Counting rooted nonseparable unicursal planar maps*

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Abstract

A map is unicursal if all its vertices are even-valent except two odd-valent vertices. This paper investigates the enumeration of rooted nonseparable unicursal planar maps and provides two functional equations satisfied by its generating functions with the number of nonrooted vertices, the number of inner faces (or the number of edges) and the valencies of the two odd vertices of maps as parameters.

Keywords: Nonseparable unicursal map; Enumerating function; Functional equation

MSC(2000): 05C45; 05C30

1. Introduction

Throughout this paper we consider the rooted maps on the plane. Definitions of terms not given here may be found in [14].

The enumeration of rooted planar maps was initiated by W.T. Tutte in the early of 1960's for attacking the Four Color Problem. His series of census papers [21–23] laid the foundation for the theory. Since then, the theory has been developed by many scholars such as Arquès [1], Brown [6], Mullin et al. [19], Bender et al. [2,3], Liskovets et al. [9–11], Bousquet-Mélou et al. [4,5], Walsh et al. [24], Mednykh et al. [20], Gao [7,8] and Liu [12–14].

A sum-free formula for the number of rooted unicursal planar maps with a given number of edges first appeared in [10]. In that paper, Liskovets and Walsh had also found a sum-free formula for the number of unicursal maps rooted in a vertex of odd valency and a formula for the number of rooted unicursal maps as a function of the odd vertex valencies. Several years later the enumeration of rooted unicursal planar maps with the valencies of the two odd vertices and the number of edges or the number of inner

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faces, the number of nonrooted vertices and the valencies of the two odd vertices of maps as parameters was investigated by Long and Cai [15,16]. Two summation-free formulae were obtained. In 2013, Long and Cai [17] treated the enumeration of 4-regular unicursal planar maps with the number of nonrooted vertices and the valencies of the two odd vertices as three parameters and obtained two sum-free formulae. In 2014, Long and Cai [18] investigated the enumeration of loopless unicursal planar maps with the valencies of the two odd vertices and the number of edges as three parameters and obtained several explicit expressions of its enumerating functions.

In this paper, we will try to enumerate nonseparable unicursal planar maps rooted in a vertex of odd valency with parameters: the number of nonrooted vertices, the number of inner faces (or the number of edges) and the valencies of the two odd vertices. Two enumerating equations will be derived.

We now define some basic concepts and terms. A *map* is a 2-cell imbedding of a connected graph on a surface, which in this article is assumed to be a sphere. *Rooting* a map means distinguishing an edge-vertex incidence pair (e, v) as the *root* (a loop is considered to be incident twice to the same vertex), the edge e as the *root-edge* and the vertex v as the *root-vertex*. The *root-face*, also called the *outer face*, is the face incident to the root-edge and on its right as seen by an observer facing away from the root-vertex. A planar map with a rooting is said to be a *rooted planar map*.

A map is called *eulerian* if all the valencies of its vertices are even and a map or graph is generally called *unicursal* if it possesses an eulerian walk, not necessarily a circuit. It is well known that a map (or connected graph) is unicursal if and only if it contains no more than two vertices of odd valency. For the sake of brevity we abuse the term and call a map unicursal if it has exactly two vertices of odd valency. An *endpoint* is a vertex of valency 1; a unicursal map evidently can have at most two such vertices.

For a rooted planar map M, a vertex v of M is said to be a *separating* vertex if there are submaps M_1 and M_2 of M such that $M = M_1 \bigcup M_2$ with $M_1 \bigcap M_2 = \{v\}$ where $\epsilon(M_1) > 0$ and $\varepsilon(M_2) > 0$ ($\varepsilon(M_i)$ denotes the number of edges of $M_i, i = 1, 2$). A *nonseparable* unicursal planar map is a unicursal planar map in which there is no separating vertex. Clearly, there is no loop in a nonseparable unicursal planar map.

For any map $M \in \mathcal{M}$, let $M - e_r(M)$ and $M \bullet e_r(M)$ be the maps obtained by deleting $e_r(M)$, the root-edge, from M and contracting $e_r(M)$ into a vertex as the new root-vertex, respectively.

