The Ramsey Numbers $R(C_{\leq n}, K_m)$*

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Abstract

Let $ex(m, C_{\leq n})$ denote the maximum size of a graph of order m and girth at least $n+1$, and $EX(m, C_{\leq n})$ be the set of all graphs of girth at least $n+1$ and size $ex(m, C_{\leq n})$. The Ramsey number $R(C_{\leq n}, K_m)$ is the smallest k such that every graph of order k contains either a cycle of order l for some $3 \leq l \leq n$ or a set of m independent vertices. It is known that $ex(2n, C_{\leq n}) = 2n + 2$ for $n \geq 4$, and the exact values of $R(C_{\leq n}, K_m)$ for $n > m$ are known. In this paper, we characterize all graphs in $EX(2n, C_{\leq n})$ for $n \geq 5$, and then obtain the exact values of $R(C_{\leq n}, K_m)$ for $m \in \{n, n+1\}$.
Keywords: *Girth; Extremal graph; Planar graph*

1. Introduction

For a simple graph G with the vertex set $V(G)$, edge set $E(G)$, and $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G, and $G \setminus S$ is the subgraph induced by the set $V(G) - S$. For $v \in V(G)$, define $N(v) = \{u : u \in V(G) \wedge uv \in E(G)\}$, $d(v) = |N(v)|$, and $N[v] = N(v) \cup \{v\}$. $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degree of G, respectively. A set of vertices $I \subseteq V(G)$ is called *independent* if $G[I]$ contains no edge. The *independence number* $\alpha(G)$ is the largest cardinality $|I|$ among all independent sets in G. $K_{m,n}$ is the complete $m \times n$ bipartite graph, P_k is a path on k vertices, and C_k is a cycle of length k. $C_{\leq m}$ is a set of cycles of length at most m, and *girth* $g(G)$ is the length of the shortest cycle in G. We refer to the regions defined by an embedding of a planar graph as

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its *faces*. A face is said to be *incident* with the vertices and edges in its boundary. The *length* of a face is the number of edges with which it is incident. If a face has length r, we say it is an *r-face*. For a planar graph G, let f denote the number of its faces.

We use $ex(m, C_{\le n})$ to denote the maximum size of a graph of order m and girth at least $n+1$. A graph of girth at least $n+1$ and size $ex(m, C_{\le n})$ is called an extremal graph, and let $EX(m, C_{\le n})$ denote the set of all corresponding graphs. It is well known that $ex(m, \{C_3\}) = \lfloor \frac{m^2}{4} \rfloor$, and the extremal graph in this case is the complete bipartite graph $K_{\lfloor m/2 \rfloor, \lceil m/2 \rceil}$. Garnick, Kwong and Lazebnik [5] obtained the exact values of $ex(m, C_{\le 4})$ for all $m \le 24$, and some lower bound constructions for $m \le 200$. Garnick and Nieuwejaar [6] determined the exact values of $ex(m, C_{\le 4})$ for $25 \le m \le 30$. Lazebnik and Wang [7] proved that $ex(2n+2, C_{\le n}) = 2n+4$ for all $n \ge 12$. Abajo and Diánez [3] presented the exact values of $ex(m, C_{\le n})$ for $n \ge 4$ and $m \le \lfloor \frac{(16n-15)}{5} \rfloor$ without proofs. They also determined all the values of the girths that extremal graphs in this interval can have. For integers $m \ge 4$ and $n \ge m+1$, Abajo and Diánez [2] obtained the bounds for $n \in \{5, 6, 7\}$ and, in several cases, even the exact values. Recently, they proved that $ex(m, C_{\le n}) = m + k$, where $m = m(k, n)$, and $1 \le k \le 7$ or $k = 15$ [1]. From their results, we know that $ex(2n, C_{\le n}) = 2n + 2$ for $n \ge 5$, and we will present all graphs in $EX(2n, C_{\le n})$ in Theorem 1.

