ON SOME REGULAR POLYHEDRONS IN THE TAXICAB 3-SPACE

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ABSTRACT. In this study, it has been researched that which Euclidean regular polyhedrons are also taxicab regular and which are not. The existence of non-Euclidean taxicab regular polyhedrons in the taxicab 3-space has also been investigated.

1. Introduction

The taxicab 3−dimensional space $\mathbb{R}^3_T$ is almost the same as the Euclidean analytical 3−dimensional space $\mathbb{R}^3$. The points, lines and planes are the same and the angles are measured in the same way, but the distance function is different. The taxicab metric is defined using the distance function as in [8, 9]

$$d_T(A, B) = |b_1 - a_1| + |b_2 - a_2| + |b_3 - a_3|.$$

Since taxicab plane and 3−dimensional space have distance function different from that in the Euclidean plane and 3−dimensional space, it is interesting to study the topics of the taxicab analogues that include the distance concept in the Euclidean plane and 3−dimensional space. Many such topics have been studied in the taxicab plane and 3−dimensional space (see [1, 2, 3, 4, 5, 6, 7, 9, 10, 11]). During the recent years, regular polygons have been studied in the taxicab plane and 3−space [12, 13]. Therefore, it can be interesting to study regular polyhedrons in the taxicab 3−dimensional space.

In Euclidean geometry, a Platonic solid is a regular, convex polyhedron with congruent faces of regular polygons and the same number of faces meeting at each vertex. Five solids meet those criteria, and each is named after its number of faces. Geometers have studied the mathematical beauty and symmetry of the Platonic solids for thousands of years. They are named for the ancient Greek philosopher Plato who theorized in his dialogue, the

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Timaeus, that the classical elements were made of these regular solids (see [10]).

Polyhedron	Number vertexes	Number faces	Number edges
Tetrahedron	4	4	6
Hexahedron	8	6	12
Octahedron	6	8	12
Dodecahedron	20	12	30
Icosahedron	12	20	30

For $\overrightarrow{V} = (v_1, v_2, v_3) \in \mathbb{R}^3$, taxicab norm of $\overrightarrow{V}$ is shown as $\left\|\overrightarrow{V}\right\|_T$ as in [6] and

$$\left\|\overrightarrow{V}\right\|_T = |v_1| + |v_2| + |v_3|.$$

Suppose that l is a line through the points P_1 and P_2. If l has direction vector $\overrightarrow{V} = (v_1, v_2, v_3)$, then these equations can be written according to [5]

$$(|v_1| + |v_2| + |v_3|)\, d_E(P_1, P_2) = \sqrt{v_1^2 + v_2^2 + v_3^2}\, d_T(P_1, P_2)$$

$$d_E(A, B) = \frac{\left\|\overrightarrow{V}\right\|}{|v_1| + |v_2| + |v_3|} d_T(A, B).$$

From now on $g(\overrightarrow{V})$ will be used in this study instead of $\dfrac{\left\|\overrightarrow{V}\right\|}{(|v_1| + |v_2| + |v_3|)}$.

Let us introduce the following abbreviations for the following Proposition 1.

$\pm x \pm y \pm z = \|\overrightarrow{w}\|_T$ plane equation is shown as $P_{\pm,\pm,\pm}$ and $\left(x \pm \frac{\|w\|_T}{3}\right)^2 + \left(y \pm \frac{\|w\|_T}{3}\right)^2 + \left(z \pm \frac{\|w\|_T}{3}\right)^2 = \frac{3\|w\|_E^2 - \|w\|_T^2}{3}$ sphere equation is shown as $S_{\pm,\pm,\pm}$.

Proposition 1. *Let $w = (x, y, z)$. The geometric locations of the w_i ($i \in \mathbb{N}$) vectors which satisfy the following equations $\|w\|_E = \|w_i\|_E$ and $\|w\|_T = \|w_i\|_T$ constitute the $\bigcup_{i=1}^{9} C_i$.*

Assume $\|w\|_E \neq \|w\|_T$.