Given two maps M_1 and M_2 with roots $r_1 = r(M_1)$ and $r_2 = r(M_2)$, respectively, we define $M = M_1 \dot{+} M_2$ to be the map obtained by identifying the root-vertices and the root-faces of M_1 and M_2 and rooting M at r_1. The operation for getting M from M_1 and M_2 is called $1v$-*addition*. Further, for two sets of maps $\mathscr{U}_1$ and $\mathscr{U}_2$, the set of maps

$$\mathscr{U}_1 \odot \mathscr{U}_2 = \{M_1 \dot{+} M_2 \mid M_i \in \mathscr{U}_i, i = 1, 2\}.$$

is said to be $1v$-*production* of $\mathscr{U}_1$ and $\mathscr{U}_2$.

Let $\mathscr{U}$ and $\widetilde{\mathscr{U}}$ denote rooted nonseparable eulerian planar maps and rooted nonseparable unicursal planar maps, respectively. Define their enumerating functions as

$$f_{\mathscr{U}}(x,y,z)=\sum_{U\in\mathscr{U}} x^{2m(U)}y^{n(U)}z^{s(M)},$$

$$f_{\widetilde{\mathscr{U}}}(x,y,z,u)=\sum_{M\in\widetilde{\mathscr{U}}} x^{2m(M)+1}y^{n(M)}z^{s(M)}u^{2t(M)+1},$$

where $2m(U), n(U)$ and $s(U)$ are respectively the root-vertex valency, the numbers of nonrooted vertices and inner faces of U, $2m(M)+1, n(M), s(M)$ and $2t(M)+1$ are the root-vertex valency, the numbers of nonrooted vertices, inner faces and the valency of the nonrooted odd-valent vertex of M, respectively.

2. Some Lemmas

In this section, some useful lemmas will be obtained.

Let O_1 and O_2 be two maps. Assume that o_1 and o_2 are the root-vertices of O_1 and O_2, respectively. Write

$$o_2=(r_2,\mathcal{J}_2r_2,\mathcal{J}_2^2r_2,\cdots,\mathcal{J}_2^{-1}r_2)=(S_1,S_2)$$

and

$$o_1=(r_1,\mathcal{J}_1r_1,\mathcal{J}_1^2r_1,\cdots,\mathcal{J}_1^{-1}r_1)=(S),$$

where S_1, S_2 and S are in linear order. Then the map $O_1+\cdot O_2$ defined to be $O_1\cup O_2$ provided that $O_1\cap O_2=\{o\}, o=o_1=o_2$ and that the root-vertex o of $O_1+\cdot O_2$ as

$$o=(S_1,S,S_2)$$

is called the inner $1v$-addition of O_1 and O_2. Define

$$O_1+\cdot O_2+\cdots+\cdot O_k=k\times\cdot O$$

when $O_1=O_2=\cdots=O_k=O$. If $k=0$, then $k\times\cdot O$ is defined to be the empty map with no vertices.

Further, for two sets of maps, $\mathcal{Q}_1$ and $\mathcal{Q}_2$, let

$$\mathcal{Q}_1\times\cdot\mathcal{Q}_2=\{O_1+\cdot O_2\mid O_1\in\mathcal{Q}_1, O_2\in\mathcal{Q}_2\}$$

be called the *inner* $1v$*-product* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Define

$$\mathcal{Q}_1\times\cdot\mathcal{Q}_2\times\cdots\times\cdot\mathcal{Q}_n=\mathcal{Q}^{\times\cdot n}$$

when $\mathcal{Q}_1=\mathcal{Q}_2=\cdots=\mathcal{Q}_n$.

Let $\mathscr{U}$ and $\widetilde{\mathscr{U}}$ be the sets of all rooted nonseparable eulerian planar maps and nonseparable unicursal planar maps rooted in a vertex of odd valency, respectively. Now, we find that $\mathscr{U}$ can be partitioned into two parts, i.e.,

$$\widetilde{\mathscr{U}}=\widetilde{\mathscr{U}}_1+\widetilde{\mathscr{U}}_2, \qquad (1)$$

where $\widetilde{\mathscr{U}_1} = \{M \mid M \in \widetilde{\mathscr{U}}$, the nonrooted end of $e_r(M)$ is an odd-valent vertex$\}$.