The Ramsey number $R(C_{\le n}, K_m)$ is the smallest k such that every graph of order k contains either a cycle of order l for some $3 \le l \le n$ or a set of m independent vertices. A *$(C_{\le n}, K_m; k)$-graph* is a graph of order k, girth greater than n and not containing any independent sets of m vertices. Spencer [9] obtained a lower bound $R(C_{\le n}, K_m) \ge c(m/\log m)^{(n-1)(n-2)}$. Erdős, Faudree, Rousseau and Schelp [4] proved that $R(C_{\le n}, K_m) = 2m-1$ for $n \ge 2m-1$, and $R(C_{\le n}, K_m) = 2m$ for $m < n < 2m-1$. For the literature on small Ramsey numbers, we refer to [8] and relevant references given in it. In this paper, based on all graphs in $EX(2n, C_{\le n})$, we extend the results of $R(C_{\le n}, K_m)$ to the cases $m \in \{n, n+1\}$ in Theorems 2 and 3.

Theorem 1. *For all $n \ge 5$, if G is any graph in $EX(2n, C_{\le n})$, then*
(a) *If n is odd, then $G \in \mathcal{F}_1 \cup \mathcal{F}_2$,*
(b) *If n is even, then $G \in \mathcal{F}_1$,*
where the graph sets $\mathcal{F}_1$ and $\mathcal{F}_2$ are as in Definition 1.

Theorem 2. *For $n \ge 5$,*

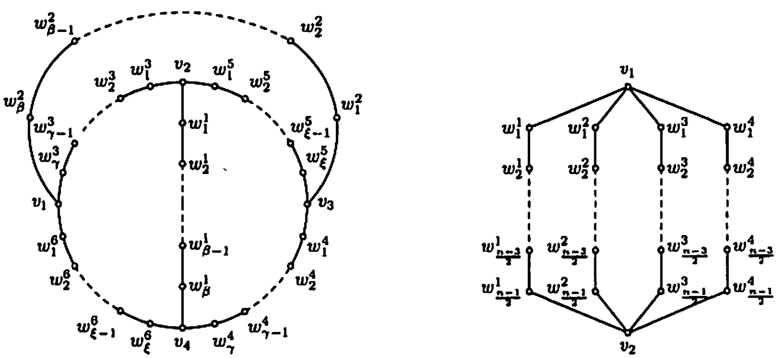
$$R(C_{\le n}, K_n) = \begin{cases} 2n, & for\ odd\ n,\ and \\ 2n+1, & for\ even\ n. \end{cases}$$

Theorem 3. *For odd $n \geq 5$, and even $n \geq 16$,*

$$R(C_{\leq n}, K_{n+1}) = 2n + 3.$$

Definition 1. *For $n \geq 2$, the graph sets $\mathcal{F}_i$, $1 \leq i \leq 2$, are defined on $2n$ vertices, and each of them is a planar graph with four $(n+1)$-faces. We describe these graphs in detail as follows.*
(1) For any graph $F_1 \in \mathcal{F}_1$, $\Delta(F_1) = 3$, and it is defined on vertices $\{v_i, w_j^1, w_j^2, w_k^3, w_k^4, w_l^5, w_l^6 : 1 \leq i \leq 4, 1 \leq j \leq \beta, 1 \leq k \leq \gamma, 1 \leq l \leq \xi, \beta \leq \gamma \leq \xi\}$ such that
(i) $\beta + \gamma + \xi = n - 2$, and
(ii) $\xi \leq \lfloor \frac{n-1}{2} \rfloor$.

In F_1, $d(v_i) = 3$ for $1 \leq i \leq 4$, and the other vertices have degree 2, see Figure 1(a). Observe that each $(n+1)$-face of F_1 is a subgraph induced by three vertices v_i, β vertices w_j^1 (or w_j^2), γ vertices w_k^3 (or w_k^4), and ξ vertices w_l^5 (or w_l^6).



(a) Family of graphs $\mathcal{F}_1$ (b) Family of graphs $\mathcal{F}_2$

Figure 1: Structure of graphs in $\mathcal{F}_1$ and $\mathcal{F}_2$

(2) For any graph $F_2 \in \mathcal{F}_2$, $\Delta(F_2) = 4$ and n is odd. F_2 is defined on vertices $\{v_i, w_j^k : 1 \leq i \leq 2, 1 \leq j \leq \frac{n-1}{2}, 1 \leq k \leq 4\}$. In F_2, $d(v_i) = 4$ for $1 \leq i \leq 2$, and the other vertices have degree 2, see Figure 1(b). Observe that each $(n+1)$-face of F_2 is a subgraph induced by two vertices v_i, $\frac{n-1}{2}$ vertices w_j^k, and $\frac{n-1}{2}$ vertices w_j^l for $1 \leq k, l \leq 4$ and $k \neq l$.

