New error-correcting pooling designs with singular linear space

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Abstract As a generalization of attenuated space, the concept of singular linear spaces was firstly introduced in [1]. In this paper, we construct a family of error-correcting pooling designs with the incidence matrix of two types of subspaces of singular linear space over finite fields, and exhibit their disjunct properties. Moreover, we show that the new construction gives better ratio of efficiency than the former ones under conditions. At last, the paper gives the brief introduction about the relationship between the columns (rows) of the matrix and the related parameters.

Keywords: Pooling designs, d^z-disjunct matrix, singular linear space, subspace

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1. Introduction

A group testing algorithm is non-adaptive if all tests could be specified without knowing the outcomes of other tests. The basic problem of non-adaptive group testing is to identify the defective parts as the subset of objects being tested. Given a set of s items with some defections, the group testing problem is asking to identify all defections with the minimum number of tests, each of which is on a subset of items, called a pool, and the test-outcome is negative when the pool does not contain any defection and positive when the pool contains a defection at least.

A mathematical model of the non-adaptive group testing design is a d-disjunct matrix, which is also called a pooling design. Designing a good error-tolerant pooling design is the central problem in the area of non-adaptive group testing.

A pooling design is usually represented by a binary matrix with columns indexed with items and rows indexed with pools. An entry at cell (i, j) is 1 if and only if the i-th pool is contained by the j-th item, and 0, otherwise. In practice, test-outcomes may contain errors, to make pooling design error tolerant, one introduced the concept of d^z-disjunct matrix (see Macula [2]). A binary matrix M is said to be d^z-disjunct if given any $d+1$ columns of M with one designated, there are $z+1$ rows with a 1 in the designated column and 0 in each of the other d columns. The concept of fully d^z-disjunct

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matrix was given in the paper [3]. A d^z-disjunct matrix is fully d^z-disjunct if it is not x^y-disjunct whenever $x > d$ or $y > z$. A d^z-disjunct matrix can be employed to discern d defections, detect z errors and correct $\lfloor \frac{z}{2} \rfloor$ errors(see [4]).

A group test is applicable to an arbitrary subset of clones. A pooling design is a specification of all tests such that they can be performed simultaneously, with the goal being to identify all positive clones with a small number of tests(see [5, 6, 7]). A pooling design can reduce the cost greatly and has many applications in molecular biology, such as DNA library screening, gene detection, nonunique probe selection, etc(see [8]). There are also several constructions of d^z-disjunct matrices in the literature. (Balding and Torney [12]; Erdös et al.[11]; Li et al.[9].)

XueMei Liu ,You Gao constructed a family of error-correcting pooling designs with the incidence matrix of two types of subspaces of singular linear space over finite fields (see [13]), as a generalization of Liu's matrix, in this paper, we use the general subspaces of type (m_1, k_1) to substitute special subspaces of type $(r, 0)$, and exhibit its disjunct properties. Moreover, we show that the new construction gives better ratio of efficiency than the others under conditions. At last, we illustrate the relationship between the columns (rows) of the matrix and the related parameters, simply.

2.The singular linear space

In order to understand the following contents better, in this section, we will introduce the concepts of the singular linear space and some counting formulas.

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power, and $\mathbb{F}_q^{(n+l)}$ be the $(n+l)$-dimensional row vector space over $\mathbb{F}_q$. The set of $(n+l) \times (n+l)$ nonsingular matrices T over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T_{11} and T_{22} are nonsingular $n \times n$ and $l \times l$ matrices, respectively, forms a group under matrix multiplication, called the singular general linear group of degree $n+l$ over $\mathbb{F}_q$ and denoted by $GL_{n+l,n}(\mathbb{F}_q)$. If $l = 0$(resp. $n = 0$), $GL_{n,n}(\mathbb{F}_q) = GL_n(\mathbb{F}_q)$ (resp. $GL_{l,0}(\mathbb{F}_q) = GL_l(\mathbb{F}_q)$) is the general linear group of degree n (resp. l)(See Wan [18]).

