r\Delta + 2$, and label all the adjacent vertices of each node in $[2, r\Delta + 1]$. And label the replacement-path P_7 as follows: $(r\Delta + 2)p0a1q(r\Delta + 2)$, where $p, q \in [2, r\Delta + 1]$, $p, q \neq a$, and a can be in [1,4]; and, label the replacement-path P_8 as follows: $(r\Delta + 2)p0ab1q(r\Delta + 2)$, where $p, q \in [2, r\Delta + 1]$, $p, q \neq a, b$, and a, b are different and can be in [1,5]. Then $\lambda_{1,1,1}(G(rP_k)) \leq r\Delta + 2$ for $7 \leq k \leq 8$. ∎

Theorem 2.4 *Suppose that G is a connected graph with degree $\Delta \geq 2$. Then $\lambda_{1,1,1}(G(rP_5)) = r\Delta + 1$ for $r \geq 1$.*

Proof. It suffices to given L(1,1,1)-labeling of $G(rP_5)$with span $r\Delta + 1$. Observe, that an L(1,1,1)-labeling of $G(rP_5)$ can be derived by the (1,1)-total-labeling of $G(rP_3)$ as follows. Label all the nodes of $G(rP_3)$ by 0, and all the inserted vertices of $G(rP_3)$ by 1. Secondly, we label all the edges of $G(rP_3)$ in $[2, \chi'(G(rP_3))+1]$, where $\chi'(\chi)$ is the edge chromatic number(the chromatic number) of the graph. Note that $G(rP_3)$ is bipartite. Then we have $\chi = 2$ and $\chi' = r\Delta$ by König's Theorem. Then we obtain a (1,1)-total-labeling of $G(rP_3)$ with span $r\Delta + 1$. Thus $\lambda_1(G(rP_5)) = r\Delta + 1$. ∎

We next consider the replacement of regular graph for $k = 6$.

A *factor* of a graph G is a spanning subgraph of G. A *k-factor* of G is a factor of G that is k-regular. Thus a 2-factor of G is a factor of G that is a disjoint union of cycles of G. A graph G is k-factorable if G is an edge-disjoint union of k-factors of G.

Theorem 2.5 *[26] Every regular graph with positive even degree has a 2-factor.*

Theorem 2.6 *Suppose that G is a regular graph with degree Δ. Let $r\Delta \geq 2$. Then $r\Delta + 1 \leq \lambda_{1,1,1}(G(rP_6)) \leq r\Delta + 2$ for even $r\Delta$ And $r\Delta + 1 \leq \lambda_{1,1,1}(G(rP_6)) \leq r\Delta + 3$ for odd $r\Delta$.*

Proof. Similarly to the proof of Theorem 2.2, it suffices to give an $L(1,1,1)$-labeling of $G(rP_6)$ with span $r\Delta + 2$ for even $r\Delta$.

For even $r\Delta$, by theorem 2.5, $G(rP_2)$ can be decomposed into $\frac{r\Delta}{2}$ 2-factors. For $k = 6$, label all the nodes of $G(rP_k)$ by $r\Delta + 2$, and, label the replacement-paths of each 2-factor from one node to the other as follows: $(r\Delta + 2)i01(i + 1)(r\Delta + 2)$, where i is even in $[2, r\Delta + 1$. Then $r\Delta + 1 \leq \lambda_{1,1,1}(G(rP_6)) \leq r\Delta + 2$ for even $r\Delta$.

For odd $r\Delta$, we structure a graph H by connecting each pair vertices x and x' in $G(rP_2), G(rP_2)'$, where $G(rP_2)'$ is the copy graph of $G(rP_2)$, and x' corresponding to x. It is easy to see that H is a regular graph with positive even degree $r\Delta + 1$. Furthermore, by above, $r\Delta + 1 \leq \lambda_{1,1,1}(G(rP_6)) \leq r\Delta + 3$ for odd $r\Delta$. ∎

Theorem 2.7 *Suppose that G is a connected graph with degree Δ. Let $r\Delta \geq 2$. Then $r\Delta + 1 \leq \lambda_{1,1,1}(G(rP_6)) \leq r\Delta + 2$ for even $r\Delta$ And $r\Delta + 1 \leq \lambda_{1,1,1}(G(rP_6)) \leq r\Delta + 3$ for odd $r\Delta$.*

Proof. Note that we can obtain a regular graph with the maximum degree $r\Delta$ such that $G(rP_2)$ is its subgraph. If there exist two vertices u and v whose degrees are less than $r\Delta$, then we add the edge uv. Lastly, we obtain a new graph G_1 in which there exists at least one vertex whose degree is less than $r\Delta$.