Further, we have

$$\widetilde{\mathscr{U}_1} = \widetilde{\mathscr{U}_{11}} + \widetilde{\mathscr{U}_{12}}, \tag{2}$$

in which $\widetilde{\mathscr{U}_{11}} = \{L_0\}$, L_0 denotes the link map. In what follows we will define an operation.

For two maps $M_1 = (\mathcal{X}_1, \mathcal{J}_1)$ and $M_2 = (\mathcal{X}_2, \mathcal{J}_2)$ with the roots r_1 and r_2, respectively, the composition of M_1 and M_2 into $M = (\mathcal{X}, \mathcal{J})$ is called the *root addition* of M_1 and M_2 and written as

$$M = M_1 + |_r\, M_2, \tag{3}$$

if $\mathcal{X} = \mathcal{X}_1 + (\mathcal{X}_2 - Kr_2)$ and $\mathcal{J}$ is induced from $\mathcal{J}_1$ and $\mathcal{J}_2$ with

$$v_r = (r_1, \mathcal{J}_2 r_2, \cdots, \mathcal{J}_2^{-1} r_2, \mathcal{J}_1 r_1, \cdots, \mathcal{J}_1^{-1} r_1)$$

and

$$v_{\beta r} = (\delta r_1, \mathcal{J}_1 \delta r_1, \cdots, \mathcal{J}_1^{-1} \delta r_1, \mathcal{J}_2 \delta r_2, \cdots, \mathcal{J}_2^{-1} \delta r_2)$$

as the root-vertex and the nonrooted end of the root-edge of M while $r = r_1$ is the root of M. It is easily checked that the operation of root addition does not satisfy the commutative law. Moreover, it can be shown that the operation satisfies the associative law. This allows us to write

$$\begin{aligned} \sum_{i=1}^{k} |_r\, M_i &= M_1 + |_r \sum_{i=2}^{k} |_r\, M_i \\ &= \sum_{i=1}^{k-1} |_r\, M_i + |_r\, M_k. \end{aligned} \tag{4}$$

For $M \in \widetilde{\mathscr{U}} - \widetilde{\mathscr{U}_{11}}$, let $r = r(M)$ be the root with the two ends of the root-edge being

$$v_r = (r, P), \tag{5}$$

where

$$P = < \mathcal{J} r, \mathcal{J}^2 r, \cdots, \mathcal{J}^{-1} r >$$

and

$$v_{\beta r} = (\delta r, Q), \tag{6}$$

where

$$Q = < \mathcal{J}\delta r, \mathcal{J}^2 \delta r, \cdots, \mathcal{J}^{-1} \delta r > .$$

If for an integer $j > 1$, there exist two integers $m \geq 1$ and $n \geq 1$ such that

$$(\mathcal{J}\delta)^m \mathcal{J}^j r = \mathcal{J}^n \delta r, \tag{7}$$

then P and Q are said to be *f-splittable* because

$$P = < P_1, P_2 >, \quad Q = < Q_2, Q_1 >,$$

where

$$P_1 = < \mathcal{J}r, \cdots, \mathcal{J}^{j-1}r >, \quad P_2 = < \mathcal{J}^j r, \cdots, \mathcal{J}^{-1}r >,$$
$$Q_2 = < \mathcal{J}\delta r, \cdots, \mathcal{J}^{n-1}\delta r >, \quad Q_1 = < \mathcal{J}^n \delta r, \cdots, \mathcal{J}^{-1}\delta r > .$$

Here, $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ are called *f-split pairs*. Of course, $j \neq 1$ because there always exists an integer l such that $(\mathcal{J}\delta)^l \mathcal{J}r = \delta r$ and $(\mathcal{J}\delta)^{l+1}\mathcal{J}r = \mathcal{J}r$. However, $j = val(v_r) - 1$ may occur, where $val(v_r)$ is the valency of root-vertex of M.

