Spanning 3-ended trees in k-connected claw-free graphs *

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Abstract. Let $\sigma_k(G)$ denote the minimum degree sum of k independent vertices of a graph G. A spanning tree with at most 3 leaves is called a spanning 3-ended tree. In this paper, we prove that for any k-connected claw-free graph G with $|G| = n$, if $\sigma_{k+3}(G) \geq n - k$, then G contains a spanning 3-ended tree.

Keywords: spanning 3-ended tree, claw-free graph, non-insertible vertex

1 Introduction

In this paper, only finite and simple graphs are considered, and we refer to [1] for notation and terminology not defined here. If a graph G has no $K_{1,3}$ induced subgraph, then G is *claw-free*. $N_H(S) = \{v : v \in V(H)$ and $uv \in E(G)$ for some vertex $u \in V(S)\}$, and $d_H(S) = |N_H(S)|$. Let $N(v) = \{u : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. We use $\sigma_k(G)$ to denote the minimum degree sum of all the independent sets with order k in G. If any two distinct vertices in a graph G can be the end vertices of a hamilton path of G, then G is a Hamilton-connected.

$P[a, b]$ (or aPb) denotes a path with end vertices a, b along the positive orientation of P. For a path $P[a, b]$, $x, y \in V(P)$, let xPy denote the subpath with endvertices x, y along the positive orientation of P, and yP^-x

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denote the subpath with endvertices y, x along the negative orientation of P. Let $w(G)$ denote the number of components of a graph G.

A spanning tree with at most m leaves is called a spanning m-ended tree. Broersma and Tuinstra[2] gave the following sufficient conditions to contain a spanning m-ended tree for a graph G.

Theorem 1.1 ([2]) *Let $m \geq 2$ and G be a connected graph of order $n \geq 2$. If $\sigma_2(G) \geq n - m + 1$, then G contains a spanning m-ended tree.*

Kyaw[6,7] gave the following sufficient conditions for a connected $K_{1,4}$-free graph to contain a spanning 3-ended tree and a spanning m-ended tree, respectively.

Theorem 1.2 ([6]) *If G is a connected $K_{1,4}$-free graph and $\sigma_4(G) \geq |G| - 1$, then G contains a spanning 3-ended tree.*

Theorem 1.3 ([7]) *Let G be a connected $K_{1,4}$-free graph.* Then

(i) *G contains a hamiltonian path if $\sigma_3(G) \geq |G|$.*

(ii) *G contains a spanning m-ended tree if $\sigma_{m+1}(G) \geq |G| - \frac{m}{2}$ for an integer $m \geq 3$.*

Kano et al.[5] gave the following sufficient conditions for a connected claw-free graph to contain spanning m-ended trees.

Theorem 1.4 ([5]) *If G is a connected claw-free graph of order n and $\sigma_{m+1}(G) \geq n - m$ ($m \geq 2$), then G contains spanning m-ended trees. And the bound is sharp.*

Recently, Chen, Chen, and Hu[3] gave the following sufficient conditions for k-connected $K_{1,4}$-free graphs to contain spanning 3-ended trees.

Theorem 1.5 ([3]) *If G is a k-connected $K_{1,4}$-free graph of order n with $k \geq 2$ and $\sigma_{k+3}(G) \geq n + 2k - 2$, then G contains a spanning 3-ended tree.*

Inspired by Theorems 1.4 and 1.5, in this paper we explore sufficient conditions for k-connected claw-free graphs to contain spanning 3-ended trees.

Theorem 1.6 *If G is a k-connected claw-free graph of order n and $\sigma_{k+3}(G) \geq n - k$, then G contains a spanning 3-ended tree.*

By Theorem 1.4, if G is a claw-free graph and $\sigma_4 \geq n - 3$, then G contains a spanning 3-ended tree and the bound is sharp. By Theorem 1.6, if G is a connected graph and $\sigma_4 \geq n-1$, then G contains a spanning 3-ended tree. It follows that the bound in Theorem 1.6 is not best possible, and here we propose the following conjecture.

Conjecture *If G is a k-connected claw-free graph of order n and $\sigma_{k+3}(G) \geq n-k-2$, then G contains spanning 3-ended trees.*

2 Preliminaries

In the proof of Theorem 1.6, we mainly use the definition and properties of insertable vertices defined in [4].

Suppose that G is a connected non-hamiltonian graph and C is a longest cycle in G with counter-clockwise direction as positive orientation. Assume that H is a component of $G-C$ and $N_C(H) = \{v_1, v_2, \cdots, v_t\}$ such that $v_1, v_2, \cdots, v_t$ are labeled in order along the positive direction of C. Let $Q_i = C(v_i, v_{i+1}]$, $1 \leq i \leq t-1$, and $Q_t = C(v_t, v_1]$. A vertex v in Q_i is an *insertible vertex* if v has consecutive neighbors u and u^+ in $C - Q_i$.