If there exists only one vertex x in G_1 whose degree is $a(< r\Delta)$, then we structure a graph H by adding all the edges between any two copies of x in $G_1, G_1^1, \cdots, G_1^{r\Delta - a + 1}$, where $G_1^1, \cdots, G_1^{r\Delta - a+1}$ is the copy graph of G_1. So we obtain a regular graph with degree $r\Delta$ such that $G(rP_2)$ is its subgraph.

By Theorem 2.6, the proof is over. ∎

Similar to the proof of Theorem 2.6, we next consider the replacement of regular graph for $k = 4$.

Theorem 2.8 *Suppose that G is a regular graph with degree Δ. Let $r\Delta \geq 3$. Then*
$r\Delta \leq \lambda_{1,1,1}(G(rP_4)) \leq \frac{5r\Delta}{2} + 5$ *for even* $r\Delta$, *and* $r\Delta \leq \lambda_{1,1,1}(G(rP_4)) \leq \frac{5(r\Delta+1)}{2} + 5$ *for odd* $r\Delta$.

Proof. Similarly to the proof of Theorem 3.2, it suffices to give an $L(1,,1,1)$-labeling of $G(rP_4)$ with span $\frac{5r\Delta}{2} + 5$ for even $r\Delta$.

For even $r\Delta$, by theorem 2.5, $G(rP_2)$ can be decomposed into $\frac{r\Delta}{2}$ 2-factors. For $k = 4$, label each replacement-path of each 2-factor as follows: $i(i+1)(i+2)(i+3)\cdots$, $i(i+1)(i+2)(i+3)(i+4)i(i+1)(i+2)(i+3)\cdots$ or $i(i+1)(i+2)(i+3)(i+4)i(i+1)(i+2)(i+3)(i+4)i(i+1)(i+2)(i+3)\cdots$, where i is in $[1, \frac{5r\Delta}{2} + 5]$ and exactly divisible by 5. Then $r\Delta \leq \lambda_{1,1,1}(G(rP_4)) \leq \frac{5r\Delta}{2} + 5$ for even $r\Delta$.

For odd $r\Delta$, similar to the proof of Theorem 2.6, $r\Delta \leq \lambda_{1,1,1}(G(rP_4)) \leq \frac{5(r\Delta+1)}{2} + 5$ for odd $r\Delta$. ∎

Theorem 2.9 *Suppose that G is a connected graph with degree Δ. Let $r\Delta \geq 3$. Then*
$r\Delta \leq \lambda_{1,1,1}(G(rP_4)) \leq \frac{5r\Delta}{2} + 5$ *for even* $r\Delta$, *and* $r\Delta \leq \lambda_{1,1,1}(G(rP_4)) \leq \frac{5(r\Delta+1)}{2} + 5$ *for odd* $r\Delta$.

Proof. Similar to the proof of Theorem 2.7, by Theorem 2.8, the proof is over. ∎

Theorem 2.10 *Suppose that G is a connected graph with degree $\Delta \geq 2$. Let $r \geq 1$ and $r\Delta \geq 3$, $\lambda_{1,1,1}(G(rP_3)) \leq r\chi' + \chi - 1$, where χ and χ' are the chromatic number and the edge chromatic number of the graph G, respectively. Furthermore, $\lambda_{1,1,1}(G(rP_3)) \leq (r+1)\Delta + r - 1$.*

Proof. Let c be a vertex colouring of G with the χ integers in $[0, \chi - 1]$. Let c' be an edge colouring of G with the χ' integers in $[0, \chi' - 1]$. Then label all the nodes of $G(rP_3))$ by c. For each $i \in \{1, 2, \cdots, r\}$, label the inserted vertices $x^{i_1}_{uv}$ as $\chi + ic'(uv) - 1$. Thus we obtain an $L(1,1,1)$-labeling of the edge-multiplicity-paths-replacement $G(rP_3)$ with span $r\chi' + \chi - 1$. So $\lambda_{1,1,1}(G(rP_3)) \leq r\chi' + \chi - 1$.

If G is neither a complete graph K_n or an odd cycle, then $\chi \leq \Delta$ by Brook's theorem. And $\chi' \leq \Delta+1$ by Vizing's theorem. Hence $\lambda_{1,1,1}(G(rP_3))$ $\leq r(\Delta + 1) + \Delta - 1 = (r+1)\Delta + r - 1$.