If $0 \le x, y, z \le \|w\|_T$ then $C_1 = S_{-,-,-} \cap P_{+,+,+}$;
If $0 \le x, z \le \|w\|_T$ and $-\|w\|_T \le y \le 0$ then $C_2 = S_{-,+,-} \cap P_{+,-,+}$;
If $0 \le z \le \|w\|_T$ and $-\|w\|_T \le x, y \le 0$ then $C_3 = S_{+,+,-} \cap P_{-,-,+}$;
If $0 \le y, z \le \|w\|_T$ and $-\|w\|_T \le x \le 0$ then $C_4 = S_{+,-,-} \cap P_{-,+,+}$;
If $0 \le x, y \le \|w\|_T$ and $-\|w\|_T \le z \le 0$ then $C_5 = S_{-,-,+} \cap P_{+,+,-}$;

If $0 \le x \le \|w\|_T$ *and* $-\|w\|_T \le y, z \le 0$ *then* $C_6 = S_{-,+,+} \cap P_{+,-,-}$;
If $-\|w\|_T \le x, y, z \le 0$ *then* $C_7 = S_{+,+,+} \cap P_{-,-,-}$;
If $0 \le y \le \|w\|_T$ *and* $-\|w\|_T \le x, z \le 0$ *then* $C_8 = S_{+,-,+} \cap P_{-,+,-}$.

Assume $\|w\|_E = \|w\|_T$.

$$C_9 = \{(\pm \|w\|_T, 0, 0), (0, \pm \|w\|_T, 0), (0, 0, \pm \|w\|_T)\}$$

(see [13]).

In this study, we answer the following question: Which Euclidean regular polyhedrons are also taxicab regular, and which are not? We also investigate the existence and nonexistence of taxicab regular polyhedrons.

2. Euclidean Regular Polyhedrons in Taxicab 3-Space

Definition 1. *Three-dimensional objects which are bounded with pieces of the polygonal plane are called polyhedrons. A face is a planar surface that forms part of the boundary of a polyhedron. An edge is a line segment joining two vertices in a polyhedron. A vertex is a corner point of a polyhedron or formed by the intersection of edges.*

Definition 2. *A polyhedron is regular if all faces are congruent regular polygons and same number of faces is assembled around each vertex.*

The points, lines and planes are the same in Euclidean and taxicab geometry and the angles are measured in the same way as in [9]. However, since the distance function is different, in order for an Euclidean regular polyhedron to be taxicab regular, all taxicab length of edges must be the same. Thus, we investigate taxicab length of edges of polyhedrons.

Corollary 1. *Let* $A_1, A_2, A_3, ..., A_n$ *be vertices and* k *be the number of edges of an Euclidean regular polyhedron. For* $k \in \{6, 12, 30\}$ *and* $1 \le i \le k$, $\overrightarrow{V_i} = (v_1, v_2, v_3)$ *are direction vectors of edges.* $A_1A_2A_3...A_n$ *is also taxicab regular if only if* $g(\overrightarrow{V_i}) = g(\overrightarrow{V_j})$ *while* $(1 \le i, j \le k)$.

Corollary 2. *Let* W *be a vector set of the edges of Euclidean regular polyhedron and* $w \in W$. *An Euclidean regular polyhedron is also taxicab regular if only if* $W \subseteq \bigcup_{i=1}^{9} C_i$ *where* $\bigcup_{i=1}^{9} C_i$ *is the same as Proposition 1.*

There exist an Euclidean and taxicab regular tetrahedrons.

Example 1. *For* $A_1(0,0,1)$, $A_2(0,1,0)$, $A_3(1,0,0)$, $A_4(\frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3})$ *vertex points,* $A_1A_2A_3A_4$ *Euclidean regular tetrahedron is also taxicab regular.*

Theorem 1. *If* 3 *different symmetry planes with at least one of which is on a different face of the Euclidean regular tetrahedron are member of*

$$\{x \mp y = k_1, x \mp z = k_2, y \mp z = k_3, x = k_4, y = k_5, z = k_6\}$$

set (for $1 \leq i \leq 6$, $k_i \in \mathbb{R}$), Euclidean regular tetrahedron is also taxicab regular.