If for every $j > 1$, there do not exist $m \geq 1$ and $n \geq 1$ such that (7) holds, then P and Q are called *f-unsplittable.*

If P and Q are f-split into

$$P = < P_1, P_2, \cdots, P_k >, \quad Q = < Q_k, \cdots, Q_2, Q_1 >$$

such that every f-split pair $\{P_i, Q_i\}$ $(i = 1, \cdots, k)$ is f-unsplittable, then the number k is called the root-index of M. Let M_i be the maximal submap of

$$M - \bigcup_{j=1, j\neq i}^{k} \{P_j, Q_j\}$$

involved with $\{P_i, Q_i\}$ for $i = 1, 2, \cdots, k$. Now, we can present our results.

Lemma 2.1. For $M \in \widetilde{\mathscr{U}_{12}}$, we have

$$M = \sum_{i=1}^{k} |_r \ M_i, \tag{8}$$

such that k is the root-index of M, there is an even number of $M_i \in \mathscr{U} - L_1$ and the others are in $\widetilde{\mathscr{U}_{12}}$ among all M_i $(i = 1, 2, \cdots, k)$, where L_1 is the loop map.

Proof. From what are just discussed, we see that M has form (8). By considering the definition of $\mathscr{U}_{12}$, P_i and Q_i of M_i have the same parity and the number of P_i with odd lengths is even. Because the root-edge of M is a link, so is that of M_i and hence those with odd length of P_i is a member of $\mathscr{U} - L_1$, where L_1 is the loop map and those with even length of P_i is an element of $\widetilde{\mathscr{U}_{12}}$. The lemma follows. □

According to (8), we see the fact that

$$M \bullet a = \sum_{i=1}^{k} \cdot M_i \bullet a_i, \tag{9}$$

where a and a_i are respectively the root-edges of M and M_i $(i = 1, 2, \cdots, k)$, and the summation with a dot is the inner $1v$-addition. Meanwhile, we have to pay attention to the fact that the root-vertices of $M \bullet a$ and $M_i \bullet a_i$ are of the forms

$$(Q, P) \text{ and } (Q_i, P_i) \tag{10}$$

for $i = 1, 2, \cdots, k$, respectively, and that the following relations

$$p = \sum_{i=1}^{k} p_i = 0 \ (mod\ 2);$$
$$q = \sum_{i=1}^{k} q_i = 0 \ (mod\ 2);$$
$$p_i = q_i \ (mod\ 2) \tag{11}$$

are satisfied, where p, q, p_i and q_i are respectively the lengths of P, Q, P_i and Q_i $(i = 1, 2, \cdots, k)$.

Lemma 2.2. Let $\widetilde{\mathscr{U}}_{(12)}(q,p) = \{M \bullet a \mid M \in \widetilde{\mathscr{U}}_{12}$, the root-vertex of $M \bullet a$ is the vertex $(Q,P)\}$. Then we have

$$\widetilde{\mathscr{U}}_{(12)}(q,p) = \sum_{k\geq 1} \sum_{(\underline{q},\underline{p})\in\Omega} \prod_{i=1}^{k} \cdot \mathscr{U}(q_i, p_i), \tag{12}$$

where q, p and $(\underline{q}, \underline{p}) = (q_1, \cdots, q_k, p_1, \cdots, p_k)$ are as described in (11),

$$\Omega = \{(\underline{q}, \underline{p}) \mid (\underline{q}, \underline{p}) \text{ satisfied by (11)}\},$$

$$\mathscr{U}(q_i, p_i) = \{M \mid M \in \mathscr{U} \text{ with the rooted vertex } (Q_i, P_i)\}$$

and the operator $\prod$ with a dot represents the inner $1v$-product.

Proof. Let the set on the right side of (12) be denoted by $\widehat{\mathscr{U}}$ for convenience.

For any $M \in \widetilde{\mathscr{U}}_{(12)}(q,p)$, we can construct a map M' obtained by splitting the root vertex into o_1 and o_2 with an edge $a' = Kr'$, where

$$o_1 = (r', P) \text{ and } o_2 = (\delta r', Q)$$

from M such that $M = M' \bullet a', M' \in \widetilde{\mathscr{U}}_{12}$. From lemma 3.1, M' has the form shown in (8) and hence M is of the form (9) satisfying (10) and (11). By the definition of $\widetilde{\mathscr{U}}_{12}$ and the nonseparability of M', any map in the terms on the right side of (9) is allowed to be in $\mathscr{U}$ which includes the loop map L_1. From the relations (10) and (11), $M \in \widehat{\mathscr{U}}$. Thus $\widetilde{\mathscr{U}}_{(12)}(q,p) \subseteq \widehat{\mathscr{U}}$.