Let A be an m-dimensional subspace of $\mathbb{F}_q^{(n+l)}$, denote also by A an $m \times (n+l)$ matrix of rank m whose rows span the subspace A and call the matrix A a matrix representation of the subspace A. Clearly, $\mathbb{F}_q^{(n+l)}$ admits an action of $GL_{n+l,n}(\mathbb{F}_q)$ defined as follows

$$\mathbb{F}_q^{(n+l)} \times GL_{n+l,n}(\mathbb{F}_q) \to \mathbb{F}_q^{(n+l)}$$

$$((x_1, \dots, x_n, x_{n+1}, \dots, x_{n+l}), T) \mapsto (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+l})T$$

The vector space $\mathbb{F}_q^{(n+l)}$ together with the above group action is called the $(n+l)$-dimensional singular linear space over $\mathbb{F}_q$. For $1 \leq i \leq n+l$, let e_i be the row vector in $\mathbb{F}_q^{(n+l)}$ whose i-th coordinate is 1 and all other coordinates are 0. Let E be the l-dimensional subspace of $\mathbb{F}_q^{(n+l)}$ generated by $e_{n+1}, e_{n+2}, \dots, e_{n+l}$. An m-dimensional subspace A of $\mathbb{F}_q^{(n+l)}$ is called a subspace of type (m,k) if $\dim(A \cap E) = k$.

Let $\mathcal{M}(m,k;n+l,n)$ denote the set of all the subspaces of type (m,k) of $\mathbb{F}_q^{(n+l)}$, and let

$$N(m,k;n+l,n) = \mid \mathcal{M}(m,k;n+l,n) \mid .$$

By Wang et al. [14], $\mathcal{M}(m,k;n+l,n)$ forms an orbit under the action of $GL_{n+l,n}(\mathbb{F}_q)$.

We begin with some useful propositions.

Proposition 2.1.(Wan [15] Corollary1.9) Let $0 \leq k \leq m \leq n$. Then the number $N'(k,m,n)$ of m-dimensional vector subspaces containing a given k-dimensional vector subspace of $\mathbb{F}_q^{(n)}$ is equal to $\begin{bmatrix} n-k \\ m-k \end{bmatrix}_q$.

Proposition 2.2 (Wang et al. [16] Lemma 2.1). $\mathcal{M}(m,k;n+l,n)$ is non-empty if and only if $0 \leq k \leq l$ and $0 \leq m-k \leq n$. Moreover, if $\mathcal{M}(m,k;n+l,n)$ is non-empty, then it forms an orbit of subspaces under $GL_{n+l,n}(\mathbb{F}_q)$ and

$$N(m,k;n+l,n) = q^{(m-k)(l-k)} \begin{bmatrix} n \\ m-k \end{bmatrix}_q \begin{bmatrix} l \\ k \end{bmatrix}_q .$$

For a fixed subspace A of type (m,k) in $\mathbb{F}_q^{(n+l)}$, let $\mathcal{M}(m_1,k_1;m,k;n+l,n)$ denote the set of all the subspaces of type (m_1,k_1) contained in A, and let $N(m_1,k_1;m,k;n+l,n) = \mid \mathcal{M}(m_1,k_1;m,k;n+l,n) \mid$.

By the transitivity of $GL_{n+l,n}(\mathbb{F}_q)$ on the set of subspaces of the same type, $N(m_1,k_1;m,$
$k;n+l,n)$ is independent of the particular choice of the subspace A of type (m,k).

Proposition 2.3.(Wang et al. [16] Lemma 2.2) $\mathcal{M}(m_1,k_1;m,k;n+l,n)$ is non-empty if and only if $0 \leq k_1 \leq k \leq l$ and $0 \leq m_1 - k_1 \leq m-k \leq n$. Moreover, if $\mathcal{M}(m_1,k_1;m,k;n+l,n)$ is non-empty, then

$$N(m_1,k_1;m,k;n+l,n) = q^{(m_1-k_1)(k-k_1)} \begin{bmatrix} m-k \\ m_1-k_1 \end{bmatrix}_q \begin{bmatrix} k \\ k_1 \end{bmatrix}_q .$$

For a fixed subspace A of type (m_1, k_1) in $\mathbb{F}_q^{(n+l)}$, let $\mathcal{M}'(m_1, k_1; m, k; n+l, n)$ denote the set of all the subspaces of type (m, k) containing A, and let $N'(m_1, k_1; m, k; n+l, n) =\mid \mathcal{M}'(m_1, k_1; m, k; n+l, n) \mid$.