Let $G(\beta, \gamma, \xi)$ denote a graph in $\mathcal{F}_1$, where β, γ and ξ are the corresponding numbers in Definition 1(1). We will show that these graphs are in $EX(2n, C_{\leq n})$. Some of the known results which will be used in our proofs are summarized in the following.

Theorem 4.[2, 3] *Let $n \geq 4$, and $0 \leq k \leq 4$ be integers, then*

$ex(v, C_{\leq n}) = v + k$ *for each* $v \in [v_k(n), v_{k+1}(n))$, *where*
$v_0(n) = n + 1$,
$v_1(n) = \lfloor 3n/2 \rfloor + 1$,
$v_2(n) = 2n$,
$v_3(n) = \begin{cases} \lceil 9n/4 \rceil, & \text{if } n \text{ is even, and} \\ \lfloor 9n/4 \rfloor, & \text{if } n \text{ is odd,} \end{cases}$
$v_4(n) = \begin{cases} \lceil (8n-2)/3 \rceil, & \text{if } n \text{ is even, and} \\ \lfloor (8n-2)/3 \rfloor, & \text{if } n \text{ is odd,} \end{cases}$
$v_5(n) = \begin{cases} 3n-2, & \text{if } n \neq 6, \text{ and} \\ 17, & \text{if } n = 6, \end{cases}$
and $v_6(5) = 14$.

If $v = 2n - 1$, then $v \in (v_1(n), v_2(n))$ for $n \geq 5$, we have $k = 1$, so $ex(2n-1, C_{\leq n}) = 2n$. Similarly, the exact values of $ex(2n, C_{\leq n})$, $ex(2n+1, C_{\leq n})$ and $ex(2n+3, C_{\leq n})$ can be determined by Theorem 4. Hence, we have the following corollary.

Corollary 1. *For* $n \geq 5$,
(a) $ex(2n-1, C_{\leq n}) = 2n$,
(b) $ex(2n, C_{\leq n}) = 2n + 2$,
(c) $ex(2n+1, C_{\leq n}) = 2n + 3$, *for even* n, *and*
(d) $ex(2n+3, C_{\leq n}) = \begin{cases} 2n+8, & \text{for } n = 5, \\ 2n+6, & \text{for } 6 \leq n \leq 13 \text{ or } n = 15, \text{ and} \\ 2n+5, & \text{for } n = 14 \text{ or } n \geq 16. \end{cases}$

2. Proof of Theorem 1

Lemma 1. *Let* G *be a graph of girth at least* $n + 1$, *then*
(a) *If* $n \geq 5$, *and* $|V(G)| = 2n$, *then* G *is planar, and*
(b) *If* $n \geq 16$, *and* $|V(G)| = 2n + 3$, *then* G *is planar.*

Proof. Assume that G is nonplanar, then by Kuratowski's Theorem, G would contain a subgraph that is a subdivision of K_5 or $K_{3,3}$. If G contains a subgraph that is a subdivision of K_5, then since $g(G) \geq n + 1$, each triangle of K_5 is subdivided by at least $n - 2$ vertices. Hence each edge of K_5 is subdivided by at least $\frac{n-2}{3}$ vertices on average, and thus $|V(G)| \geq \lceil 10 \times \frac{n-2}{3} \rceil + 5 = \lceil \frac{10n-5}{3} \rceil$. Similarly, if G contains a subgraph that is a subdivision of $K_{3,3}$, then since $g(G) \geq n+1$, each quadrilateral of $K_{3,3}$ is subdivided by at least $n - 3$ vertices. Hence each edge of $K_{3,3}$ is subdivided by at least $\frac{n-3}{4}$ vertices on average, and thus $|V(G)| \geq \lceil 9 \times \frac{n-3}{4} \rceil + 6 = \lceil \frac{9n-3}{4} \rceil$. If $n \geq 5$, then $|V(G)| > 2n$, a contradiction. If $n \geq 16$, then $|V(G)| > 2n + 3$, a contradiction. Hence, G is planar with the hypothesis of (a) or (b), and the lemma holds. □