Conversely for any $M \in \widehat{\mathscr{U}}$, since M has the form as shown in (9) in the replacement of $M \bullet a$ and $M_i \bullet a_i$ by M and M_i respectively with (10) and (11), we may construct maps M' and M_i' from M and M_i by splitting the root-vertex $o = (Q, P)$ into $o'_1 = (r', P)$ and $o'_2 = (\delta r', Q)$ with an edge $a' = Kr'$, and the root-vertex $o_i = (Q_i, P_i)$ into $o'_{i1} = (r'_i, P_i)$ and $o'_{i2} = (\delta r'_i, Q_i)$ with an edge Kr'_i for $i = 1, 2, \cdots, k$, respectively. From the eulerianity and nonseparability of M_i with the relations (10) and (11), $M \in \widetilde{\mathscr{U}_{(12)}}$. Since $M = M' \bullet a'$, we have $M \in \widetilde{\mathscr{U}_{(12)}}(q, p)$. Hence $\widetilde{\mathscr{U}} \subseteq \widetilde{\mathscr{U}_{(12)}}(q, p)$. □

Similarly, we have

Lemma 2.3. For $M \in \widetilde{\mathscr{U}_2}$, we have

$$M = \sum_{i=1}^{k} |_r \, M_i \tag{13}$$

such that k is the root-index of M, there is only one map M_j $(1 \leq j \leq k)$ with exactly two odd vertices, one of them being not incident with $e_r(M_j)$ and the other being nonrooted end (or rooted end) of $e_r(M_j)$, there is an odd (or even) number of $M_i \in \mathscr{U}$ and the others are in $\widetilde{\mathscr{U}_{12}}$ among M_i $(i = 1, 2, \cdots, k)$.

Proof. From what are discussed before, we see that M has form (13). By considering the definition of $\widetilde{\mathscr{U}_2}$, there is only one map M_j $(1 \leq j \leq k)$ with exactly two odd vertices, one of them being not incident with $e_r(M_j)$ and the other being the nonrooted end (or rooted end) of $e_r(M_j)$, P_i and Q_i of M_i $(i \neq j)$ have the same parity and the number of P_i with odd lengths is odd (or even). Because the root-edge of M is a link, so is that of M_i and hence those with odd length of P_i $(i \neq j)$ is an element of $\mathscr{U} - L_1$ and those with even length of P_i $(i \neq j)$ is a member of $\widetilde{\mathscr{U}_{12}}$. The lemma follows. □

By (13) we also see the fact that

$$M \bullet a = \sum_{i=1}^{k} \cdot M_i \bullet a_i, \tag{14}$$

where a and a_i are the root-edges of M and $M_i, i = 1, 2, \cdots, k$, respectively, and the summation with a dot is the inner $1v$-addition. Meanwhile, we have to pay attention to the fact that the root-vertices of $M \bullet a$ and $M_i \bullet a_i$ are respectively of the forms

$$(Q, P) \; and \; (Q_i, P_i) \tag{15}$$

for $i = 1, 2, \cdots, k$ and that the following relations

$$p = \sum_{i=1}^{k} p_i = 0 \; (mod \; 2);$$

$$q = \sum_{i=1}^{k} q_i = 1 \; (mod \; 2);$$

$$(p_j + q_j) = 1 \ (mod\ 2);$$
$$p_i = q_i \ (mod\ 2) \ (i \neq j) \tag{16}$$

are satisfied, where p, q, p_i and q_i are the lengths of P, Q, P_i and Q_i ($i = 1, 2, \cdots, k$), respectively.