By the transitivity of $GL_{n+l,n}(\mathbb{F}_q)$ on the set of subspaces of the same type, $N'(m_1, k_1; m, k; n+l, n)$ is independent of the particular choice of the subspace A of type (m_1, k_1).

Proposition 2.4.(Wang et al. [16] Lemma 2.3) $\mathcal{M}'(m_1, k_1; m, k; n+l, n)$ is non-empty if and only if $0 \le k_1 \le k \le l$ and $0 \le m_1 - k_1 \le m - k \le n$. Moreover, if $\mathcal{M}'(m_1, k_1; m, k; n+l, n)$ is non-empty, then

$$N'(m_1, k_1; m, k; n+l, n) = q^{(l-k)(m-k-m_1+k_1)} \begin{bmatrix} n-(m_1-k_1) \\ (m-k)-(m_1-k_1) \end{bmatrix}_q \begin{bmatrix} l-k_1 \\ k-k_1 \end{bmatrix}_q .$$

Proposition 2.5.(see [13]) Given integers $0 \le k \le l$ and $0 \le m-k \le n$, the sequence $N(m, k; n+l, n)$ is unimodal and gets its peak at $m = \lfloor \frac{n+l+k}{2} \rfloor$.

3.Constructing Pooling designs

In this section, the paper provides the construction of inclusion matrix associated with subspaces of $\mathbb{F}_q^{(n+l)}$, and show its disjunct properties.

Definition 3.1 Given integers $0 \le k_1 \le k \le l$, $k \ge 2$, and $0 \le m_1 - k_1 \le m - k \le n$. Let $M(m_1, k_1; m, k; n+l, n)$ be the binary matrix whose rows(resp.columns) are indexed by $\mathcal{M}(m_1, k_1; n+l, n)$ (resp.$\mathcal{M}(m, k; n+l, n)$). We also order elements of these sets lexicographically. $M(m_1, k_1; m, k; n+l, n)$ has a 1 in row i and column j if and only if the i-th subspace of $\mathcal{M}(m_1, k_1; n+l, n)$ is a subspace of the j-th subspace of $\mathcal{M}(m, k; n+l, n)$.

By Proposition 2.2, 2.3 and 2.4, we know that $M(m_1, k_1; m, k; n+l, n)$ is a $N(m_1, k_1; n+l, n) \times N(m, k; n+l, n)$ matrix, whose constant column(resp.row) weight is $N(m_1, k_1; m, k; n+l, n)$(resp.$N'(m_1, k_1; m, k; n+l, n)$).

Theorem 3.2 Let $0 \le k_1 < k-1 < l-1$, $k \ge 2$, $0 \le m_1 - k_1 < m-k-1 < n-2$, and $a = N(m_1, k_1; m, k; n+l, n)$, $b = N(m_1, k_1; m-1, k; n+l, n)$, $c = N(m_1, k_1; m-1, k-1; n+l, n)$, $e = N(m_1, k_1; m-2, k; n+l, n)$, $f = N(m_1, k_1; m-2, k-1; n+l, n)$, $g = N(m_1, k_1; m-2, k-2; n+l, n)$, $\mu = \max\{b-e, b-f, b-g, c-e, c-f, c-g\}$. If $1 \le d \le \lfloor \frac{a-\max\{b,c\}-1}{\mu} \rfloor + 1$, then $M(m_1, k_1; m, k; n+l, n)$ is d^z-disjunct, where $z = a - \max\{b, c\} - (d-1)\mu - 1$. Moreover, if $1 \le d \le \min\{\lfloor \frac{a-\max\{b,c\}-1}{\mu} \rfloor + 1, q+1\}$, then $M(m_1, k_1; m, k; n+l, n)$ is full d^z-disjunct.