Proof of Theorem 1. If $G \in EX(2n, C_{\leq n})$, then $|E(G)| = 2n + 2$ by

Corollary 1(b). Since $C_l \not\subseteq G$ for $3 \le l \le n$, G is planar by Lemma 1(a). By Euler's formula, $f = |E(G)| - |V(G)| + 2 = 4$. Note that $|E(G)| = 2n+2$, so we have $g(G) = n+1$.

If $\delta(G) \ge 3$, then $|E(G)| > 2n+2$ for $n \ge 5$, a contradiction. If $\delta(G) \le 1$, let v be a vertex of degree $\delta(G)$, then $G \setminus \{v\}$ is a graph of order $2n-1$ and $|E(G\setminus\{v\})| \ge 2n+1$, which contradicts Corollary 1(a). Hence, we can assume that $\delta(G) = 2$.

If $\Delta(G) = 2$, then G has $2n$ edges, a contradiction. Suppose that $\Delta(G) \ge 5$, let v be a vertex of degree $\Delta(G)$ and $N_i(v)$ be the neighborhood of v at distance i. Assume that $w_i^j \in N_i(v)$ for $1 \le i \le k$, $1 \le j \le 5$, and $w_s^j w_{s+1}^j \in E(G)$ for $1 \le s \le k-1$. Since G contains no C_l for $3 \le l \le n$ and $\delta(G) = 2$, we have $k \ge \lceil \frac{n-1}{2} \rceil$. Hence $|V(G)| \ge \lceil \frac{n-1}{2} \rceil \times 5 + 1 > 2n$, a contradiction, and thus $3 \le \Delta(G) \le 4$. For $2 \le i \le 4$, let n_i denote the number of vertices of degree i, so $n_2 + n_3 + n_4 = 2n$. There are two cases depending on $\Delta(G)$.

Case 1. Suppose that $\Delta(G) = 3$, then $n_2 + n_3 = 2n$. Since $3n_3 + 2n_2 = 4n+4$, we have $n_3 = 4$. Since every graph G can be constructed from a 3-regular planar multigraph of order 4 and 4 faces by subdividing its edges until $g(G)$ becomes as large as desired, we first consider such multigraphs. It is easy to show that there are exactly two such multigraphs MG_1 and MG_2, see Figure 2, where all edges of MG_1 (or MG_2) are subdivided by x_i vertices respectively in order to construct G, $1 \le i \le 6$.

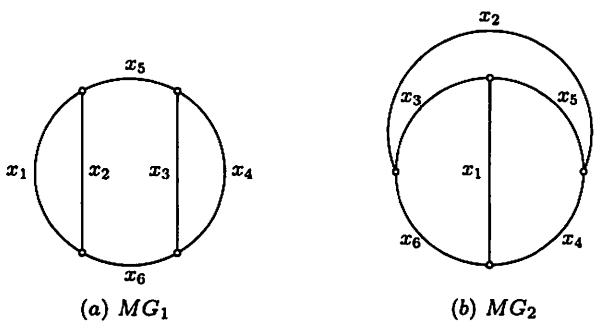


Figure 2: The 3-regular planar multigraphs of order 4

Assume that G is a subdivision of MG_1. Since $g(G) = n+1$,

$$\begin{cases} x_1 + x_2 = n-1, \\ x_3 + x_4 = n-1, \\ x_1 + x_4 + x_5 + x_6 = n-3, \\ x_2 + x_3 + x_5 + x_6 = n-3. \end{cases}$$

It follows that $x_5 + x_6 = -2$, a contradiction. Now assume that G is the subdivision of MG_2. Similarly,