Proof. Since a reflection according to $x \mp y = 0$, $x \mp z = 0$, $y \mp z = 0$, $x = 0$, $y = 0$, $z = 0$ planes is a taxicab isometry and every Euclidean translation of $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3_T$ [9], a reflection according to $x \mp y = k_1$, $x \mp z = k_2$, $y \mp z = k_3$, $x = k_4$, $y = k_5$, $z = k_6$ planes is taxicab isometry. Let $A_1A_2A_3A_4$ be an Euclidean regular tetrahedron and let two symmetry planes belong to $A_2A_3A_4$ triangle, one symmetry plane belong to triangle $A_1A_2A_4$ (see Fig. 1).

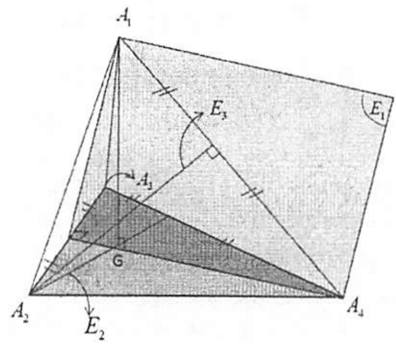

Figure 1

Let symmetry planes E_1, E_2 belong to triangle $A_2A_3A_4$ and E_3 symmetry plane belong to triangle $A_1A_2A_4$.

Since E_1 is a symmetry plane; $$\begin{array}{l} d_T(A_2,A_4) = d_T(A_3,A_4), \\ d_T(A_1,A_2) = d_T(A_1,A_3). \end{array} \tag{2.1}$$

Since E_2 is a symmetry plane; $$\begin{array}{l} d_T(A_2,A_4) = d_T(A_2,A_3), \\ d_T(A_1,A_4) = d_T(A_1,A_3). \end{array} \tag{2.2}$$

Since E_3 is a symmetry plane; $$\begin{array}{l} d_T(A_1,A_2) = d_T(A_2,A_3), \\ d_T(A_1,A_4) = d_T(A_3,A_4). \end{array} \tag{2.3}$$

So it is clear from the 2.1, 2.2 and 2.3 equalities that;

$$\begin{aligned} d_T(A_1,A_2) &= d_T(A_2,A_3) = d_T(A_3,A_4) = d_T(A_1,A_4) \\ &= d_T(A_1,A_3) = d_T(A_2,A_4) \end{aligned}$$

Thus, $A_1A_2A_3A_4$ is a taxicab regular tetrahedron at the same time. If we use the other symmetry planes, we can find the same result. □

Example 2. *For $A_1(1,1,1)$, $A_2(-1,1,1)$, $A_3(-1,-1,1)$, $A_4(1,-1,1)$, $A_5(1,1,-1)$, $A_6(-1,1,-1)$, $A_7(-1,-1,-1)$, $A_8(1,-1,-1)$ vertex points, $A_1A_2A_3A_4A_5A_6A_7A_8$ in an Euclidean regular hexahedron is also taxicab regular.*

Corollary 3. *If Euclidean regular hexahedron's faces are parallel to $x = k_1$, $y = k_2$, $z = k_3$ planes (for $k_i \in \mathbb{R}$), it is also taxicab regular.*

Proof. Euclidean regular hexahedron's faces are square. In the $x = k_1$, $y = k_2$, $z = k_3$ planes, every Euclidean square is taxicab square [12, 13]. So taxicab lengths of sides are the same. Thus, Euclidean regular hexahedron is also taxicab regular. □

Corollary 4. *Let $\overrightarrow{V} \in \{(1,0,0),(0,1,0),(0,0,1)\}$. If there exists a rotation whose rotation axis is $\overrightarrow{V}$ between three edges at the same vertex of an Euclidean regular hexahedron, it is also taxicab regular.*

Proof. It is clear that three edges at the same vertex are enough to create a taxicab regular hexahedron because the other edges are parallel to these three edges. Since the angles between three edges at the same vertex are $\frac{\pi}{2}$, the rotation angle is $\frac{\pi}{2}$. Thus, the rotation is a taxicab isometry [9]. Then hexahedron is also taxicab regular. □

There exists an Euclidean and taxicab regular octahedron.