Lemma 2.4. Let $\widetilde{\mathscr{U}}_{(2)}(q,p) = \{M \bullet a \mid M \in \widetilde{\mathscr{U}_2}$ with the root-vertex $(Q,P)\}$. Then we have

$$\widetilde{\mathscr{U}}_{(2)}(q,p) = \sum_{k\geq 1} \sum_{(\underline{q},\underline{p})\in\Omega} \prod_{i=1}^{k} \cdot \mathscr{M}(q_i, p_i), \tag{17}$$

where q, p and $(\underline{q}, \underline{p}) = (q_1, \cdots, q_k, p_1, \cdots, p_k)$ are as described in (16),

$$\Omega = \{(\underline{q}, \underline{p}) \mid (\underline{q}, \underline{p}) \ satisfied\ by\ (16)\};$$

$$\begin{aligned}\mathscr{M}(q_i, p_i) =&\{M \mid M \in \widetilde{\mathscr{U}} \ with\ the\ rooted\ vertex\ (Q_j, P_j)\} \cup \\ &\{M \mid M \in \mathscr{U} \ with\ the\ rooted\ vertex\ (Q_i, P_i)\ (i \neq j)\}\end{aligned}$$

and the operator $\prod$ with a dot is the inner $1v$-product.

Proof. Let the set on the right side of (17) be denoted by $\widehat{\mathscr{M}}$ for convenience.

For any $M \in \widetilde{\mathscr{U}}_{(2)}(q,p)$, we can construct a map M' obtained by splitting the rooted vertex into o_1 and o_2 with an edge $a' = Kr'$, where

$$o_1 = (r', P) \ and\ o_2 = (\delta r', Q)$$

from M such that $M = M' \bullet a', M' \in \widetilde{\mathscr{U}_2}$. From lemma 2.3, M' has the form shown in (13) and therefore M is of the form (14) satisfying (15) and (16). By the definition of $\widetilde{\mathscr{U}_2}$ and the nonseparability of M', any map in the terms on the right side of (14) is allowed to be in $\mathscr{U}$ which includes the loop map L_1 except for one map in $\widetilde{\mathscr{U}}$. From the relations (15) and (16), $M \in \widehat{\mathscr{M}}$. Thus, $\widetilde{\mathscr{U}}_{(2)}(q,p) \subseteq \widehat{\mathscr{M}}$.

Conversely for any $M \in \widehat{\mathscr{M}}$, since M has the form as shown in (14) in the replacement of $M \bullet a$ and $M_i \bullet a_i$ by M and M_i respectively with (15) and (16), we may construct maps M' and M_i' from M and M_i by splitting the rooted vertex $o = (Q,P)$ into $o_1' = (r', P)$ and $o_2' = (\delta r', Q)$ with an edge $a' = Kr'$, and the rooted vertex $o_i = (Q_i, P_i)$ into $o_{i1}' = (r_i', P_i)$ and $o_{i2}' = (\delta r_i', Q_i)$ with an edge Kr_i' for $i = 1, 2, \cdots, k$, respectively. From $M_j \in \widetilde{\mathscr{U}}$ and the eulerianity and nonseparability of $M_i (i \neq j)$ with the relations (15–16), $M \in \widetilde{\mathscr{U}}_{(2)}$. Since $M = M' \bullet a'$, we have $M \in \widetilde{\mathscr{U}}_{(2)}(q,p)$. Therefore $\widehat{\mathscr{M}} \subseteq \widetilde{\mathscr{U}}_{(2)}(q,p)$. □

3. Main results

Theorem 2.1. The enumerating function $f = f_{\widetilde{\mathscr{U}}}(x, y, z, u)$ satisfies the following equation:

$$f = \frac{xyu[u^2(1+f_0) - x^2(1+f_0^*)]}{u^2(1+f_0)^2 - x^2(1+f_0^*)^2} + \frac{x^2y(xh-f)[(1+f_0)(1-f_0+2h_0) - x^2(1+h_0)^2]}{[(1+f_0)^2 - x^2(1+h_0)^2]^2}, \qquad (18)$$

where $f_0 = f_{\mathscr{U}}(x, y, z)$, $f_0^* = f_{\mathscr{U}}(u, y, z)$, $h_0 = h_{\mathscr{U}}(y, z) = f_{\mathscr{U}}(1, y, z)$ and $h = h_{\widetilde{\mathscr{U}}}(y, z, u) = f_{\widetilde{\mathscr{U}}}(1, y, z, u)$.

Proof. First, the contribution of $\widetilde{\mathscr{U}_{11}}$ is

$$f_{\widetilde{\mathscr{U}_{11}}} = xyu, \qquad (19)$$

because there is only one map, the link map in $\widetilde{\mathscr{U}_{11}}$.