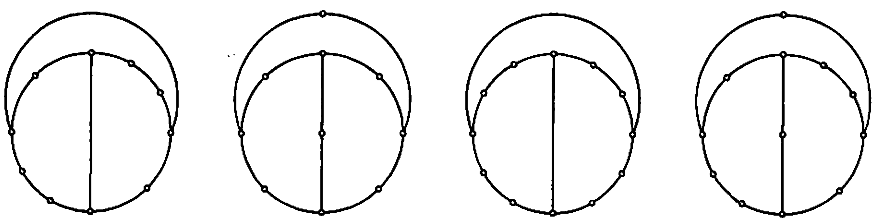
$$\begin{cases} x_1 + x_3 + x_6 = n-2, \\ x_1 + x_4 + x_5 = n-2, \\ x_2 + x_3 + x_5 = n-2, \\ x_2 + x_4 + x_6 = n-2. \end{cases}$$

It follows that $x_1 = x_2$, $x_3 = x_4$ and $x_5 = x_6$. Assume that $x_1 \le x_3 \le x_5$. Since $g(G) = n+1$, we have $x_1 + x_2 + x_3 + x_4 \ge n-3$, that is

$$\begin{cases} x_2 + x_3 + x_5 = n-2, \\ 2(x_2 + x_3) \ge n-3. \end{cases}$$

Hence, $x_5 \le \lfloor \frac{n-1}{2} \rfloor$.

Setting $\beta = x_1, \gamma = x_3$ and $\xi = x_5$, the theorem holds. Taking $n = 5, 6$ as examples, the extremal graphs in $EX(2n, C_{\le n})$ with $\Delta(G) = 3$ are shown in Figure 3.



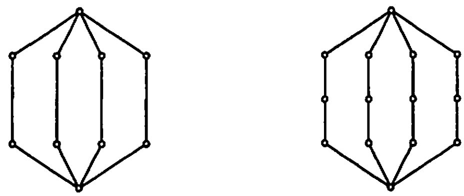
(a) $G(0,1,2)$ (b) $G(1,1,1)$ (c) $G(0,2,2)$ (d) $G(1,1,2)$

Figure 3: The graphs in $EX(2n, C_{\le n})$ with $\Delta(G) = 3$ for $n = 5, 6$

Case 2. Suppose that $\Delta(G) = 4$. Since $2n_2 + 3n_3 + 4n_4 = 4n + 4$, we have

$$n_3 + 2n_4 = 4. \tag{1}$$

Let v_1 be a vertex of degree $\Delta(G)$ and $N_i(v_1)$ be the neighborhood of v_1 at distance i, $w_i^j \in N_i(v_1)$ for $1 \le i \le k$, $1 \le j \le 4$, and $w_s^j w_{s+1}^j \in E(G)$ for $1 \le s \le k-1$. Similar to the proof for $\Delta(G) < 5$, we have $k \ge \lceil \frac{n-1}{2} \rceil$. If $k > \lceil \frac{n-1}{2} \rceil$, then $|V(G)| \ge 1 + 4 \times (\lceil \frac{n-1}{2} \rceil + 1) > 2n$, a contradiction. Hence $k = \lceil \frac{n-1}{2} \rceil$. If n is even, $|V(G)| \ge 1 + 4 \times \lceil \frac{n-1}{2} \rceil > 2n$, a contradiction, Hence n has to be odd, that is, $k = \frac{n-1}{2}$. Notice that there are $2n-2$ vertices w_i^j for $1 \le i \le \frac{n-1}{2}, 1 \le j \le 4$, so we have $n_3 = 0$ and $n_4 = 2$ by equality (1). Hence each vertex w_k^j for $1 \le j \le 4$ has to be adjacent to the other vertex of degree 4, denoted by v_2. Thus the theorem holds for odd n. Taking $n = 5, 7$ as examples, the extremal graphs in $EX(2n; C_{\leq n})$ with $\Delta(G) = 4$ are shown in Figure 4. □



(a) $n = 5$ (b) $n = 7$

Figure 4: The graphs in $EX(2n, C_{\leq n})$ with $\Delta(G) = 4$ for $n = 5, 7$

Lemma 2. *For $n \geq 5$, if G is any $(C_{\leq n}, K_n; 2n)$-graph, then $G \in \mathcal{F}_1 \cup \mathcal{F}_2$.*