Example 3. *For $A_1(0,0,1)$, $A_2(0,1,0)$, $A_3(1,0,0)$, $A_4(0,-1,0)$, $A_5(-1,0,0)$, $A_6(0,0,-1)$ vertex points, $A_1A_2A_3A_4A_5A_6$ Euclidean regular octahedron is also taxicab regular.*

All octahedrons are made up of two pyramids whose bases are the same. Let $A_1A_2A_3A_4A_5A_6$ be an Euclidean regular octahedron. It is made up of $A_1A_2A_3A_4A_5$ and $A_6A_2A_3A_4A_5$ pyramids and their bases are $A_1A_5A_6A_3$. For the next theorem, let us call the base of the pyramid as the base of the octahedron.

Theorem 2. *Let the base of octahedron be on a plane P_1. Let two consecutive symmetry planes of octahedron base be P_2 and P_3 one of which is on the vertexes of base and the other one is on the middle of the base edges. If P_1, P_2 and P_3 are members of*

$$\{x \mp y = k_1, x \mp z = k_2, y \mp z = k_3, x = k_4, y = k_5, z = k_6\}$$

set (for $1 \le i \le 6$, $k_i \in \mathbb{R}$), an Euclidean regular octahedron is also taxicab regular.

Proof. Suppose A_1 and A_2 are vertex points of an Euclidean regular octahedron $A_1A_2A_3A_4A_5A_6$ then $A_2A_3A_4A_5$ square is the base of the octahedron (see Fig. 2,3). Let $A_2A_3A_4A_5$ square be on $z = 0$ plane which is a member of

$$\{x \mp y = k_1, x \mp z = k_2, y \mp z = k_3, x = k_4, y = k_5, z = k_6\}$$

set.

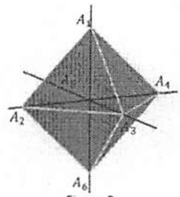

Figure 2

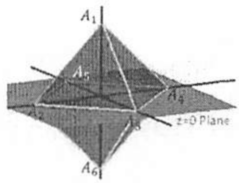

Figure 3

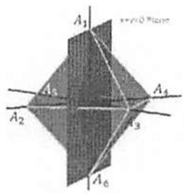

Figure 4

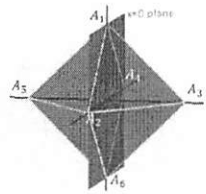

Figure 5

Since reflections with respect to $z = 0$ plane is a taxicab isometry,

$$\begin{aligned} \|A_1A_2\|_T = \|A_2A_6\|_T \,, \|A_1A_3\|_T = \|A_3A_6\|_T \,, \\ \|A_1A_4\|_T = \|A_4A_6\|_T \,, \|A_1A_5\|_T = \|A_5A_6\|_T \end{aligned} \tag{2.4}$$

Besides, the angle of two consecutive symmetry planes of $A_2A_3A_4A_5$ square is 45^o. Let these planes be $x + y = 0$ and $x = 0$. Since reflections about $x + y = 0$ plane is a taxicab isometry,

$$\begin{aligned} &\|A_1A_3\|_T = \|A_1A_2\|_T \,, \|A_1A_4\|_T = \|A_1A_5\|_T \,, \|A_2A_6\|_T = \|A_3A_6\|_T \\ &\|A_4A_6\|_T = \|A_5A_6\|_T \,, \|A_2A_5\|_T = \|A_3A_4\|_T \end{aligned} \tag{2.5}$$

Since reflections with respect to $x = 0$ plane is a taxicab isometry,

$$\begin{aligned} \|A_1A_3\|_T = \|A_1A_5\|_T \ \,, \|A_5A_6\|_T = \|A_3A_6\|_T \,, \\ \|A_2A_5\|_T = \|A_2A_3\|_T \,, \|A_4A_5\|_T = \|A_4A_3\|_T \end{aligned} \tag{2.6}$$