Proof Let $P, P_1, P_2 \cdots P_d$ be $d+1$ distinct columns of $M(m_1, k_1; m, k; n+l, n)$. To obtain the maximum numbers of subspaces of type(m_1, k_1) in

$$P \bigcap (\bigcup_{i=1}^{d} P_i) = \bigcup_{i=1}^{d} (P \bigcap P_i).$$

We may assume that $\dim(P \cap P_i) = m-1$ and $\dim(P \cap P_i \cap P_j) = m-2$ for any two distinct i and j, where $1 \leq i, j \leq d$, we have that $(P \cap P_i)$ is an $(m-1)$-dimensional subspace in P, in the same, $(P \cap P_i \cap P_j)$ is an $(m-2)$-dimensional subspace in $(P \cap P_i)$. Since P is a subspace of type (m, k), by Proposition 2.3, $P \cap P_i$ is a subspace of type $(m-1, k)$ or$(m-1, k-1)$. Thus, there are two cases to be considered for $(P \cap P_i \cap P_j)$.

Case 1 : Let $P \cap P_i$ be a subspace of type $(m-1, k)$ in $\mathbb{F}_q^{(n+l)}$, by Proposition 2.2, $0 \leq k \leq l$ and $0 \leq m-k-1 \leq n$. Suppose that $(P \cap P_i \cap P_j)$ is a subspace of type $(m-2, x)$, by Proposition 2.3, $x = k$ or $k-1$, i.e., $(P \cap P_i \cap P_j)$ is a subspace of type $(m-2, k)$ or type $(m-2, k-1)$.

Case 2 : Let $P \cap P_i$ be a subspace of type $(m-1, k-1)$ in $\mathbb{F}_q^{(n+l)}$, by Proposition 2.2, we have $0 \leq k-1 \leq l$, $0 \leq m-k \leq n$. Suppose that $(P \cap P_i \cap P_j)$ is a subspace of type $(m-2, y)$, by Proposition 2.3, $y = k-1$ or $k-2$, i.e., $(P \cap P_i \cap P_j)$ is a subspace of type $(m-2, k-1)$ or type $(m-2, k-2)$.

In both cases $(P \cap P_i \cap P_j)$ is a subspace of type $(m-2, k)$, type $(m-2, k-1)$ or type $(m-2, k-2)$, Note that $a = N(m_1, k_1; m, k; n+l, n)$, $b = N(m_1, k_1; m-1, k; n+l, n)$, $c = N(m_1, k_1; m-1, k-1; n+l, n)$, $e = N(m_1, k_1; m-2, k; n+l, n)$, $f = N(m_1, k_1; m-2, k-1; n+l, n)$, $g = N(m_1, k_1; m-2, k-2; n+l, n)$, $\mu = \max\{b-e, b-f, b-g, c-e, c-f, c-g\}$. Therefore the subspaces of type (m_1, k_1) of P not covered by $P_1, P_2, \cdots, P_d$ is at least

$$a - d \times \max\{b, c\} + (d-1) \times \min\{e, f, g\},$$

i.e., $a - \max\{b, c\} - (d-1)\mu$. Hence $z = a - \max\{b, c\} - (d-1)\mu - 1$. Since $z \geq 0$, we have $1 \leq d \leq \lfloor \frac{a - \max\{b,c\} - 1}{\mu} \rfloor + 1$.

Now we show that the maximal dimensions of $P \bigcap (\bigcup_{i=1}^{d} P_i)$ can be deserved by an explicit construction. For $P \cap P_i$, by Proposition 2.3, $N(m_1, k_1; m-2, k; n+l, n) \geq 1$, $N(m_1, k_1; m-2, k-1; n+l, n) \geq 1$ and $N(m_1, k_1; m-2, k-2; n+l, n) \geq 1$. Hence there exists a $(m-2)$-dimensional subspace contained in $P \cap P_i$, denoted by Q, such that the number of m_1-dimensional subspaces contained in Q is equal to $\min\{e, f, g\}$. By Proposition 2.1, the number of $(m-1)$-dimensional subspaces containing Q and contained in P is equal to $q+1$, moreover, each of these subspaces is a subspace of type $(m-1,k)$ or type $(m-1,k-1)$. Since $1 \le d \le \min\{\lfloor \frac{a-\max\{b,c\}-1}{\mu} \rfloor + 1, q+1\}$, we can choose d distinct $(m-1)$-dimensional subspaces containing Q and contained in P, denoted by $Q_i (1 \le i \le d)$. By Proposition 2.4, if assume that $N(m-1,x;m,k;n+l,n) \neq 0$, where $x = k$ or $k-1$, then $N'(m-1,x;m,k;n+l,n) \ge 2$. For each Q_i, we can choose a subspace of type $(m-1,x)$ denoted by P_i,where $x = k$ or $k-1$, such that $P \cap P_i = Q_i$. Hence each pair of P_i and P_j satisfy $P_i \cap P_j = Q$, therefore the maximal dimensions of $P \bigcap (\bigcup_{i=1}^{d} P_i)$ can be achieved.