Proof. Since $ex(2n, C_{\leq n}) = 2n + 2$ by Corollary 1(b), we have $\delta(G) \leq 2$. If $\delta(G) \leq 1$, let v be a vertex of degree $\delta(G)$ and $H = G[V(G) \setminus N[v]]$, then $|V(H)| \geq 2n - 2$. Since $R(C_{\leq n}, K_{n-1}) = 2n - 2$ [4], the appropriate $n - 1$ vertices of $V(H)$ together with v would be an independent set of n vertices in G, a contradiction. Hence, $\delta(G) = 2$. By the same argument as in the proof of Theorem 1, we have $\Delta(G) \leq 4$. If $\Delta(G) = 2$, then since $g(G) \geq n+1$, we have $G \cong C_{2n}$, a contradiction with $\alpha(G) \leq n-1$. Hence $3 \leq \Delta(G) \leq 4$.

Since G has order $2n$ and girth at least $n + 1$ for $n \geq 5$, G is a planar graph by Lemma 1(a). Therefore, since $\delta(G) = 2$, each vertex of G has to lie on at least one cycle. Let r_i denote the length of the cycle forming the boundary of the i-th face for $1 \leq i \leq f$. Since each cycle is of length at least $n + 1$,

$$2|E(G)| = \sum_{i=1}^{f} r_i \geq f(n + 1),$$

and by Euler's formula,

$$2n = |E(G)| - f + 2 \geq \frac{f}{2}(n + 1) - f + 2 = \frac{f}{2}(n - 1) + 2,$$

that is, $\frac{f}{2}(n - 1) + 2 \leq 2n$. Hence $f \leq 4$. Since $G \ncong C_{2n}$, we have $f > 2$. If $f = 3$, then G is a theta graph. Let v be a vertex which belongs to every cycle of G. Since $G \setminus \{v\}$ is a tree of order $2n - 1$ and it contains an independent set of at least $\lceil \frac{2n-1}{2} \rceil$ vertices, we have $\alpha(G) \geq n$, a contradiction. Hence $f = 4$, and thus $|E(G)| \geq 2n + 2$. Note that $ex(2n, C_{\leq n}) = 2n + 2$, so $|E(G)| = 2n + 2$. By Theorem 1, we have $G \in \mathcal{F}_1 \cup \mathcal{F}_2$. □

3. Proof of Theorem 2

For even β, γ and ξ, we will determine the independence numbers of $G(\beta,\gamma,\xi)$ of order $2n$ in the following lemma,

Lemma 3. *If β, γ and ξ are even, then $\alpha(G(\beta,\gamma,\xi)) = n-1$.*

Proof. Let $G \cong G(\beta,\gamma,\xi)$ and S be an independent set in G, and $S = \{v_1, v_2\} \cup \{w_2^1, w_4^1, \ldots, w_\beta^1\} \cup \{w_1^2, w_3^2, \ldots, w_{\beta-1}^2\} \cup \{w_2^3, w_4^3, \ldots, w_{\gamma-2}^3\} \cup \{w_1^4, w_3^4, \ldots, w_{\gamma-1}^4\} \cup \{w_2^j, w_4^j, \ldots, w_\xi^j : 5 \le j \le 6\}$, then $|S| = 2 + \frac{\beta}{2} \times 2 + (\frac{\gamma}{2} - 1) + \frac{\gamma}{2} + \frac{\xi}{2} \times 2 = n-1$. Hence $\alpha(G) \ge n-1$. We will prove that $\alpha(G) \le n-1$. Assume that $\alpha(G) \ge n$, and I is a maximum independent set of G, then $|I| \ge n$.