It is clear from the 2.4, 2.5 and 2.6 equalities that

$$\begin{aligned} \|A_1A_2\|_T &= \|A_1A_3\|_T = \|A_1A_4\|_T = \|A_1A_5\|_T = \|A_2A_6\|_T \\ &= \|A_3A_6\|_T = \|A_4A_6\|_T = \|A_5A_6\|_T \end{aligned}$$

Thus, $A_1A_2A_3A_4A_5A_6$ Euclidean regular octahedron is also taxicab regular. □

Theorem 3. *There do not exist any Euclidean and taxicab regular dodecahedrons.*

Proof. As it is known that an Euclidean regular dodecahedron is created by twelve regular pentagons (see Fig. 6).

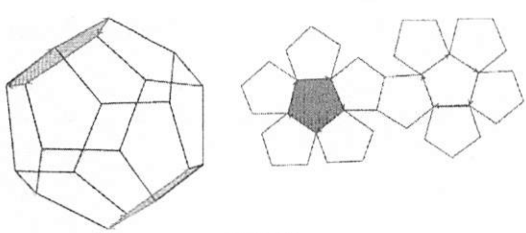

Figura 6

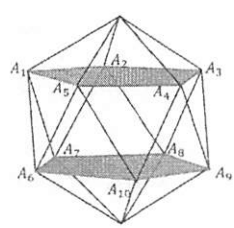

Figure 7

According to Theorem 4 [13], there do not exist any Euclidean and taxicab regular pentagons. Furthermore, all the edges of Euclidean regular pentagon's taxicab lengths are not the same. Thus, there are not any Euclidean and taxicab regular dodecahedrons. □

Theorem 4. *There do not exist any Euclidean and taxicab regular icosahedrons.*

Proof. As we know, there are five Euclidean regular triangles at the every vertex of icosahedron and these five triangles can create a Euclidean regular pentagon (see Fig. 7). According to Theorem 4 [13], there do not exist any Euclidean and taxicab regular pentagons in $\mathbb{R}^3_T$. Thus, all the edges of Euclidean regular pentagon's taxicab lengths cannot be the same. As a result, there do not exist any Euclidean and taxicab regular icosahedrons. □

3. Taxicab Regular Polyhedrons in Taxicab 3-Space

Theorem 5. *There do not exist any taxicab regular tetrahedrons, octahedrons and icosahedrons which are not Euclidean.*

Proof. There does not exist taxicab regular triangle which is not Euclidean by Theorem 6 [13]. As we know, faces of tetrahedrons, octahedrons and icosahedrons are made up of triangles. Thus, there do not exist none taxicab regular tetrahedrons, octahedrons and icosahedrons which are not Euclidean. □

Theorem 6. *There exist taxicab regular hexahedrons which are not Euclidean.*

Proof. A hexahedron can be identified by three orthogonal sides whose taxicab lengths are the same and the sides intersect at the same vertex but their Euclidean length are not the same. Since every Euclidean translation is a taxicab isometry [9], the other sides can be created by translating these three orthogonal sides. Draw taxicab sphere $ABCDEF$ centered at the origin with radius 1 (see Fig. 8).

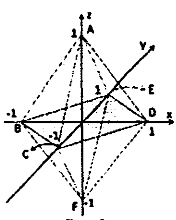

Figure 8

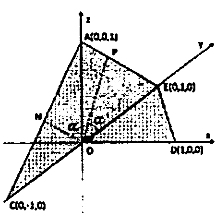

Figure 9

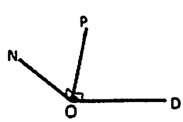

Figure 10

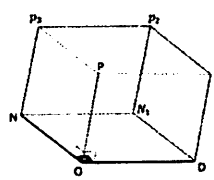

Figure 11

For $P \in]AE[$ line segment, it is clear that $d_T(O,P) = d_T(O,D) = 1$ and $[OP] \perp [OD]$ (see Fig. 9). Let $m(\angle POE) = \alpha$. If $N \in]AC[$ is chosen to be $m(\angle AON) = \alpha$, $m(\angle PON) = 90^o$ and $d_T(O,P) = d_T(O,N)$.