Then, by Lemma 2.2, the contribution of $\widetilde{\mathscr{U}_{12}}$ is

$$\begin{aligned} f_{\widetilde{\mathscr{U}_{12}}} &= \sum_{M\in\widetilde{\mathscr{U}_{12}}} x^{2m(M)+1}y^{n(M)}z^{s(M)}u^{2t(M)+1} \\ &= xyu \sum_{M\in\widetilde{\mathscr{U}}_{(12)}} \sum_{r=1}^{m(M)-1} x^{2m(M)-2r}y^{n(M)}z^{s(M)}u^{2r} \\ &= xyu \sum_{k\geq 1}\sum_{l=0}^{\lfloor \frac{k}{2}\rfloor} \binom{k}{2l} \Delta_1^{2l}\Delta_2^{k-2l}, \end{aligned}$$

where

$$\Delta_1 = \sum_{M\in\mathscr{U}} \sum_{i=1}^{m(M)} x^{2m(M)-(2i-1)}y^{n(M)}z^{s(M)}u^{2i-1},$$

$$\Delta_2 = \sum_{M\in\mathscr{U}} \sum_{i=1}^{m(M)-1} x^{2m(M)-2i}y^{n(M)}z^{s(M)}u^{2i}.$$

Since

$$\Delta_1 = \sum_{M\in\mathscr{U}} \frac{xu[u^{2m(M)} - x^{2m(M)}]}{u^2 - x^2} y^{n(M)}z^{s(M)} = \frac{xu(f_0^* - f_0)}{u^2 - x^2},$$

$$\Delta_2 = \sum_{M\in\mathscr{U}} \frac{x^2u^{2m(M)} - u^2x^{2m(M)}}{u^2 - x^2} y^{n(M)}z^{s(M)} = \frac{x^2f_0^* - u^2f_0}{u^2 - x^2},$$

where $f_0^* = f_0(u, y, z)$; so we have

$$f_{\widetilde{\mathscr{U}}_{12}} = \frac{1}{2} xyu \sum_{k \geq 1} \left[(\Delta_2 + \Delta_1)^k + (\Delta_2 - \Delta_1)^k \right]$$
$$= \frac{xyu \left[(1 - \Delta_2)\Delta_2 + (\Delta_1)^2 \right]}{(1 - \Delta_2)^2 - (\Delta_1)^2}$$
$$= \frac{xyu[x^2 f_0^*(1 + f_0^*) - u^2 f_0(1 + f_0)]}{u^2(1 + f_0)^2 - x^2(1 + f_0^*)^2}. \tag{20}$$

Finally, by Lemma 2.4, the contribution of $\widetilde{\mathscr{U}_2}$ is

$$f_{\widetilde{\mathscr{U}_2}} = \sum_{M \in \widetilde{\mathscr{U}_2}} x^{2m(M)+1} y^{n(M)} z^{s(M)} u^{2t(M)+1}$$
$$= xy \sum_{M \in \widetilde{\mathscr{U}}_{(2)}} \sum_{r=0}^{m(M)-1} x^{2m(M)-2r} y^{n(M)} z^{s(M)} u^{2t(M)+1}$$
$$= xy \left\{ \left[\sum_{k \geq 2} \sum_{l=0}^{\lfloor \frac{k-2}{2} \rfloor} k \binom{k-1}{2l+1} \Delta_1'^{2l+1} \Delta_2'^{k-2l-2} \right] \right.$$
$$\times \left[\sum_{M \in \widetilde{\mathscr{U}}} \sum_{i=1}^{m(M)} x^{2m(M)-2i+1} y^{n(M)} z^{s(M)} u^{2t(M)+1} \right]$$
$$+ \left[\sum_{k \geq 1} \sum_{l=0}^{\lfloor \frac{k-1}{2} \rfloor} k \binom{k-1}{2l} \Delta_1'^{2l} \Delta_2'^{k-2l-1} \right]$$
$$\left. \times \left[\sum_{M \in \widetilde{\mathscr{U}}} \sum_{i=0}^{m(M)-1} x^{2m(M)-2i} y^{n(M)} z^{s(M)} u^{2t(M)+1} \right] \right\},$$

where

$$\Delta_1' = \sum_{M \in \mathscr{U}} \sum_{i=1}^{m(M)} x^{2m(M)-(2i-1)} y^{n(M)} z^{s(M)},$$
$$\Delta_2' = \sum_{M \in \mathscr{U}} \sum_{i=1}^{m(M)-1} x^{2m(M)-2i} y^{n(M)} z^{s(M)}.$$

