Explicit Constructions of Cyclic Packing and Their Related OOCs *†

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Abstract: Since its desirable features, variable-weight optical orthogonal codes (VWOOCs) have found wide ranges of applications in various optical networks and systems. In recent years, optimal 2-CP$(W,1,Q;n)$s are used to construct optimal VWOOCs. So far, some works have been done on optimal 2-CP$(W,1,Q;n)$s with $w_{\max} \leq 6$, where $w_{\max}$ =max$\{w : w \in W\}$. As far as the authors are aware of, little is known for explicit constructions of optimal 2-CP$(W,1,Q;n)$s with $w_{\max} \geq 7$ and $|W| = 3$. In this paper, two explicit constructions of 2-CP$(\{3,4,7\},1,Q;n)$s are given, and two new infinite classes of optimal VWOOCs are obtained.

Keywords: Variable-weight optical orthogonal codes, constant-weight optical orthogonal codes, quadratic residues, cyclic packing, combinatorial design.

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1 INTRODUCTION

To meet the requirements of multiple quality-of-service (QoS) in the CDMA network, Yang [27] introduced the variable-weight optical orthogonal codes (variable-weight OOCs) in 1996. Due to its desirable features, recently variable-weight OOCs have found wide ranges of applications in various optical networks and systems, such as soft limiting in the number of possible subscribers, efficiency in bursty traffic, dynamic bandwidth assignment, and allowing many simultaneous users to access the same optical channel asynchronously [13], [23],[28]. For these reasons, there has been a recent upsurge of interest in constructing optimal OOCs [9, 11, 12, 21, 19, 24, 26] etc..

To facilitate readers, throughout this paper, we will use the symbols and definitions of OOCs and cyclic Packing based on [5] and [25].

Suppose that $W = \{w_1, w_2, \cdots, w_r\}$ is an ordering of a set of r integers greater than 1, and $\Lambda_a = \{\lambda_a^{(1)}, \lambda_a^{(2)}, \cdots, \lambda_a^{(r)}\}$ is an r-tuple (*auto-correlation sequence*) of positive integers. Moreover, suppose that λ_c is a positive integer (*cross-correlation parameter*), and $Q = \{q_1, q_2, \cdots, q_r\}$ is an r-tuple (*weight distribution sequence*) of positive rational numbers whose sum is 1. Then an $(n, W, \Lambda_a, \lambda_c, Q)$ *optical orthogonal code* (OOC) (briefly, $(n, W, \Lambda_a, \lambda_c, Q)$-OOC) is a set $\mathcal{C}$ of subsets (called *codeword-sets*) of Z_n with sizes (*weights*) from W satisfying the following three properties:

1) *Weight distribution property*: The number of codewords with weight w_i is exactly $q_i|\mathcal{C}|$, $1 \le i \le r$, and $\sum_{i=1}^{r} q_i|\mathcal{C}| = |\mathcal{C}|$, where $\sum_{i=1}^{r} q_i = 1$;

2) *The auto-correlation property*: $|C \cap (C + t)| \le \lambda_a^i$ for any $C = \{x_1, x_2, \cdots, x_{w_i}\} \in \mathcal{C}$ with weight w_i and $t \in Z_n \setminus 0$;

3) *The cross-correlation property*: $|C' \cap (C + t)| \le \lambda_c$ for any $C' = \{x_1, x_2, \cdots, x_{w_i}\} \in \mathcal{C}$, $C = \{y_1, y_2, \cdots, y_{w_j}\} \in \mathcal{C}$ with $C' \neq C$ and $t \in Z_n$.

Generally, when $|W| = 1$ OOC is called constant, otherwise, is variable. If $\lambda_a^{(1)} = \lambda_a^{(2)} = \cdots = \lambda_a^{(r)} = \lambda_c = \lambda$, then the notation (n, W, λ, Q)-OOC denotes $(n, W, \Lambda_a, \lambda_c, Q)$-OOC. Q is *normalized* if it is written in the form $Q = \{\frac{a_1}{b}, \frac{a_2}{b}, \cdots, \frac{a_r}{b}\}$ with $\gcd(a_1, a_2, \cdots, a_r) = 1$. And an $(n, W, 1, Q)$-

OOC is said *balanced* if $Q = \{\frac{1}{r}, \frac{1}{r}, \cdots, \frac{1}{r}\}$.

Let s be the size of an $(n, W, 1, Q)$-OOC. Then an upper bound on s is given by the following lemma.

Lemma 1 ([5]). *If C is an $(n, W, 1, Q)$-OOC of size s with $W = \{w_1, w_2, \cdots, w_r\}$ and normalized weight distribution sequence $Q = \{\frac{a_1}{b}, \frac{a_2}{b}, \cdots, \frac{a_r}{b}\}$, then we have*

$$s \leq b \left\lfloor \frac{n-1}{\sum_{i=1}^{r} a_i w_i (w_i - 1)} \right\rfloor .$$

An $(n, W, 1, Q)$-OOC is *optimal* if the size of C reaches the upper bound.

A 2-CP$(W, 1, Q; n)$ is a key tool for constructing an $(n, W, 1, Q)$-OOC, and was introduced in [25] as a generalization of cyclic packing CP$(w, 1; n)$ [29] or cyclic 2-$(n, w, 1)$ packing [14].