In [4], Chen and Schelp gave the following two lemmas. We will use them in the proofs of Theorem 1.6.

Lemma 2.1 ([4]) *For each Q_i, there is a non-insertible vertex in $Q_i - \{v_{i+1}\}$.*

For each Q_i, let w_i be the first non-insertible vertex in $Q_i - \{v_{i+1}\}$. Then the following lemma holds.

Lemma 2.2 ([4]) *Let $1 \leq i < j \leq t$. Then for $x_i \in C[v_i^+, w_i]$ and $x_j \in C[v_j^+, w_j]$, the following properties hold:*

(1) *There does not exist a path $P[x_i, x_j]$ in G such that $P[x_i, x_j] \cap V(C) = \{x_i, x_j\}$.*

(2) *For every $v \in C[x_i^+, x_j^-]$, if $vx_i \in E(G)$, then $v^- x_j \notin E(G)$. Similarly, for every $u \in C[x_j^+, x_i^-]$, if $ux_j \in E(G)$, then $u^- x_i \notin E(G)$.*

(3) *For every $v \in C[x_i, x_j]$, if $vx_i, vx_j \in E(G)$, then $v^- v^+ \notin E(G)$.*

Suppose for some $i \in [1, t]$, $N(w_i) \cap V(G-C-H) \neq \emptyset$ and w_i' is the second non-insertible vertex in $Q_i - \{v_{i+1}\}$. Then Chen, Chen and Hu[3] gave the following result.

Lemma 2.3 ([3]) *Let $1 \leq i < j \leq t$, $x_i \in C[w_i^+, w_i']$ and $x_j \in C[v_j^+, w_j]$. Then*

(1) *there does not exist a path $P[x_i, x_j]$ in G such that $P[x_i, x_j] \cap$*

$V(C) = \{x_i, x_j\}$.

(2) *for every* $v \in C[x_i^+, x_j^-]$, *if* $vx_i \in E(G)$, *then* $v^-x_j \notin E(G)$. *Similarly, for every* $u \in C[x_j^+, x_i^-]$, *if* $ux_j \in E(G)$, *then* $u^-x_i \notin E(G)$.

(3) *for every* $v \in C[x_i, x_j]$, *if* $vx_i, vx_j \in E(G)$, *then* $v^-v^+ \notin E(G)$.

3 Proof of Theorem 1.6

Let G be a graph satisfying the conditions of Theorem 1.6. Suppose to the contrary, any spanning tree in G contains more than 3 leaves. Let $P = P[a, b]$ be a longest path in G such that P satisfies the following two conditions:

(T1) $w(G - P)$ is minimum;

(T2) subject to (T1), $|P[a, v]|$ is minimum, where v is the first vertex of P with $N(v) \cap V(G - P) \neq \emptyset$.

Let G' denote a graph with $V(G') = V(G) \cup \{v_0\}, E(G') = E(G) \cup \{v_0v : v \in V(G)\}$. Then the cycle $C = v_0P[a, b]v_0$ is a maximum cycle of G'. We define the counter-clockwise orientation as the positive direction of C. Let $N_P(H) = \{v_1, v_2, \cdots, v_t\}$, $Q_i = C(v_i, v_{i+1}]$ for $0 \leq i \leq t - 1$ and $Q_t = C(v_t, v_0)$. By Lemma 2.1, let w_i denote the first non-insertible vertex in $Q_i - \{v_{i+1}\}$ for $0 \leq i \leq t$ and $W = \{w_0, w_1, \cdots, w_t\}$. By Lemma 2.2(1), W is an independent set.

Obviously, C can be divided into disjoint intervals $T = C[c, d]$ with $c, d^+ \notin N(W)$ and $C[c^+, d] \subseteq N(W)$. We call the intervals *W-segments.* If $c = d$, then $C[c^+, d] = \emptyset$, i.e., if $|T| = 1$, then $d_W(T) = 0$. By the definition of W-segment, for any W-segment T, there exists $l \in [0, t]$ such that $T \subseteq C[w_l, w_{l+1}^-]$ (subscripts expressed modulo $t + 1$).