Since $[OD]$ line segment is orthogonal to the CAE plane, $[ON] \perp [OD]$. Besides $d_T(O,P)$, $d_T(O,N) < d_T(O,D) = 1$. Thus, $[ON]$, $[OP]$ and $[OD]$ are orthogonal to each other, their taxicab lengths are the same but their Euclidean lengths are not the same. Besides every Euclidean translation is a taxicab isometry. As a result, by translating $[ON]$, $[OP]$ and $[OD]$ we can create $ODN_1NP_3PP_1P_2$ non-Euclidean taxicab regular hexagon (see Fig. 10,11). □

Example 4. *Let* $A_1\,(0,0,0)$, $A_2\,(2,0,0)$, $A_3(2,\text{-}1,1)$, $A_4(0,\text{-}1,1)$, $A_5(0,1,1)$, $A_6\,(2,1,1)$, $A_7\,(2,0,2)$, $A_8\,(0,0,2)$ *be vertex points of* $A_1A_2A_3A_4A_5A_6A_7A_8$ *hexagon. Since all the taxicab lengths of edges of hexagon are the same and all the faces are also taxicab squares, it is a taxicab regular hexagon. However, all the edges of hexagon's Euclidean lengths are not the same.* $d_E\,(A_1,A_2) = 2$ *and* $d_E\,(A_1,A_5) = \sqrt{2}$ *so* $d_E\,(A_1,A_2) \neq d_E\,(A_1,A_5)$. *As a result,* $A_1A_2A_3A_4A_5A_6A_7A_8$ *is not an Euclidean regular.*

REFERENCES

[1] H. MINKOWSKI, Gesammelte Abhandlungen, Chelsa Publishing Co., New York, 1967.

[2] Menger, K. : You Will Like Geometry, Guildbook of the Illinois Institute of Technology Geometry Exhibit, Museum of Science and Industry, Chicago, III, 1952.

[3] E.F. KRAUSE, Taxicab Geometry; An Adventure in Non-Euclidean Geometry, Dover Publications, Inc., New York, 1986.

[4] Reynolds, B.E. : Taxicab Geometry, Pi Mu Epsilon Journal, 7 (1980), 77-88.

[5] Akca, Z. - Kaya, R. : On the Distance Formulae in Three Dimensional Space, Hadronic Journal 27, 521-532 (2004).

[6] Ekici, C. - Kocayusufoglu, I. - Akca, Z. : The Norm in Taxicab Geometry, Tr.J. of Mathematics, 22, No 3 (1998) 295-307.

[7] Akca, Ziya, and Rüstem Kaya. "On the norm in higher dimensional taxicab spaces." Hadronic Journal Supplement 19.5 (2004): 491-501.

[8] Divjak, Blaženka. "Notes on taxicab geometry." KoG. Croat. Soc. Geom. Graph 5 (2000): 5-9.

[9] Gelişgen, Ö., and R. Kaya. "The taxicab space group." Acta Mathematica Hungarica 122.1-2 (2008): 187-200.

[10] Atiyah, Michael, and Paul Sutcliffe. "Polyhedra in physics, chemistry and geometry." Milan Journal of Mathematics 71.1 (2003): 33-58.

[11] S. S. SO, Recent Developments in Taxicab Geometry, Cubo Matematica Educacional, Vol. 4 (2002), No. 2, 76-96.

[12] Colakoglu, H. BARIS, and Rüstem Kaya. "Regular Polygons in the Taxicab Plane." Scientific and Professional Information Journal of Croatian Society for Constructive Geometry and Computer Graphics (KoG) 12 (2008): 27-33.

[13] Yuksel,S.,Özcan, M., On Some Regular Polygons in the Taxicab Space, it is submitted.

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