Now we have showed that $M(m_1,k_1;m,k;n+l,n)$ is d^z-disjunct. Moreover, by the assumption of z, we have that $M(m_1,k_1;m,k;n+l,n)$ is d^z-disjunct but not d^{z+1}-disjunct. On the other hand , we assume that $M(m_1,k_1;m,k;n+l,n)$ is $(d+1)^{z'}$-disjunct. By the maximality of z, we infer that $z' \le a - \max\{b,c\} - (d+1-1)\mu - 1 < a - \max\{b,c\} - (d-1)\mu - 1 = z$. Hence $M(m_1,k_1;m,k;n+l,n)$ is not $(d+1)^z$-disjunct. Therefore, $M(m_1,k_1;m,k;n+l,n)$ is fully d^z-disjunct.

Hence, this completes the proof. □

4.Comparision of test efficiency

We know that the smaller the value of t is, the better the design is, and the larger the value of s, the more perfectible the pooling design is. Now we take $\frac{t}{s}$ as a measure of the design is, where t denotes the number of tests, i.e., the number of rows of inclusion matrix, s denotes the number of detected items, i.e., the number of columns of the inclusion matrix.
In this paper, we can know the test efficiency $\frac{t}{s}$, i.e.,

$$\frac{t}{s} = \frac{|M(m_1,k_1;n+l,n)|}{|M(m,k;n+l,n)|} = \frac{N(m_1,k_1;n+l,n)}{N(m,k;n+l,n)}$$

$$= q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)} \frac{\prod_{i=m_1-k_1+1}^{m-k}(q^i-1)\prod_{i=k_1}^{k}(q^i-1)}{\prod_{i=n-(m-k)+1}^{n-(m_1-k_1)}(q^i-1)\prod_{i=l-k+1}^{l-k_1}(q^i-1)}.$$

Macula(1996) constructed a specifical inclusion matrix. Let $V = \{1,2,\cdots,\nu\}$ be the base set, each of the rows(resp.column) is labeled by a r (resp.u) subset of V, where $r < u < \nu$, the$(i,j) = 1$ if and only if the label of column j contain the label of row i. Similarly, let $\nu = n+l$, $r = m_1$, and $u = m$, then the test efficiency is $\frac{t_1}{s_1}$, thus

$$\frac{t_1}{s_1} = \frac{m\cdots(m_1+1)}{(n+l-m_1)\cdots(n+l-m+1)}.$$

D'yachkov et al.(2005) constructed with subspaces of $\mathbb{F}_q$, where q is a prime power, and the row(resp.column) of it is denoted by an r(resp.u)-dimensional subspace of $\mathbb{F}_q^{\nu}$, where $r < u < \nu$, the $(i,j) = 1$ if and only if the i-th label is contained in the j-th label. For comparing with $\frac{t}{s}$, let

$\nu = n + l$, $r = m_1$, and $u = m$, and assume that the test efficiency is $\frac{t_2}{s_2}$, then

$$\frac{t_2}{s_2} = \frac{\prod_{i=m_1+1}^{m}(q^i - 1)}{\prod_{i=n+l-m+1}^{n+l-m_1}(q^i - 1)}.$$