Claim 1. *$a = w_0$ and $b \notin N(w_i)$ for $i \in [0, t - 1]$.*

Proof. Suppose that a is an insertable vertex such that there exists a vertex $v \in C - Q_0$ with $av, av^+ \in E(G)$. If $v \neq b$, then we can get a path $P' = P[a^+, v]aP[v^+, b]$. If $v = b$, then we can get a path $P' = P[a^+, b]a$. In any case, $|V(P')| = |V(P)|$ and $|P[a^+, v_1]| < |P[a, v_1]$, a contradiction with (T2). Thus $a = w_0$.

Suppose $w_ib \in E(G)$, for some $i \in [0, t - 1]$. Obviously, $v_0 = b^+$ and $v_0w_i \in E(G)$. It follows that w_i is an insertable vertex, a contradiction. Thus $b \notin N(w_i)$ for $i \in [0, t - 1]$. □

Claim 2. *$d_W(v_i) = 0$, for any vertex v_i and any integer $i \in [1, t]$.*

Proof. Suppose $i = 1, w_j \in N_W(v_1)$ and $y \in N_H(v_1)$, $j \in [0,t]$. Since $G[v_1, w_j, v_1^-(v_1^+), y] \neq K_{1,3}$ and $yv_1^-, yv_1^+, w_jy \notin E(G)$, $w_jv_1^-, w_jv_1^+ \in E(G)$. Since $v_1w_j, w_jv_1^- \in E(G)$ and w_j is a non-insertible vertex, $w_j = w_0$. Then by $w_jv_1^+ \in E(G)$, we can get a cycle $C' = v_0yC^-[v_1, w_0]C[v_1^+, v_0]$ longer than C, a contradiction. Thus $d_W(v_1) = 0$.

Suppose $i \in [2,t], w_j \in N_W(v_i)$ and $y \in N_H(v_i)$, $j \in [0,t]$. By the proof of preceding case $i = 1$, $w_j \in C(v_{i-1}, v_i)$, i.e., $w_j = w_{i-1}$, and $w_{i-1}v_i^+ \in E(G)$. Obviously, all the vertices in $C(v_{i-1}, w_{i-1})$ can be inserted into $C(v_i, v_{i-1})$. Let $v_iP_1v_{i-1}$ denote the path with $V(P_1) = C(v_{i-1}, w_{i-1}) \cup C[v_i, v_{i-1}]$ obtained by the inserting process, and $v_{i-1}P_Hv_i$ denote a path connecting v_{i-1} and v_i with internal vertices in H. Then we can get a cycle $C' = C^-[v_i, w_{i-1}]v_i^+P_1v_{i-1}P_Hv_i$ longer than C, a contradiction. Thus $d_W(v_i) = 0$ for $i \in [2,t]$, and then by the case $i = 1$, the claim holds. □

Claim 3. *For any vertex $u \in T$, $d_W(u) \leq 1$ and $d_W(T) = |T| - 1$, where $T \subseteq C[w_j, w_{j+1}^-]$ for any integer $j \in [0, t-1]$.*

Proof. Suppose $|T| = 1$ and $T = \{u\}$. Then $d_W(u) = 0$ and $d_W(T) = |T| - 1 = 0$. Suppose $|T| \geq 2$, and $T = \{x, x_1, x_2, \cdots, x_h\}$, where $x, x_1, x_2, \cdots, x_h$ are labeled in order along the positive direction of C. Then $x \notin N(W)$, $\{x_1, x_2, \cdots, x_h\} \subseteq N(W)$. Assume that there exists a vertex $x_i \in T$ such that $d_W(x_i) \geq 2$, for some $i \in [1,h]$. Suppose $\{w_{j_1}, w_{j_2}\} \subseteq N_W(x_i)$ and $0 \leq j_1 < j_2 \leq t$. Then by Lemma 2.2(3), $x_i^-x_i^+ \notin E(G)$. Since $G[x_i, w_{j_1}, x_i^-, x_i^+] \neq K_{1,3}$, $w_{j_1}x_i^- \in E(G)$ or $w_{j_1}x_i^+ \in E(G)$. Similarly, $w_{j_2}x_i^- \in E(G)$ or $w_{j_2}x_i^+ \in E(G)$. Since $j_1 < j_2$, $x_i^-w_{j_1}, x_i^+w_{j_2} \in E(G)$ by Lemma 2(2). Since $x_iw_{j_1}, x_iw_{j_2} \in E(G)$ and w_{j_1}, w_{j_2} are non-insertible vertices, $x_i = v_{j+1}$, $w_{j_1} = w_j$, $w_{j_2} = w_{j+1}$, a contradiction to Claim 2. Thus Claim 3 holds. □

Claim 4. *If $T \subseteq C[w_t, w_0^-]$*, then $d_W(u) \leq 1$ for any vertex $u \in T - \{v_0\}$.