Suppose that G is an Abelian group, and $\mathcal{B} = \{B_j : B_j \subseteq G, 1 \leq j \leq t\}$. Define $\Delta B_j = \{x - y : x, y \in B_j, x \neq y\}$, $1 \leq j \leq t$, and $\Delta\mathcal{B} = \bigcup_{i=1}^{t} \Delta B_j$, where ΔB_j, $1 \leq j \leq t$ and $\Delta\mathcal{B}$ are multisets.

Assume that $\mathcal{B} = \{B_j : B_j \subseteq Z_n, 1 \leq j \leq t\}$, $|B_j| \in W = \{w_1, w_2, \cdots, w_r\}$, $1 \leq j \leq t$. A 2-CP$(W, 1; n)$ is a family $\mathcal{B}$ of subsets of Z_n with sizes from W (called base blocks), such that difference list $\Delta\mathcal{B}$ covers each element of $Z_n \setminus \{0\}$ at most once. Let $Q = \{q_1, q_2, \cdots, q_r\}$ be an r-tuple of positive rational numbers whose sum is 1, that is, $\sum_{i=1}^{r} q_i = 1$. A 2-CP$(W, 1, Q; n)$ is a 2-CP$(W, 1; n)$ with the property that the number of blocks size w_i is $q_i|\mathcal{B}|$, where $q_i \in Q$, $1 \leq i \leq r$.

A 2-CP$(W, 1; gv)$ is g-regular (called also *Relative difference families*[5]) if $\Delta\mathcal{B} = \bigcup_{i=1}^{t} \Delta B_j$ covers each element of $Z_{gv} \setminus vZ_{gv}$ exactly once, and each element of vZ_{gv} is not covered.

The following results reflect the equivalence between a 2-CP$(W, 1, Q; n)$ and an $(n, W, 1, Q)$-OOC.

Lemma 2 ([25]). *An optimal 2-CP$(W, 1, Q; n)$ is equivalent to an optimal $(n, W, 1, Q)$-OOC.*

Lemma 3 ([25]). *Let $w = \sum_{i=1}^{r} a_i w_i (w_i - 1)$, where $w_i \in W$. If $1 \leq g \leq w$, then a g-regular 2-CP$(W, 1, Q; gv)$ is optimal.*

Most existing works focus on 2-$(n,w,1)$ packing, see [1, 3, 4, 6, 7, 8, 10, 14, 15, 16, 20, 29] for examples. For 2-CP$(W,1;n)$ with $|W| = 2$ and $w_{\max} \leq 6$, where $w_{\max} = \max\{w : w \in W\}$, some works have been obtained, see [25, 30, 31] for examples. When $|W| = 3$, there are only few results. For examples, [5, 18] presented explicit results with $W \in \{\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\}\}$. In [17], basing on the existence of $(g,W,1)$-DF (or perfect $(g,W,1)$-DF), the authors deduced many optimal 2-CP$(W,1;n)$ with arbitrarily large base blocks and $|W| \geq 3$, but they are not explicit constructions. Moreover, constructing a $(g,K,1)$-DFs (or perfect $(g,W,1)$-DFs) with $w_{\max} \geq 7$ is still a hard work. So far, as we are aware of, there is no explicit result of 2-CP$(W,1,Q;n)$ with $|W| = 3$ and $w_{\max} \geq 7$.

In this paper, by further investigating elementary conclusions of the quadratic residues and the construction method in [5], two explicit constructions of optimal 2-CP$(\{3,4,7\},1,Q;n)$s are given, and the corresponding optimal VWOOCs are obtained. The following are the results.

Theorem 1. *For any prime* $p \equiv 3 \pmod 4$ *and* $p \geq 7$, *there exists an optimal* 2−*CP*$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\},30p)$, *and an optimal* $(30p,\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\})$-*OOC.*

Theorem 2. *For any prime* $p \equiv 3 \pmod 4$ *and* $p \geq 7$, *there exists an optimal* 2−*CP*$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\},33p)$, *and an optimal* $(33p,\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\})$-*OOC.*

This paper is organized as follows. In Section 2, some results on the quadratic residues in Z_p are given. In Section 3, we present explicit constructions of the optimal 2-CP$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p)$s, and the corresponding optimal VWOOCs are obtained. In Section 4, we obtain optimal 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$s and the optimal $(33p,\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\})$-OOCs.

2 Preliminaries

In this section, some notations and results on the quadratic residues in Z_p will be given.

Suppose that p is a prime, and θ is a primitive element of Z_p. Then we denote $C_0^2 = \{\theta^{2i} : 0 \leq i \leq \frac{p-1}{2}\}$ and $C_1^2 = \theta C_0^2$ as the quadratic residues and the quadratic nonresidues of Z_p, respectively. Assume that $A = \{(a_1, j_1), (a_2, j_2), \ldots, (a_k, j_k)\}$ is a k-subset of $Z_p \times Z_m$. Let K be a non-empty subset of Z^+ and each element of it is greater than 1. Suppose that $\mathcal{F} = \{A \mid A = \{(a_1, j_1), (a_2, j_2), \ldots, (a_k, j_k)\} \subset Z_p \times Z_m, k \in K\}$.

Define

(1) $x \cdot A = \{(xa_1, j_1), (xa_2, j_2), \ldots, (xa_k, j_k)\}$, $x \in Z_p$;

(2) $B \cdot A = \{b \cdot A \mid b \in B\}$, $B \subseteq Z_p$.

Define the difference lists:

$L_i = \{a_l - a_s : \{(a_l, j_l), (a_s, j_s)\} \subset A \in \mathcal{F}, i \equiv j_l - j_s \pmod{m}, 1 \leq l, s \leq k, k \in K\}$, $i = 0, \cdots, m-1$.