Suppose now that G is a complete graph K_n on n vertices. Note that $\chi \leq \Delta+1$. If n is even then $\chi' \leq \Delta$. So $\lambda_{1,1,1}(K_n(rP_3)) \leq r(\Delta+1)+\Delta-1$. If n is odd then $\chi' \leq \Delta+1$. Let c' be an edge colouring of G with n colours. And let M_j, $1 \leq j \leq n$, be the matchings corresponding to the colour classes. Note that each vertex is in every M_j but one, and M_j contains all the vertices but one v_j. For $1 \leq i \leq r$, suppose that x^{i_j} is the inserted vertex corresponding to the edges in M_j. For $1 \leq j \leq n$ and $1 \leq i \leq r$, label the vertex v_j with $n - j$, and the inserted vertex x^i_j with $n + ij - 2$. Then we obtain an $L(1,1,1)$-labeling of $K_n(rP_3)$ in $[0, rn + n - 2]$. So $\lambda_{1,1,1}(K_n(rP_3)) \leq (r+1)\Delta + r - 1$, since $\Delta = n - 1$. ∎

3 L(1,1)-edge-labeling of $G(rP_k)$ with $r \geq 1$ and $k \geq 3$

For $r\Delta = 2$, note that $G(rP_k)$ is $P_n(P_k)(\cong P_{(k-1)n-k+2})$, $C_n(P_k)(\cong C_{(k-1)n})$, or $P_2(2P_k)(\cong C_{2k-2})$. Then we have the following theorem.

Theorem 3.1 *$\lambda_1^{'}(P_n(P_k)) = 2$ and $2 \leq \lambda_1^{'}(C_n(P_k)) \leq 3$, $2\lambda_1^{'}(P_2(2P_k)) \leq 3$ for $k \geq 3$.*

We next consider the case $r\Delta \geq 3$ and $k \geq 3$. Note that $\lambda_1^{'}(G) \geq \Delta$ except that G is the star.

Theorem 3.2 *Suppose that G is a connected graph with degree Δ. Then $\lambda_1'(G(P_k)) = r\Delta$ for $k \geq 7$ and $r\Delta \geq 3$.*

Proof. It is easy to see that $\lambda_1(G(rP_k)) \geq r\Delta$, since there $G(rP_k) \not\cong k_{1,r\Delta}$. Thus it suffices to give an L(1,1)-edge-labeling of $G(rP_k)$ with span $r\Delta$ for $k \geq 7$. We first give L(1,1)-edge-labelings of $G(rP_k)$ with span $r\Delta$ for $k = 7, 8, 9$ as follows. Label all the edges incident to each node of $G(rP_k)$ in [1,$r\Delta$], and label their the adjacent edges by 0. For $k = 7$, the replacement-path P_7 is labeled as follows: $p0ab0q$, where $p, q \in [1, r\Delta]$, a, b are different and can be in [1,3]; For $k = 8$, the replacement-path P_8 is labeled as follows: $p0abc0q$, where $p, q \in [1, r\Delta]$, and, $a = q$, $c = p$ and $b \in [1, 3]$ for $p \neq q$, otherwise $b = p = q$ and a, b are the different two in $[1, 3]$; For $k = 8$, the replacement-path P_9 is labeled as follows: $p0abcd0q$, where $p, q \in [1, r\Delta]$, and, $a = q$, $c = p$ and b, c are the different two in $[1, 4]$ for $p \neq q$, otherwise $b = p = q$ and a, b, c are the different two in $[1, 4]$.

Note that the L(1,1)-edge-labeling of $G(rP_k)$ with span $r\Delta$ for $k \geq 10$ can be obtained by repeating the labels of P_4 in the replacement-path P_k for $k = 7, 8, 9$. Then $\lambda_1'(G(rP_k)) = r\Delta$ for $k \geq 7$. ∎

Theorem 3.3 *Suppose that G is a connected graph with degree Δ. Let $5 \leq k \leq 6$ and $r\Delta \geq 3$. Then $r\Delta \leq \lambda_1'(G(rP_k)) \leq r\Delta + 1$.*