Proof. Suppose $b \notin T$. Then by the proof of Claim 3, for any vertex $u \in T$, $d_W(u) \leq 1$. Suppose $b \in T$ and $b \in N(W)$. Then by Claim 1, $N_W(b) = \{w_t\}$. By Lemma 2.2(2), $N_W(v) \subseteq \{w_t\}$ and then $d_W(v) \leq 1$ for any vertex $v \in T - \{v_0\}$. If $b \in T$ and $b \notin N(W)$, then $T = \{b, v_0\}$ and $d_W(b) = 0$. □

Claim 5. $\sum\limits_{i=0}^{t} d_P(w_i) \leq |P| - 2t - 1$.

Proof. By Claim 2, $d_P(W)$ is maximal if and only if $u \in N(W)$, for any vertex $u \in V(P) - \{v_1, v_2, \cdots, v_t\} \cup W$. By Claim 3 and Claim 4, $\sum\limits_{i=0}^{t} d_P(w_i) \leq |P| - 2t - 1$. □

Claim 6. *If $z_1, z_2 \in V(G-P)$ and $z_1z_2 \notin E(G)$, then $N_P(z_1) \cap N_P(z_2) = \emptyset$.*

Proof. Suppose $v_i \in N_P(z_1) \cap N_P(z_2), i \in [1,t]$. Since $v_i^- z_1, v_i^- z_2, z_1z_2 \notin E(G)$, $G[v_i, v_i^-, z_1, z_2] = K_{1,3}$, a contradiction. Thus $N_P(z_1) \cap N_P(z_2) = \emptyset$. □

Claim 7. *For any component H of $G-P$, $|N_P(H)| = k$.*

Proof. By Lemma 2.2(1), for $0 \le i \ne j \le t$, $N_{G-P}(w_i) \cap N_{G-P}(w_j) = \emptyset$, and then $\sum_{i=0}^{t} d_{G-P}(w_i) \le n - |P| - |H|$. Since G is k-connected, $t \ge k$. If $t \ge k+2$, then $\{w_0, w_1, \cdots, w_t\}$ is an independent set with order at least $k+3$. By Claim 5, $\sum_{i=0}^{t} d(w_i) = \sum_{i=0}^{t} d_P(w_i) + \sum_{i=0}^{t} d_{G-P}(w_i) \le (|P| - 2t - 1) + (n - |P| - |H|) = n - 2t - 1 - |H|$, a contradiction to $\sigma_{k+3}(G) \ge n-k$. Suppose $t = k+1$ and $u \in V(H)$. Then $\{u, w_0, w_1, \cdots, w_t\}$ is an independent set with order $k+3$. Since $N(u) \subseteq \{v_1, v_2, \cdots, v_t\} \cup (H - \{u\})$, $d(u) \le t + |H| - 1 = k + |H|$. By Claim 5, $\sum_{i=0}^{k+1} d(w_i) + d(u) = \sum_{i=0}^{k+1} d_P(w_i) + \sum_{i=0}^{k+1} d_{G-P}(w_i) + d(u) \le (|P| - 2(k+1) - 1) + (n - |P| - |H|) + k + |H| = n-k-3$, a contradiction to $\sigma_{k+3}(G) \ge n-k$. Thus $t = k$. □

By Claim 6 and Claim 7, we can get the following result.

Claim 8. *Suppose $z_1, z_2 \in V(G-P)$ and $z_1z_2 \notin E(G)$. Then $d_P(z_1) + d_P(z_2) \le k$.*

Claim 9. *For any component H of $G-P$, H is hamiltonian-connected.*

Proof. Suppose that H is not hamiltonian-connected. Then by Ore's theorem in [8], there exist two nonadjacent vertices z_1 and z_2 such that $d_H(z_1) + d_H(z_2) \le |H|$. By Claim 8, $d_P(z_1) + d_P(z_2) \le k$. Notice that $\{z_1, z_2, w_0, w_1, \cdots, w_k\}$ is an independent set with order $k+3$, and by Claim 5, we have that $\sum_{i=0}^{k} d(w_i) + d(z_1) + d(z_2) = \sum_{i=0}^{k} d_P(w_i) + \sum_{i=0}^{k} d_{G-P}(w_i) + d_P(z_1) + d_P(z_2) + d_H(z_1) + d_H(z_2) \le (|P| - 2k - 1) + (n - |P| - |H|) + k + |H| = n - k - 1$, a contradiction to $\sigma_{k+3}(G) \ge n-k$. □

Claim 10. *For any two distinct vertices $v_i, v_j \in N_P(H)$, $|N_H(v_i) \cup N_H(v_j)| \ge 2, i, j \in [1,t]$.*

Proof. Suppose there exist two distinct vertices $v_i, v_j \in N_P(H)$ and a vertex $u \in V(H)$ such that $N_H(v_i) \cup N_H(v_j) = \{u\}$. Then $N_P(H) \cup \{u\} - \{v_i, v_j\}$ is a vertex cut of order $k-1$ of G, a contradiction to the connectedness of G. □

Suppose $w(G-P) = 1$, then by Claims 9-10, G contains a spanning 3-ended tree. Thus we assume that $w(G-P) \geq 2$ and H' is a component in $G-P-H$.