By the *Construction I* in [5], assume that $p \equiv 3 \pmod 4$, $p \nmid m$, and $|L_i \bigcap C_k^2| = 1$, $0 \leq i \leq m-1$, $k = 0, 1$, and $\mathcal{A} = \{C_0^2 \cdot A : A \in \mathcal{F}\}$. Then $\Delta\mathcal{A}$ satisfies the following properties: It does cover every element in $Z_p \times Z_m \setminus (\{0\} \times Z_m)$ exactly once, and each element of $\{0\} \times Z_m$ is not covered. So $\mathcal{A}$ forms an m-regular 2-CP$(W, 1, Q; mp)$.

Next, to construct and prove our results easily, some elementary conclusions of the quadratic residues will be listed as follows. Interested readers can refer to [22].

Lemma 4. *Suppose that* $p \equiv 3 \pmod 4$ *is a prime. We have:*

(1) $2 \in C_0^2, 3 \in C_0^2, 5 \in C_0^2, 7 \in C_0^2$, *if and only if* $p \equiv 71, 191, 239, 359, 431, 599 \pmod{840}$;

(2) $2 \in C_0^2, 3 \in C_0^2, 5 \in C_0^2, 7 \in C_1^2$, *if and only if* $p \equiv 311, 479, 551, 671, 719, 839 \pmod{840}$;

(3) $2 \in C_1^2, 3 \in C_1^2, 5 \in C_1^2, 7 \in C_0^2$, *if and only if* $p \equiv 43, 67, 163, 403, 547, 667 \pmod{840}$;

(4) $2 \in C_1^2, 3 \in C_1^2, 5 \in C_1^2, 7 \in C_1^2$, *if and only if* $p \equiv 187, 283, 307, 523, 643, 787 \pmod{840}$*;*

(5) $2 \in C_0^2, 3 \in C_0^2, 5 \in C_1^2, 7 \in C_0^2$, *if and only if* $p \equiv 23, 263, 407, 527, 743, 767 \pmod{840}$*;*

(6) $2 \in C_0^2, 3 \in C_0^2, 5 \in C_1^2, 7 \in C_1^2$, *if and only if* $p \equiv 47, 143, 167, 383, 503, 647 \pmod{840}$*;*

(7) $2 \in C_1^2, 3 \in C_1^2, 5 \in C_0^2, 7 \in C_0^2$, *if and only if* $p \equiv 211, 331, 379, 499, 571, 739 \pmod{840}$*;*

(8) $2 \in C_1^2, 3 \in C_1^2, 5 \in C_0^2, 7 \in C_1^2$, *if and only if* $p \equiv 19, 139, 451, 619, 691, 811 \pmod{840}$*;*

(9) $2 \in C_0^2, 3 \in C_1^2, 5 \in C_0^2, 7 \in C_0^2$, *if and only if* $p \equiv 79, 151, 319, 631, 751, 799 \pmod{840}$*;*

(10) $2 \in C_0^2, 3 \in C_1^2, 5 \in C_0^2, 7 \in C_1^2$, *if and only if* $p \equiv 31, 199, 271, 391, 439, 559 \pmod{840}$*;*

(11) $2 \in C_1^2, 3 \in C_0^2, 5 \in C_1^2, 7 \in C_0^2$, *if and only if* $p \equiv 107, 323, 347, 443, 683, 827 \pmod{840}$*;*

(12) $2 \in C_1^2, 3 \in C_0^2, 5 \in C_1^2, 7 \in C_1^2$, *if and only if* $p \equiv 83, 227, 467, 563, 587, 803 \pmod{840}$*;*

(13) $2 \in C_1^2, 3 \in C_0^2, 5 \in C_0^2, 7 \in C_0^2$, *if and only if* $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$*;*

(14) $2 \in C_1^2, 3 \in C_0^2, 5 \in C_0^2, 7 \in C_1^2$, *if and only if* $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$*;*

(15) $2 \in C_0^2, 3 \in C_1^2, 5 \in C_1^2, 7 \in C_0^2$, *if and only if* $p \equiv 127, 247, 463, 487, 583, 823 \pmod{840}$*;*

(16) $2 \in C_0^2, 3 \in C_1^2, 5 \in C_1^2, 7 \in C_1^2$, *if and only if* $p \equiv 103, 223, 367, 607, 703, 727 \pmod{840}$.

Proof. Let $B_y^i = \{p : y \in C_i^2\}, i = 0, 1; y \in \{2, 3, 5, 7\}$. By [22], it is not difficult to see that

$p \in B_2^0 \Longleftrightarrow p \equiv \pm 1 \pmod{8}, p \in B_2^1 \Longleftrightarrow p \equiv \pm 3 \pmod{8}$;

$p \in B_3^0 \Longleftrightarrow p \equiv \pm 1 \pmod{12}, p \in B_3^1 \Longleftrightarrow p \equiv \pm 5 \pmod{12}$;

$p \in B_5^0 \Longleftrightarrow p \equiv \pm 1 \pmod{5}, p \in B_5^1 \Longleftrightarrow p \equiv \pm 2 \pmod{5}$;

$p \in B_7^0 \iff p \equiv \pm5, \pm11, \pm13 \pmod{28}, p \in B_7^1 \iff p \equiv \pm1, \pm3, \pm9 \pmod{28}$.