Zhang et al.(2008) also constructed a d^z-disjunct matrix of two types of subspaces of the dual space of the symplectic space $\mathbb{F}_q^{2\nu}$, where q is also a prime power. In the same way, each of the rows(resp. columns) are labeled by subspaces of type $(r, 0)$ (resp.$(u, 0)$), which are contained in $P_0^{\perp}$ and containing P_0, where $m_0 < r < u < \nu$ and P_0 is a given subspace of type $(m_0, 0)$, and the $(i, j) = 1$ if and only if the column j contains the row i. Let $n + l$ be even, $\nu - m_0 = n + l$, $r - m_0 = m_1$ and $u - m_0 = m$, and the test efficiency is $\frac{t_3}{s_3}$, then we have

$$\frac{t_3}{s_3} = \frac{\prod_{i=m_1+1}^{m}(q^i - 1)}{\prod_{i=\frac{n+l}{2}-m+1}^{\frac{n+l}{2}-m_1}(q^{2i} - 1)}.$$

Theorem 4.1 Let $0 \leq k_1 < k-1 < l-1$, $k \geq 2$, $0 \leq m_1 - k_1 < m-k-1 < n-2$, and $\theta = (l-k-k_1-1)(k-k_1) + (n+l-m-1)(m-k) - (n+l-m_1-1)(m_1-k_1)$, if $n+l-m_1 \leq m$ and $\theta \geq 0$, then $\frac{\frac{t}{s}}{\frac{t_1}{s_1}} < \frac{1}{q^\theta}$; if $n+l-m_1 > m$ and $\theta \geq 0$, and $q > (\frac{n+l-m+1}{m_1+1})^{\frac{m-m_1}{\theta}}$, then $\frac{\frac{t}{s}}{\frac{t_1}{s_1}} < \frac{(\frac{n+l-m+1}{m_1+1})^{m-m_1}}{q^\theta}$.

Proof If $n + l - m_1 \leq m$ and $\theta \geq 0$, then

$$\begin{aligned}
\frac{\frac{t}{s}}{\frac{t_1}{s_1}} &= q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)} \frac{\prod_{i=m_1-k_1+1}^{m-k}(q^i-1)\prod_{i=k_1}^{k}(q^i-1)}{\prod_{i=n-(m-k)+1}^{n-(m_1-k_1)}(q^i-1)\prod_{i=l-k+1}^{l-k_1}(q^i-1)} \\
&\quad \times \frac{(n+l-m_1)\cdots(n+l-m+1)}{m(m-1)\cdots(m_1+1)} \\
&< q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)} \frac{\prod_{i=m_1-k_1+1}^{m-k}(q^i-1)\prod_{i=k_1}^{k}(q^i-1)}{\prod_{i=n-(m-k)+1}^{n-(m_1-k_1)}(q^i-1)\prod_{i=l-k+1}^{l-k_1}(q^i-1)} \\
&< q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)} \frac{\prod_{i=m_1-k_1+1}^{m-k} q^i \prod_{i=k_1}^{k} q^i}{\prod_{i=n-(m-k)+1}^{n-(m_1-k_1)} q^{i-1} \prod_{i=l-k+1}^{l-k_1} q^{i-1}} \\
&= \frac{q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)}}{q^{(m_1-k_1)^2-(m-k)^2+(n-1)[(m-k)-(m_1-k_1)]+k_1^2-k^2+(l-1)(k-k_1)}} \\
&= \frac{1}{q^\theta}.
\end{aligned}$$

If $n + l - m_1 > m$ and $\theta \geq 0$, then

$$\frac{\frac{t}{s}}{\frac{t_1}{s_1}} < \frac{1}{q^\theta} \times \frac{(n+l-m_1)\cdots(n+l-m+1)}{m(m-1)\cdots(m_1+1)}$$

$< \frac{1}{q^{\theta}} \times (\frac{n+l-m+1}{m_1+1})^{m-m_1}$

$= \frac{(\frac{n+l-m+1}{m_1+1})^{m-m_1}}{q^{\theta}}$. □

Example4.1.1 Let $n = l = 10$, $k = 4$, $k_1 = 2$, $q = 13$, $m = 8$, $m_1 = 4$, and $\theta = 20 > 0$, $n + l - m_1 = 16 > 8 = m$, then $\frac{\frac{t}{s}}{\frac{t_1}{s_1}} < \frac{1}{13^{16} \cdot 5^4}$.