Proof. It is easy to see that $\lambda_1'(G(rP_k)) \geq r\Delta$, since there exists a star $k_{1,r\Delta}$ in $G(rP_k)$. So, it suffices to give an L(1,1)-edge-labeling of $G(rP_k)$ with span $r\Delta + 1$ for $r\Delta \geq 3$. Label all the edges incident to each node of $G(rP_k)$ in [2,$r\Delta$+1]. For $k = 5$, the replacement-path P_5 is labeled as follows: $p01q$, where $p, q \in [2, r\Delta + 1]$; For $k = 6$, the replacement-path P_6 is labeled as follows: $p0a1q$, where $p, q \in [2, r\Delta + 1]$ and a can be in [2,4]. Then $r\Delta \leq \lambda_1'(G(rP_k)) \leq r\Delta + 1$ for $r\Delta \geq 3$. ∎

Similar to the proof of Theorem 2.6, we next consider the replacement of regular graph for $k = 3, 4$.

Theorem 3.4 *Suppose that G is a regular graph with degree Δ. Let $r\Delta \geq 3$. Then*

$\lambda_1'(G(rP_4)) = r\Delta$ for even $r\Delta$, and $r\Delta \leq \lambda_1'(G(rP_4)) \leq r\Delta + 1$ for odd $r\Delta$.

$r\Delta \leq \lambda_1'(G(rP_3)) \leq 2r\Delta + 3$ for even $r\Delta$, and $r\Delta \leq \lambda_1'(G(rP_3)) \leq 2r\Delta + 5$ for odd $r\Delta$.

Proof. For even $r\Delta$, by theorem 2.5, $G(rP_2)$ can be decomposed into $\frac{r\Delta}{2}$ 2-factors. For $k = 4$, label each replacement-path of each 2-factor as follows: $i0(i+1)$, where i is odd in $[1,r\Delta]$. For $k = 3$, label each 2-factor as follows: $i(i+1)(i+2)\cdots$, $i(i+1)(i+2)(i+3)i(i+1)(i+2)\cdots$ or $i(i+1)(i+2)(i+3)i(i+1)(i+2)(i+3)i(i+1)(i+2)\cdots$, where i is in $[1, \frac{4r\Delta}{2}+3]$ and exactly divisible by 4.

Then $\lambda_1'(G(rP_4)) = r\Delta$ and $r\Delta \leq \lambda_1'(G(rP_3)) \leq 2r\Delta + 3$ for even $r\Delta$.

For odd $r\Delta$, similar to the proof of Theorem 2.6, we have $r\Delta \leq \lambda_1'(G(rP_4)) \leq r\Delta + 1$ and $r\Delta \leq \lambda_1'(G(rP_3)) \leq 2r\Delta + 5$ for odd $r\Delta$. ∎

Theorem 3.5 *Suppose that G is a connected graph with degree Δ. Let $r\Delta \geq 3$. Then*

$\lambda_1'(G(rP_4)) = r\Delta$ for even $r\Delta$, and $r\Delta \leq \lambda_1'(G(rP_4)) \leq r\Delta + 1$ for odd $r\Delta$.

$r\Delta \leq \lambda_1'(G(rP_3)) \leq 2r\Delta + 3$ for even $r\Delta$, and $r\Delta \leq \lambda_1'(G(rP_3)) \leq 2r\Delta + 5$ for odd $r\Delta$.

Proof. Similar to the proof of Theorem 2.7, by Theorem 3.4, proof is over. ∎

4 $tndi_{\sum}(G(rP_k))$ with $r \geq 1$ and $k \geq 3$

Note that the incident graph of $G(rP_k)$ is $G(rP_{2k-1})$, furthermore, for $k \geq 3$, the total-neighbor-distinguishing coloring of $G(rP_k)$ can be derived from by the $L(1,1,1)$-labelings of $G(rP_{2k-1})$ in the proofs of Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 2.1. So we have:

Theorem 4.1 *Suppose $k \geq 3$. $tndi_{\sum}(P_n(P_k)) = 4$ for $n \geq 2$, and $4 \leq tndi_{\sum}(C_n(P_k)) \leq 5$ for $n \geq 3$.*

Theorem 4.2 *Suppose that G is a connected graph with degree Δ. Let $r\Delta \geq 3$. Then $tndi_{\sum}(G(rP_k)) = r\Delta + 2$ for $k \geq 5$, $tndi_{\sum}(G(rP_4)) \leq r\Delta + 3$, and, $tndi_{\sum}(G(rP_3)) = r\Delta + 2$.*

5 $ndi_{\sum}(G(rP_k))$ with $r \geq 1$ and $k \geq 3$

For $k \geq 3$, the neighbor-distinguishing coloring of $G(rP_k)$ can be derived from by the $L(1,1)$-edge-labelings of $G(rP_k)$. in the proofs of Theorem 3.2 and Theorem 3.3.