Let $S(i,j,k,s) = \{p : p \equiv 3 \pmod 4, p \in B_2^i \cap B_3^j \cap B_5^k \cap B_7^s,\ 0 \leq i,j,k,s \leq 1\} \neq \phi$. We have

$$2 \in C_i^2, 3 \in C_j^2, 5 \in C_k^2, 7 \in C_s^2 \iff p \in S(i,j,k,s),\ 0 \leq i,j,k,s \leq 1.$$

This completes the proof. □

3 Proof of Theorem 1

In this section, we will prove Theorem 1. For each prime $p \equiv 3 \pmod 4$, and $p > 7$, it is clear that $\gcd(30,p) = 1$, and hence Z_{30p} is isomorphic to $Z_p \times Z_{30}$.

3.1 The case: $p \equiv 71, 191, 239, 359, 431, 599, 311, 479, 551, 671, 719, 839 \pmod{840}$

Lemma 5. *Suppose that* $p \equiv 71, 191, 239, 359, 431, 599, 311, 479, 551, 671, 719, 839 \pmod{840}$ *is a prime,* $\xi = \min\{x : x \in C_1^2\}$*. Then* $\mathcal{A}=\{C_0^2 \cdot A_{1,j} : j = 1,2,3\}$ *forms a* $2-CP$ $(\{3,4,7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}, 30p)$*, where*

$A_{1,1} = \{(0,0),(5,0),(1,7),(\xi,19),(3,21),(2,25),(4,29)\}$,
$A_{1,2} = \{(0,0),(-3,14),(-1,17),(-2,20)\}$,
$A_{1,3} = \{(-1,0),(0,15),(-2,28)\}$.

Proof We compute the difference lists L_i, $0 \leq i \leq 29$ from $A_{1,1}$, $A_{1,2}$ and $A_{1,3}$. It is not difficult to see that $L_s = -L_{30-s}$, $16 \leq s \leq 29$. So we only need to compute the difference lists L_i for $0 \leq i \leq 15$.

$L_0 = \{5,-5\}$, $L_1 = L_7 = \{1,-4\}$, $L_2 = \{3-\xi, 1\}$, $L_3 = -L_4 = -L_{13} = \{-1,2\}$, $L_5 = -L_9 = -L_{14} = \{3,-2\}$, $L_6 = \{1, 2-\xi\}$, $L_8 = \{-3,1\}$, $L_{10} = \{4-\xi, 2\}$, $L_{11} = \{-\xi, 5-\xi\}$, $L_{12} = \{-1, 1-\xi\}$, $L_{15} = \{1,-1\}$.

According to (1) and (2) of Lemma 4, we know that $2 \in C_0^2$, $3 \in C_0^2$ and $5 \in C_0^2$. And because of $\xi - l \in C_0^2$, $1 \leq l < \xi$, it is easy to check that $|L_i \bigcap C_k^2| = 1$, $0 \leq i \leq 29$, $k = 0,1$. Hence, $\Delta\mathcal{A}$ satisfies the following properties: It does cover every element in $Z_p \times Z_{30} \setminus (\{0\} \times Z_{30})$ exactly once, and each element of $\{0\} \times Z_{30}$ is not covered.

So $\mathcal{A}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p)$. □

3.2 The case: $p \equiv 43, 67, 163, 403, 547, 667, 187, 283, 307, 523, 643, 787 \pmod{840}$

Lemma 6. *Suppose that* $p \equiv 43,67,163,403,547,667,187,283,307,523,643,787 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C_0^2 \cdot A_{2,j} : j = 1,2,3\}$ *forms a* 2-*CP*$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p)$, *where*

$A_{2,1} = \{(1,0),(2,0),(3,7),(-3,19),(0,21),(6,25),(5,29)\}$,

$A_{2,2} = \{(0,0),(2,14),(1,17),(5,20)\}$, $A_{2,3} = \{(1,0),(0,15),(4,28)\}$.

Proof According to (3) and (4) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 29$ from $A_{2,1}$, $A_{2,2}$ and $A_{2,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p)$. □

3.3 The case: $p \equiv 211, 331, 379, 499, 571, 739, 19, 139, 451, 619, 691, 811 \pmod{840}$

Lemma 7. *Suppose that* $p \equiv 211,331,379,499,571,739,19,139,451,619,691,811 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C_0^2 \cdot A_{3,j} : j = 1,2,3\}$ *forms is a* 2-*CP*$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p)$, *where*

$A_{3,1} = \{(0,0),(2,0),(1,7),(4,19),(-2,21),(3,25),(-1,29)\}$,

$A_{3,2} = \{(0,0),(1,14),(5,17),(4,20)\}$, $A_{3,3} = \{(2,0),(0,15),(3,28)\}$.

Proof According to (7) and (8) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 29$ from $A_{3,1}$, $A_{3,2}$ and $A_{3,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p)$. □

3.4 The case: $p \equiv 23, 263, 407, 527, 743, 767 \pmod{840}$

Lemma 8. *Suppose that* $p \equiv 23,263,407,527,743,767 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C_0^2 \cdot A_{4,j} : j = 1,2,3\}$ *forms a* 2-*CP*$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};$

$30p$), *where*
$A_{4,1}=\{(9,0),(5,0),(7,7),(0,19),(6,21),(4,25),(8,29)\}$,
$A_{4,2}=\{(0,0),(2,14),(8,17),(7,20)\}$, $A_{4,3}=\{(1,0),(0,15),(2,28)\}$.