Theorem 4.2 Let $0 \leq k_1 < k - 1 < l - 1$, $k \geq 2$, $0 \leq m_1 - k_1 < m - k - 1 < n - 2$, and $\theta = k_1(l - m_1) + k(m - l)$, if $\theta \geq 0$, then $\frac{\frac{t}{s}}{\frac{t_2}{s_2}} < \frac{1}{q^{\theta}}$.

Proof Let $0 \leq k_1 \leq k < l$, $k \geq 2$, $1 < m_1 - k_1 < m - k + 1 < n$, then we have

$$\frac{\frac{t}{s}}{\frac{t_2}{s_2}} = q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)} \frac{\prod_{i=m_1-k_1+1}^{m-k}(q^i-1)\prod_{i=k_1}^{k}(q^i-1)}{\prod_{i=n-(m-k)+1}^{n-(m_1-k_1)}(q^i-1)\prod_{i=l-k+1}^{l-k_1}(q^i-1)}$$

$$\times \frac{\prod_{i=n+l-m+1}^{n+l-m_1}(q^i-1)}{\prod_{i=m_1+1}^{m}(q^i-1)}$$

$$< q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)} \frac{\prod_{i=m_1-k_1+1}^{m-k} q^i \prod_{i=k_1}^{k} q^i}{\prod_{i=n-(m-k)+1}^{n-(m_1-k_1)} q^{i-1} \prod_{i=l-k+1}^{l-k_1} q^{i-1}}$$

$$\times \frac{\prod_{i=n+l-m+1}^{n+l-m_1} q^i}{\prod_{i=m_1+1}^{m} q^{i-1}}$$

$$= \frac{q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)}}{q^{-k^2+k_1^2+2mk-2m_1k_1-lm+lm_1+(n-1)(k-k_1)}}$$

$$= \frac{1}{q^{\theta}}.$$

Therefore,the proof of Theorem 4.2 is completed. □

Example 4.2.1 Let $n = l = 10$, $m = 11$, $k = 6$, $m_1 = 5$, $k_1 = q = 3$, and $\theta = 21 > 0$, then we have $\frac{\frac{t}{s}}{\frac{t_2}{s_2}} < \frac{1}{3^{21}}$.

Theorem 4.3 Assume that $n + l$ be even, let $0 \leq k_1 < k - 1 < l - 1$, $k \geq 2$, $0 \leq m_1 - k_1 < m - k - 1 < n - 2$, and $\theta = \frac{1}{2}(m^2 - m_1^2) + k_1^2 - k^2 - m_1k_1 + mk - n(k - k_1) - \frac{5}{2}(m - m_1)$, if $\theta \geq 0$, then $\frac{\frac{t}{s}}{\frac{t_3}{s_3}} < \frac{1}{q^{\theta}}$.

Proof Since $0 \leq k_1 < k-1 < l-1$, $k \geq 2$, $0 \leq m_1-k_1 < m-k-1 < n-2$, we have

$$\frac{\frac{t}{s}}{\frac{t_3}{s_3}} = q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)} \frac{\prod_{i=m_1-k_1+1}^{m-k}(q^i-1)\prod_{i=k_1}^{k}(q^i-1)}{\prod_{i=n-(m-k)+1}^{n-(m_1-k_1)}(q^i-1)\prod_{i=l-k+1}^{l-k_1}(q^i-1)}$$

$$\times \frac{\prod_{i=\frac{n+l}{2}-m+1}^{\frac{n+l}{2}-m_1}(q^{2i}-1)}{\prod_{i=m_1+1}^{m}(q^i-1)}$$

$$< q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)} \frac{\prod_{i=m_1-k_1+1}^{m-k} q^i \prod_{i=k_1}^{k} q^i}{\prod_{i=n-(m-k)+1}^{n-(m_1-k_1)} q^{i-1} \prod_{i=l-k+1}^{l-k_1} q^{i-1}}$$