Since

$$\Delta_1' = \sum_{M\in\mathscr{U}} \frac{x(1-x^{2m(M)})}{1-x^2} y^{n(M)} z^{s(M)} = \frac{x(h_0 - f_0)}{1-x^2},$$

$$\Delta_2' = \sum_{M\in\mathscr{U}} \frac{x^2-x^{2m(M)}}{1-x^2} y^{n(M)} z^{s(M)} = \frac{x^2 h_0 - f_0}{1-x^2},$$

where $h_0 = f_0(1, y, z)$, we get

$$\begin{aligned} f_{\widetilde{\mathscr{U}}_2} &= \frac{1}{2}xy\left\{\frac{xh-f}{1-x^2}\sum_{k\geq 2}\left[k(\Delta_2'+\Delta_1')^{k-1} - k(\Delta_2'-\Delta_1')^{k-1}\right]\right. \\ &\quad \left. + \frac{x^2h - xf}{1-x^2}\sum_{k\geq 1}\left[k(\Delta_2'+\Delta_1')^{k-1} + k(\Delta_2'-\Delta_1')^{k-1}\right]\right\} \\ &= \frac{xy(xh-f)\{2\Delta_1'(1-\Delta_2') + x[\Delta_1'^2 + (1-\Delta_2')^2]\}}{(1-x^2)[(1-\Delta_2')^2 - \Delta_1'^2]^2} \\ &= \frac{x^2y(xh-f)[(1+f_0)(1-f_0+2h_0) - x^2(1+h_0)^2]}{[(1+f_0)^2 - x^2(1+h_0)^2]^2}, \end{aligned} \tag{21}$$

where $h = f(1, y, z, u)$.

Now, by $(19-21)$, we obtain

$$\begin{aligned} f =& f_{\widetilde{\mathscr{U}}_{11}} + f_{\widetilde{\mathscr{U}}_{12}} + f_{\widetilde{\mathscr{U}}_2} \\ =& xyu + \frac{xyu[x^2 f_0^*(1+f_0^*) - u^2 f_0(1+f_0)]}{u^2(1+f_0)^2 - x^2(1+f_0^*)^2} \\ &+ \frac{x^2y(xh-f)[(1+f_0)(1-f_0+2h_0) - x^2(1+h_0)^2]}{[(1+f_0)^2 - x^2(1+h_0)^2]^2}, \end{aligned}$$

which is the statement of the theorem. □

Let $u = 1$ and $x = y$ in (18). Then we have

Theorem 2.2. The enumerating function $g = g_{\widetilde{\mathscr{U}}}(x, y) = f_{\widetilde{\mathscr{U}}}(x, y, y, 1)$ satisfies the following equation:

$$\begin{aligned} g =& \frac{xy[(1+g_0) - x^2(1+H_0)]}{(1+g_0)^2 - x^2(1+H_0)^2} \\ &+ \frac{x^2y(xH-g)[(1+g_0)(1-g_0+2H_0) - x^2(1+H_0)^2]}{[(1+g_0)^2 - x^2(1+H_0)^2]^2}, \end{aligned} \tag{22}$$

where $g_0 = f_{\mathscr{U}}(x, y, y), H_0 = h_{\mathscr{U}}(y, y) = f_{\mathscr{U}}(1, y, y)$ and $H = h_{\widetilde{\mathscr{U}}}(y, y, 1) = f_{\widetilde{\mathscr{U}}}(1, y, y, 1)$.

From the equation (18), one may see that enumerating function of rooted nonseparable unicursal planar maps f has a close relationship with that of rooted nonseparable Eulerian planar maps f_0. And the enumerating function f_0 satisfies a cubic equation which has no easy way to solve up to now [14]. This implies that further work can also be done on how to find an explicit solution of the equation.

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