Proof According to (5) of Lemma 4, we compute the difference lists L_i, $0\le i\le 29$ from $A_{4,1}$, $A_{4,2}$ and $A_{4,3}$. It is not difficult to see that $L_s=-L_{30-s}$, $16\le s\le 29$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP($\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p$). □

3.5 The case: $p\equiv 47,143,167,383,503,647 \pmod{840}$

Lemma 9. *Suppose that* $p\equiv 47,143,167,383,503,647 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C_0^2\cdot A_{5,j}: j=1,2,3\}$ *forms a* 2-*CP*($\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p$), *where*
$A_{5,1}=\{(9,0),(5,0),(7,7),(0,19),(6,21),(2,25),(8,29)\}$,
$A_{5,2}=\{(0,0),(2,14),(8,17),(-10,20)\}$, $A_{5,3}=\{(1,0),(0,15),(2,28)\}$.

Proof According to (6) of Lemma 4, we compute the difference lists L_i, $0\le i\le 29$ from $A_{5,1}$, $A_{5,2}$ and $A_{5,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP($\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p$). □

3.6 The case: $p\equiv 79,151,319,631,751,799,31,199,271,391,439,559 \pmod{840}$

Lemma 10. *Suppose that* $p\equiv 79,151,319,631,751,799,31,199,271,391,439,559 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C_0^2\cdot A_{6,j}: j=1,2,3\}$ *forms a* 2-*CP*($\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p$), *where*
$A_{6,1}=\{(2,0),(0,0),(1,7),(6,19),(-3,21),(3,25),(5,29)\}$,
$A_{6,2}=\{(0,0),(4,14),(5,17),(3,20)\}$, $A_{6,3}=\{(2,0),(0,15),(1,28)\}$.

Proof According to (9) and (10) of Lemma 4, we compute the difference lists L_i, $0\le i\le 29$ from $A_{6,1}$, $A_{6,2}$ and $A_{6,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP($\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p$). □

3.7 The case: $p \equiv 107, 323, 347, 443, 683, 827 \pmod{840}$

Lemma 11. *Suppose that* $p \equiv 107, 323, 347, 443, 683, 827 \pmod{840}$ *is a prime. Then* $C_{0P} \cdot A^0_{61}$, $\mathcal{A}=\{C^2_0 \cdot A_{7,j} : j = 1, 2, 3\}$ *forms a* 2-*CP*($\{3, 4, 7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p$), *where*
$A_{7,1} = \{(9,0), (5,0), (8,7), (0,19), (7,21), (6,25), (4,29)\}$,
$A_{7,2} = \{(0,0), (1,14), (3,17), (4,20)\}$, $A_{7,3} = \{(2,0), (0,15), (3,28)\}$.

Proof According to (11) of Lemma 4, we compute the difference lists L_i, $0 \leq i \leq 29$ from $A_{7,1}$, $A_{7,2}$ and $A_{7,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP($\{3, 4, 7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p$). □

3.8 The case: $p \equiv 83, 227, 467, 563, 587, 803 \pmod{840}$

Lemma 12. *Suppose that* $p \equiv 83, 227, 467, 563, 587, 803 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C^2_0 \cdot A_{8,j} : j = 1, 2, 3\}$ *forms a* 2-*CP*($\{3, 4, 7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p$), *where*
$A_{8,1} = \{(9,0), (5,0), (8,7), (0,19), (7,21), (6,25), (4,29)\}$,
$A_{8,2} = \{(0,0), (1,14), (3,17), (4,20)\}$, $A_{8,3} = \{(4,0), (0,15), (3,28)\}$.

Proof According to (12) of Lemma 4, we compute the difference lists L_i, $0 \leq i \leq 29$ from $A_{8,1}$, $A_{8,2}$ and $A_{8,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP($\{3, 4, 7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p$). □

3.9 The case: $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$

Lemma 13. *Suppose that* $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C^2_0 \cdot A_{9,j} : j = 1, 2, 3\}$ *forms a* 2-*CP*($\{3, 4, 7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p$), *where*
$A_{9,1} = \{(3,0), (8,0), (4,7), (0,19), (9,21), (7,25), (2,29)\}$,
$A_{9,2} = \{(0,0), (1,14), (2,17), (4,20)\}$, $A_{9,3} = \{(1,0), (0,15), (3,28)\}$.

Proof According to (13) of Lemma 4, we compute the difference lists L_i, $0 \leq i \leq 29$ from $A_{9,1}$, $A_{9,2}$ and $A_{9,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP($\{3, 4, 7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p$). □

3.10 The case: $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$

Lemma 14. *Suppose that* $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C_0^2 \cdot A_{10,j} : j = 1,2,3\}$ *forms a* 2-*CP*$(\{3,4,7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p)$, *where*
$A_{10,1} = \{(3,0),(8,0),(4,7),(0,19),(9,21),(7,25),(2,29)\}$,
$A_{10,2} = \{(0,0),(2,14),(3,17),(5,20)\}$, $A_{10,3} = \{(0,0),(2,15),(4,28)\}$.

Proof According to (14) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 29$ from $A_{10,1}$, $A_{10,2}$ and $A_{10,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP$(\{3,4,7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p)$. □

3.11 The case: $p \equiv 127, 247, 463, 487, 583, 823 \pmod{840}$

Lemma 15. *Suppose that* $p \equiv 127, 247, 463, 487, 583, 823 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C_0^2 \cdot A_{11,j} : j = 1,2,3\}$ *forms a* 2-*CP*$(\{3,4,7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p)$, *where*
$A_{11,1} = \{(7,0),(4,0),(6,7),(9,19),(12,21),(2,25),(5,29)\}$,
$A_{11,2} = \{(3,0),(0,14),(1,17),(4,20)\}$, $A_{11,3} = \{(3,0),(0,15),(1,28)\}$.