Theorem 5.1 *Suppose $k \geq 3$. $ndi_{\sum}(P_n(P_k)) = 3$ for $n \geq 2$, and $3 \leq ndi_{\sum}(C_n(P_k)) \leq 4$ for $n \geq 3$.*

Theorem 5.2 *Suppose that G is a connected graph with degree Δ. Let $r\Delta \geq 3$. Then $ndi_{\sum}(G(rP_4)) = r\Delta + 1$ for even $r\Delta$, $ndi_{\sum}(G(rP_4)) = r\Delta + 2$ for odd $r\Delta$, $ndi_{\sum}(G(rP_k)) = r\Delta + 1$ for $k \geq 5$.*

By the methods as follows, we obtain:

Theorem 5.3 *Suppose that G is a connected graph with degree $\Delta \geq 2$. Let and $r\Delta \geq 3$. Then $ndi_{\sum}(G(rP_3)) = r\Delta$ for for even $r\Delta$, and $ndi_{\sum}(G(rP_3)) \leq r\Delta + 1$ for odd $r\Delta$.*

Proof. Note that G is a subgraph of a Δ-regular graph H. For even $r\Delta$, by theorem 2.5, $H(rP_2)$ can be decomposed into $\frac{r\Delta}{2}$ 2-factors. For $k = 3$, label the replacement of each 2-factor as follows: 1Δ, $i(\Delta - i)$, where i is in $[2, r\Delta]$. Then it is easy to see that the labeling is a neighbor-distinguishing coloring of $H(rP_3)$. Then $ndi_{\sum}(H(rP_3)) = r\Delta$ for even $r\Delta$.

For odd $r\Delta$, similar to the proof of Theorem 2.6, we have $ndi_{\sum}(H(rP_3)) \leq r\Delta + 1$ for odd $r\Delta$.

The neighbor-distinguishing coloring as above cannot work on $G(rP_3)$. If there exists a path P_s whose length is larger than 3 in $G(rP_3)$, then We modify the neighbor-distinguishing coloring of $H(rP_3)$ as follows. Suppose the label of the path P_s is $\underline{ab}ab$. For $a, b \neq 1, 2$, modify the label of the path P_s as follows: $\underline{a12}a1b$ for $s-1 \equiv 0 \pmod 3$ and $s \geq 7$; $\underline{a12}b$ for $s-1 \equiv 1 \pmod 3$ and $s \geq 5$; $\underline{a12}ab12b$ for $s-1 \equiv 2 \pmod 3$ and $s \geq 9$. We next consider the case that one in $\{a, b\}$ is 1 or 2, the other is larger than 2.

Without loss of generality, suppose $a = 1$. modify the label of the path P_s as follows: $\underline{12b}$ for $s-1 \equiv 0 \pmod 3$ and $s \geq 7$; $\underline{132}b$ for $s-1 \equiv 1 \pmod 3$ and $s \geq 5$; $\underline{12b}1231b$ for $s-1 \equiv 2 \pmod 3$ and $s \geq 9$.

Similarly, without loss of generality, suppose $a = 2$. modify the label of the path P_s as follows: $\underline{21b}$ for $s-1 \equiv 0 \pmod 3$ and $s \geq 7$; $\underline{231}b$ for $s-1 \equiv 1 \pmod 3$ and $s \geq 5$; $\underline{21b}2132b$ for $s-1 \equiv 2 \pmod 3$ and $s \geq 9$.

It is easy to see that the labeling is a neighbor-distinguishing coloring of $G(rP_3)$. Thus, for $3 \leq k \leq 4$, $ndi_{\sum}(G(rP_3)) \leq r\Delta$ for for even $r\Delta$, and $ndi_{\sum}(G(rP_3)) \leq r\Delta + 1$ for odd $r\Delta$. ∎

6 Note

By Theorem 4.1 and 4.2, and $tndi_S(G(rP_k)) \leq tndi_{\sum}(G(rP_k))$, we close by noticing that Conjecture 1.2 and 1.3 are true for the graphs $G(rP_k)$ with $k \geq 3$ and $r \geq 1$.

By Theorem 5.1, Theorem 5.2 and Theorem 5.3, we close by noticing that Conjecture 1.1 is true for the graphs $G(rP_k)$ with $k \geq 3$ and $r \geq 1$.

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