$$\times \frac{\prod_{i=\frac{n+l}{2}-m+1}^{\frac{n+l}{2}-m_1} q^{2i}}{\prod_{i=m_1+1}^{m} q^{i-1}}$$

$$= \frac{q^{(m_1-k_1)(l-k_1)-(m-k)(l-k)}}{q^{\frac{1}{2}(m^2-m_1^2)+2k(m-k)-2k_1(m_1-k_1)+(l+\frac{5}{2})(m_1-m)+(l-n)(k-k_1)}}$$

$$= \frac{1}{q^{\theta}}.$$

This completes the proof. □

Example4.3.1 Let $n = l = 6$, $m = 9$, $m_1 = 5$, $k = 5$, $k_1 = 3$, $q = 2$, and $\theta = 20 > 0$, then we have $\frac{\frac{t}{s}}{\frac{t_3}{s_3}} < \frac{1}{2^{20}}$.

5.The discussion on design parameters

From Proposition 2.5, we know the change tendency of the sequence $N(m, k; n+l, n)$ with m. The change tendency of the sequence $N(m, k; n+l, n)$ with k is given by the following Theorem 5.1.

Theorem 5.1 Given integers $0 \le k_1 < k-1 < l-1$, $k \ge 2$, $0 \le m_1 - k_1 < m-k-1 < n-2$, then we have

(i) when $k \le \lfloor \frac{m-n-3}{2} \rfloor$, the sequence $N(m, k; n+l, n)$ increases with k, if n, l, m are fixed.

(ii) when $k \ge \lceil \frac{m-n+1}{2} \rceil$, the sequence $N(m, k; n+l, n)$ decreases with k, if n, l, m are fixed.

Proof By Proposition 2.2, if n, l, m is fixed, then we have

$$\frac{N(m, k; n+l, n)}{N(m, k+1; n+l, n)} = \frac{q^{(m-k)(l-k)} \begin{bmatrix} n \\ m-k \end{bmatrix}_q \begin{bmatrix} l \\ k \end{bmatrix}_q}{q^{(m-k-1)(l-k-1)} \begin{bmatrix} n \\ m-k-1 \end{bmatrix}_q \begin{bmatrix} l \\ k+1 \end{bmatrix}_q}$$

$$= q^{m+l-2k-1} \cdot \frac{q^{n-m+k+1}-1}{q^{m-k}-1} \cdot \frac{q^{k+1}-1}{q^{l-k}-1}$$

If $k \le \lfloor \frac{m-n-3}{2} \rfloor$, then

$$
\begin{aligned}
\frac{N(m,k;n+l,n)}{N(m,k+1;n+l,n)} &= q^{m+l-2k-1}\cdot\frac{q^{n-m+k+1}-1}{q^{m-k}-1}\cdot\frac{q^{k+1}-1}{q^{l-k}-1}\\
&< q^{m+l-2k-1}\cdot\frac{q^{n-m+k+1}}{q^{m-k-1}}\cdot\frac{q^{k+1}}{q^{l-k-1}}\\
&= \frac{1}{q^{m-(n+2k+3)}}\\
&\le 1
\end{aligned}
$$

If $k \ge \lceil\frac{m-n+1}{2}\rceil$, then

$$
\begin{aligned}
\frac{N(m,k;n+l,n)}{N(m,k+1;n+l,n)} &= q^{m+l-2k-1}\cdot\frac{q^{n-m+k+1}-1}{q^{m-k}-1}\frac{q^{k+1}-1}{q^{l-k}-1}\\
&> q^{m+l-2k-1}\cdot\frac{q^{n-m+k}}{q^{m-k}}\cdot\frac{q^{k}}{q^{l-k}}\\
&= q^{n-m+2k-1}\\
&\ge 1
\end{aligned}
$$

Therefore, we complete the proof of Theorem 5.1. □

From Proposition 2.5, we have the following conclusion.

Theorem 5.2 Given integers $0 \le k_1 < k-1 < l-1$, $k \ge 2$, $0 \le m_1 - k_1 < m-k-1 < n-2$, the sequence $N(m_1,k_1;n+l,n)$ is unimodal and gets its peak at $m_1 = \lfloor\frac{n+l+k_1}{2}\rfloor$.

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