Proof According to (15) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 29$ from $A_{11,1}$, $A_{11,2}$ and $A_{11,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP$(\{3,4,7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p)$. □

3.12 The case: $p \equiv 103, 223, 367, 607, 703, 727 \pmod{840}$

Lemma 16. *Suppose that* $p \equiv 103, 223, 367, 607, 703, 727 \pmod{840}$ *is a prime. Then* $\mathcal{A}=\{C_0^2 \cdot A_{12,j} : j = 1,2,3\}$ *forms a* 2-*CP*$(\{3,4,7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p)$, *where*
$A_{12,1} = \{(7,0),(4,0),(5,7),(9,19),(2,21),(6,25),(12,29)\}$,
$A_{12,2} = \{(1,0),(0,14),(5,17),(-1,20)\}$, $A_{12,3} = \{(1,0),(0,15),(2,28)\}$.

Proof According to (16) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 29$ from $A_{12,1}$, $A_{12,2}$ and $A_{12,3}$ as the proof of Lemma 5. It is not difficult to see that $\mathcal{A}$ forms a 2-CP$(\{3,4,7\}, 1, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}; 30p)$. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1 For each prime $p \equiv 3 \pmod 4$, and $p > 7$, a 2-CP$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p)$ exists from Lemmas 5-16. For $p = 7$, let $\mathcal{A} = \{\{0,1,3,7,12,20,30\},\{0,14,35,50,66,88,120\},\{0,24,49,75,108,$ $142,170\},\{0,37,76,119\},\{0,41,83,138\},\{0,44,100,145\},\{0,46,94\},$ $\{0,47,107\},\{0,57,130\}\}$.
Then, $\mathcal{A}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};210)$.

By Lemmas 2 and 3, we obtain an optimal 2-CP$(\{3,4,7\},1,\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\};30p)$s for any prime $p \equiv 3 \pmod 4$ and $p \geq 7$, and the corresponding optimal VWOOC are obtained. This completes the proof of Theorem 1. □

4 Proof of Theorem 2

In this section, we will prove Theorem 2. For each prime $p \equiv 3 \pmod 4$, and $p > 11$, it is clear that $\gcd(33,p) = 1$, and hence Z_{33p} is isomorphic to $Z_p \times Z_{33}$.

4.1 The case: $p \equiv 71,191,239,359,431,599,311,479,551,$ $671,719,839 \pmod{840}$

Lemma 17. *Suppose that* $p \equiv 71,191,239,359,431,599,311,479,551,$ $671,719,839 \pmod{840}$ *is a prime,* $\xi = min\{x : x \in C_{1P}\}$. *Then* $\mathcal{B}=\{C_0^2 \cdot B_{1,j} : j = 1,2,3,4\}$ *forms a* 2-*CP*$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};$ $33p)$, *where*
$B_{1,1} = \{(0,0),(5,0),(1,1),(\xi,3),(3,5),(2,9),(4,20)\}$,
$B_{1,2} = \{(0,0),(3,6),(1,14),(2,21)\}$,
$B_{1,3} = \{(1,0),(0,7),(2,17)\}$,
$B_{1,4} = \{(1,0),(0,10),(-2,21)\}$.

Proof We compute the difference lists L_i, $0 \leq i \leq 32$ from $B_{1,1}$, $B_{1,2}$, $B_{1,3}$ and $B_{1,4}$. It is not difficult to see that $L_s = -L_{33-s}$, $17 \leq s \leq 32$. So we only need to compute the difference lists L_i for $0 \leq i \leq 16$.

$L_0 = \{5, -5\}$, $L_1 = L_{13} = \{1, -4\}$, $L_2 = \{3-\xi, \xi-1\}$, $L_3 = \{\xi-5, \xi\}$, $L_4 = -L_8 = L_{10} = \{2, -1\}$, $L_5 = -L_9 = L_{12} = \{3, -2\}$, $L_6 = \{3, 2-\xi\}$, $L_7 = L_{15} = \{1, -1\}$, $L_{11} = \{-2, 2\}$, $L_{14} = \{-3, 1\}$, $L_{16} = \{\xi-4, -1\}$.

According to (1) and (2) of Lemma 4, we know that $2 \in C_0^2$, $3 \in C_0^2$ and $5 \in C_0^2$. And because of $\xi - l \in C_0^2$, $1 \le l < \xi$, it is easy to check that $|L_i \bigcap C_k^2| = 1$, $0 \le i \le 32$, $k = 0, 1$. Hence, $\Delta\mathcal{B}$ satisfies the following properties: It does cover every element in $Z_p \times Z_{33} \setminus (\{0\} \times Z_{33})$ exactly once, and each element of $\{0\} \times Z_{33}$ is not covered. So $\mathcal{B}$ forms a 2-CP$(\{3,4,7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$. □

4.2 The case: $p \equiv 43, 67, 163, 403, 547, 667, 187, 283, 307, 523, 643, 787 \pmod{840}$

Lemma 18. *Suppose that* $p \equiv 43, 67, 163, 403, 547, 667, 187, 283, 307, 523, 643, 787 \pmod{840}$ *is a prime. Then* $\mathcal{B}=\{C_0^2 \cdot B_{2,j} : j = 1, 2, 3, 4\}$ *forms a* 2-*CP*$(\{3,4,7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$, *where*

$B_{2,1} = \{(1,0), (2,0), (-3,1), (0,3), (6,5), (5,9), (3,20)\}$,

$B_{2,2} = \{(0,0), (1,6), (-2,14), (4,21)\}$,

$B_{2,3} = \{(1,0), (0,7), (2,17)\}$,

$B_{2,4} = \{(2,0), (0,10), (5,21)\}$.