Claim 11. *$N(w_i) \cap V(H') \neq \emptyset$ for some $1 \leq i \leq k$.*

Proof. By Claim 1, $N(w_0) \cap V(H') = \emptyset$. Suppose $N(w_i) \cap V(H') = \emptyset$ for any $i \in [1, k]$. Let $z_1 \in V(H), z_2 \in V(H')$. Then $\{z_1, z_2, w_0, w_1, \cdots, w_k\}$ is an independent set of order $k+3$. By Claim 8, $d_P(z_1) + d_P(z_2) \leq k$. By Lemma 2.2(1), $\sum_{i=0}^{k} d_{G-P}(w_i) \leq n - |P| - |H| - |H'|$. Then $\sum_{i=0}^{k} d(w_i) + d(z_1) + d(z_2) = \sum_{i=0}^{k} d_P(w_i) + \sum_{i=0}^{k} d_{G-P}(w_i) + d_P(z_1) + d_P(z_2) + d_H(z_1) + d_{H'}(z_2) \leq (|P| - 2k - 1) + (n - |P| - |H| - |H'|) + k + |H| - 1 + |H'| - 1 = n - 3 - k$, a contradiction to $\sigma_{k+3}(G) \geq n-k$. □

By Claim 11, we assume $N(w_i) \cap V(H') \neq \emptyset$ for some $i \in [1, k]$. By Lemma 2.2(1), $N(w_j) \cap V(H') = \emptyset$ for any $j \in [0, k] - \{i\}$.

Claim 12. *There exists a second non-insertible vertex w_i' in $Q_i - \{v_{i+1}\}$ and $w_i' \notin N(H')$.*

Proof. Suppose $Q_i - \{v_{i+1}\}$ contains only one non-insertible vertex w_i. Let $v_{i+1} P_1 v_i$ denote the path with $V(P_1) = V(C) - \{w_i\}$ obtained by inserting all the vertices in $Q_i - \{w_i, v_{i+1}\}$ into $C[v_{i+1}, v_i]$. Suppose $H = \{u\}$. Then $v_i, v_{i+1} \in N(u)$, and we can get a path $C' = v_{i+1} P_1 v_i u v_{i+1}$. Let $P' = C' - \{v_0\}$, then $w(G - P' - H) < w(G - P - H)$, a contradiction to (T1). Suppose $|H| \geq 2$. By Claims 9-10, assume $v_i H v_{i+1}$ is a hamilton path of $H \cup \{v_{i+1}, v_i\}$. Then we can get a cycle $C' = v_{i+1} P_1 v_i H v_{i+1}$ longer than C, a contradiction. □

By Claim 12 and Lemma 2.3(1), $W' = \{w_0, \cdots, w_{i-1}, w_i', w_{i+1}, \cdots, w_k\}$ is an independent set. By the preceding proof and Lemma 2.3, W' has the same properties as W. Then by Claim 5, $\sum_{w \in W'} d_P(w) \leq |P| - 2k - 1$.

Now, we complete the proof of Theorem 1.6. Let $z_1 \in V(H), z_2 \in V(H')$. Then $W' \cup \{z_1, z_2\}$ is an independent set of order $k+3$ in G. By Lemma 2.3(1), $\sum_{w \in W'} d_{G-P}(w) \leq n - |P| - |H| - |H'|$. By Claim 8,

$d_P(z_1) + d_P(z_2) \leq k$. Obviously, $d_H(z_1) \leq |H| - 1, d_{H'}(z_2) \leq |H'| - 1$. Thus $\sum_{w\in W'} d(w) + d(z_1) + d(z_2) = \sum_{w\in W'} d_P(w) + \sum_{w\in W'} d_{G-P}(w) + d_P(z_1) + d_P(z_2) + d_H(z_1) + d_{H'}(z_2) \leq (|P| - 2k - 1) + (n - |P| - | H| - | H'|) + k + (|H| - 1) + (|H'| - 1) = n - 3 - k$, a contradiction to $\sigma_{k+3}(G) \geq n - k$. It follows that Theorem 1.6 holds. □

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