Proof According to (3) and (4) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 32$ from $B_{2,1}$, $B_{2,2}$, $B_{2,3}$ and $B_{2,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3,4,7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$. □

4.3 The case: $p \equiv 211, 331, 379, 499, 571, 739, 19, 139, 451, 619, 691, 811 \pmod{840}$

Lemma 19. *Suppose that* $p \equiv 211, 331, 379, 499, 571, 739, 19, 139, 451, 619, 691, 811 \pmod{840}$ *is a prime. Then* $\mathcal{B}=\{C_0^2 \cdot B_{3,j} : j = 1, 2, 3, 4\}$ *forms a* 2-*CP*$(\{3,4,7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$, *where*

$B_{3,1} = \{(1,0), (3,0), (2,1), (4,3), (5,5), (6,9), (0,20)\}$,

$B_{3,2} = \{(0,0), (1,6), (4,14), (5,21)\}$,

$B_{3,3} = \{(1,0),(0,7),(5,17)\}$,
$B_{3,4} = \{(1,0),(0,10),(4,21)\}$.

Proof According to (7) and (8) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 32$ from $B_{3,1}$, $B_{3,2}$, $B_{3,3}$ and $B_{3,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$. □

4.4 The case: $p \equiv 23,263,407,527,743,767 \pmod{840}$

Lemma 20. *Suppose that* $p \equiv 23,263,407,527,743,767 \pmod{840}$ *is a prime. Then* $\mathcal{B}=\{C_0^2 \cdot B_{4,j} : j = 1,2,3,4\}$ *forms a* 2-*CP*$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$, *where*
$B_{4,1} = \{(9,0),(5,0),(7,1),(0,3),(4,5),(6,9),(8,20)\}$,
$B_{4,2} = \{(0,0),(5,6),(6,14),(2,21)\}$,
$B_{4,3} = \{(-1,0),(0,7),(4,17)\}$,
$B_{4,4} = \{(6,0),(0,10),(5,21)\}$.

Proof According to (5) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 32$ from $B_{4,1}$, $B_{4,2}$, $B_{4,3}$ and $B_{4,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$. □

4.5 The case: $p \equiv 47,143,167,383,503,647 \pmod{840}$

Lemma 21. *Suppose that* $p \equiv 47,143,167,383,503,647 \pmod{840}$ *is a prime. Then* $\mathcal{B}=\{C_0^2 \cdot B_{5,j} : j = 1,2,3,4\}$ *forms a* 2-*CP*$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$, *where*
$B_{5,1} = \{(9,0),(5,0),(0,1),(7,3),(8,5),(2,9),(6,20)\}$,
$B_{5,2} = \{(0,0),(5,6),(1,14),(6,21)\}$,
$B_{5,3} = \{(-1,0),(0,7),(1,17)\}$,
$B_{5,4} = \{(6,0),(0,10),(5,21)\}$.

Proof According to (6) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 32$ from $B_{5,1}$, $B_{5,2}$, $B_{5,3}$ and $B_{5,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$. □

4.6 The case: $p \equiv 79, 151, 319, 631, 751, 799, 31, 199, 271, 391, 439, 559 \pmod{840}$

Lemma 22. *Suppose that* $p \equiv 79, 151, 319, 631, 751, 799, 31, 199, 271,$ $391, 439, 559 \pmod{840}$ *is a prime. then* $\mathcal{B}$=$\{C_0^2 \cdot B_{6,j} : j = 1, 2, 3, 4\}$ *forms a* 2-*CP*$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\};$
$33p)$, *where*

$B_{6,1} = \{(2, 0), (0, 0), (-3, 1), (6, 3), (5, 5), (3, 9), (1, 20)\},$
$B_{6,2} = \{(0, 0), (3, 6), (5, 14), (4, 21)\},$
$B_{6,3} = \{(0, 0), (1, 7), (4, 17)\},$
$B_{6,4} = \{(3, 0), (0, 10), (2, 21)\}.$

Proof According to (9) and (10) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 32$ from $B_{6,1}$, $B_{6,2}$, $B_{6,3}$ and $B_{6,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$. □

4.7 The case: $p \equiv 107, 323, 347, 443, 683, 827, 83, 227, 467, 563, 587, 803 \pmod{840}$

Lemma 23. *Suppose that* $p \equiv 107, 323, 347, 443, 683, 827, 83, 227,$ $467, 563, 587, 803 \pmod{840}$ *is a prime. Then* $\mathcal{B}$=$\{C_0^2 \cdot B_{7,j} : j = 1, 2, 3, 4\}$ *forms a* 2-*CP*$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$, *where*

$B_{7,1} = \{(5, 0), (1, 0), (4, 1), (3, 3), (6, 5), (0, 9), (2, 20)\},$
$B_{7,2} = \{(0, 0), (3, 6), (1, 14), (4, 21)\},$
$B_{7,3} = \{(1, 0), (0, 7), (4, 17)\},$
$B_{7,4} = \{(1, 0), (0, 10), (3, 21)\}.$

Proof According to (11) and (12) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 32$ from $B_{7,1}$, $B_{7,2}$, $B_{7,3}$ and $B_{7,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$. □

4.8 The case: $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$

Lemma 24. *Suppose that* $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$ *is a prime. Then* $\mathcal{B}$=$\{C_0^2 \cdot B_{8,j} : j = 1, 2, 3, 4\}$ *forms a* 2-*CP*$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$, *where*

$B_{8,1} = \{(4,0), (9,0), (5,1), (8,3), (10,5), (1,9), (3,20)\}$,
$B_{8,2} = \{(0,0), (1,6), (5,14), (4,21)\}$,
$B_{8,3} = \{(2,0), (0,7), (3,17)\}$,
$B_{8,4} = \{(1,0), (0,10), (3,21)\}$.

Proof According to (13) of Lemma 4, we compute the difference lists L_i, $0 \leq i \leq 32$ from $B_{8,1}$, $B_{8,2}$, $B_{8,3}$ and $B_{8,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$. □

4.9 The case: $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$

Lemma 25. *Suppose that* $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$ *is a prime. Then* $\mathcal{B}$=$\{C_0^2 \cdot B_{9,j} : j = 1, 2, 3, 4\}$ *forms a* 2-*CP*$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$, *where*

$B_{9,1} = \{(4,0), (9,0), (5,1), (8,3), (10,5), (1,9), (3,20)\}$,
$B_{9,2} = \{(0,0), (2,6), (5,14), (4,21)\}$,
$B_{9,3} = \{(2,0), (0,7), (3,17)\}$,
$B_{9,4} = \{(1,0), (0,10), (3,21)\}$.

Proof According to (14) of Lemma 4, we compute the difference lists L_i, $0 \leq i \leq 32$ from $B_{9,1}$, $B_{9,2}$, $B_{9,3}$ and $B_{9,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$. □

4.10 The case: $p \equiv 127, 247, 463, 487, 583, 823 \pmod{840}$

Lemma 26. *Suppose that* $p \equiv 127, 247, 463, 487, 583, 823 \pmod{840}$ *is a prime. Then* $\mathcal{B}$=$\{C_0^2 \cdot B_{10,j} : j = 1, 2, 3, 4\}$ *forms a* 2-*CP*$(\{3, 4, 7\}, 1, \{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\}; 33p)$, *where*

$B_{10,1} = \{(5,0), (2,0), (0,1), (7,3), (3,5), (4,9), (10,20)\}$,
$B_{10,2} = \{(0,0), (5,6), (3,14), (4,21)\}$,

$B_{10,3} = \{(1,0),(0,7),(8,17)\}$,
$B_{10,4} = \{(1,0),(0,10),(4,21)\}$.

Proof According to (15) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 32$ from $B_{10,1}$, $B_{10,2}$, $B_{10,3}$ and $B_{10,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$. □

4.10.1 The case: $p \equiv 103, 223, 367, 607, 703, 727 \pmod{840}$

Lemma 27. *Suppose that* $p \equiv 103, 223, 367, 607, 703, 727 \pmod{840}$ *is a prime. Then* $\mathcal{B}=\{C_0^2 \cdot B_{11,j} : j=1,2,3,4\}$ *forms a* 2-*CP*$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};$ $33p)$, *where*
$B_{11,1} = \{(7,0),(4,0),(9,1),(2,3),(5,5),(6,9),(0,20)\}$,
$B_{11,2} = \{(0,0),(5,6),(3,14),(4,21)\}$,
$B_{11,3} = \{(1,0),(0,7),(3,17)\}$,
$B_{11,4} = \{(5,0),(0,10),(3,21)\}$.

Proof According to (16) of Lemma 4, we compute the difference lists L_i, $0 \le i \le 32$ from $B_{11,1}$, $B_{11,2}$, $B_{11,3}$ and $B_{11,4}$ as the proof of Lemma 17. It is not difficult to see that $\mathcal{B}$ forms a 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$. □

We are now in a position to prove Theorem 2.

Proof of Theorem 2 For each prime $p \equiv 3 \pmod 4$, and $p > 11$, a 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$ exists from Lemmas 17-27.

Let
$\mathcal{B}_7 = \{\{0,14,35,50,66,88,112\},\{0,33,72,106\},\{0,61,124\},\{0,64,129\},$
$\{0,68,149\},\{0,1,3,7,12,20,30\},\{0,40,84,127\},\{0,45,92,140\},$
$\{0,49,105\},\{0,55,113\},\{0,57,116\},\{0,25,51,79,111,148,189\}\}$.
$\mathcal{B}_{11} = \{\{0,1,3,7,12,20,30\},\{0,14,35,50,66,88,112\},$
$\{0,25,51,79,111,144,178\},\{0,114,248\},\{0,105,214\},\{0,49,106,167\},$
$\{0,58,122,193\},\{0,43,85,125,166,205,242\},\{0,63,138,211\},$
$\{0,72,155,233\},\{0,84,174\},\{0,113,223\},\{0,91,186\},\{0,96,200\},$
$\{0,97,204\},\{0,101,209\},\{0,70,235,304\},\{0,87,220\},\{0,94,226\},$
$\{0,44,89,136,191,239,307\}\}$.
Then, $\mathcal{B}_p$ forms a 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$ for $p=7,11$.

By Lemmas 2 and 3, we obtain an optimal 2-CP$(\{3,4,7\},1,\{\frac{2}{4},\frac{1}{4},\frac{1}{4}\};33p)$ for any prime $p \equiv 3 \pmod 4$ and $p \geq 7$, and the corresponding optimal VWOOC are obtained. This completes the proof of Theorem 2. □

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