Let $T_1 = \{v_i, w_1^j, w_2^j, \ldots w_\beta^j,: 1 \le i \le 4, 1 \le j \le 2\}$ and $T_2 = \{w_1^j, w_2^j, \ldots, w_\gamma^j : 3 \le j \le 4\} \cup \{w_1^j, w_2^j, \ldots w_\xi^j : 5 \le j \le 6\}$, then $V(G) = T_1 \cup T_2$. Let $I_i = I \cap T_i$ for $i = 1$ and 2. Since $G[T_1]$ is isomorphic to $2P_{\beta+2}$ and $\alpha(P_k) = \lceil \frac{k}{2} \rceil$, we have $|I_1| \le \beta + 2$. If $|I_1| < \beta + 2$, then $|I_2| \ge \gamma + \xi + 1$. However, since $G[T_2] \cong 2P_\gamma \cup 2P_\xi$, it contains an independent set of at most $\gamma + \xi$ vertices, a contradiction. Hence $|I_1| = \beta + 2$. Note that $G[T_1] \cong 2P_{\beta+2}$, there is at least one vertex from $\{v_1, v_3\}$, and one vertex from $\{v_2, v_4\}$ in I_1. By symmetry, there are three cases:

Case 1. Suppose that $v_i \in I_1$ for $1 \le i \le 4$. Let $X = \{w_\gamma^3, w_1^6, w_1^3, w_1^5, w_1^4, w_\xi^5, w_\gamma^4, w_\xi^6\}$, and $H = G[T_2 - X]$. Since $H \cong 2P_{\gamma-2} \cup 2P_{\xi-2}$, we have $\alpha(H) = \gamma + \xi - 4$, and thus $|I| = \beta + \gamma + \xi - 2 = n - 4$, a contradiction.

Case 2. Suppose that $v_i \in I_1$ for $1 \le i \le 3$. Let $X = \{w_\gamma^3, w_1^6, w_1^3, w_1^5, w_1^4, w_\xi^5\}$, and $H = G[T_2 - X]$. Since $H \cong P_{\gamma-2} \cup P_{\gamma-1} \cup P_{\xi-2} \cup P_{\xi-1}$, we have $\alpha(H) = \gamma + \xi - 2$, and thus $|I| = \beta + \gamma + \xi = n - 2$, a contradiction.

Case 3. Suppose that $v_i \in I_1$ for $1 \le i \le 2$. Let $X = \{w_\gamma^3, w_1^6, w_1^3, w_1^5\}$, and $H = G[T_2 - X]$. Since $H \cong P_{\gamma-2} \cup P_\gamma \cup 2P_{\xi-1}$, we have $\alpha(H) = \gamma + \xi - 1$, and thus $|I| = \beta + \gamma + \xi + 1 = n - 1$, a contradiction.

Cases 1-3 imply that $\alpha(G) \le n-1$, and thus the lemma holds. □

Proof of Theorem 2. (1) $R(C_{\le n}, K_n) = 2n+1$ for even $n \ge 6$.

For odd $\frac{n}{2}$, the graphs $G(0, \frac{n}{2} - 1, \frac{n}{2} - 1)$ show that $R(C_{\le n}, K_n) \ge 2n+1$ by Lemma 3. For even $\frac{n}{2}$, the graphs $G(2, \frac{n}{2} - 2, \frac{n}{2} - 2)$ show that $R(C_{\le n}, K_n) \ge 2n+1$ by Lemma 3. Assume that there exists a $(C_{\le n}, K_n; 2n+1)$-graph G for even n. If $\delta(G) \ge 3$, then $|E(G)| > 2n+3$, a contradiction with Corollary 1(c). If $\delta(G) \le 2$, let v be a vertex of degree $\delta(G)$, then $|V(G) - N[v]| \ge 2n-2$. Since $R(C_{\le n}, K_{n-1}) = 2n-2$ [4], the subgraph $G[V(G) - N[v]]$ contains an independent set of $n-1$ vertices.

These $n-1$ vertices together with v would be an independent set of n vertices in G, a contradiction. Hence $R(C_{\leq n}, K_n) \leq 2n+1$ for even $n \geq 6$, and the theorem holds.

(2) $R(C_{\leq n}, K_n) = 2n$ for odd $n \geq 5$.

$\alpha(C_{2n-1}) = n-1$ shows that $R(C_{\leq n}, K_n) \geq 2n$. We will prove that $R(C_{\leq n}, K_n) \leq 2n$. Assume that there exists a $(C_{\leq n}, K_n; 2n)$-graph G for odd n. By Lemma 2, we have $G \in \mathcal{F}_1 \cup \mathcal{F}_2$. Let S be an independent set in G.

Case 1. Suppose that $G \in \mathcal{F}_1$. Since $\beta+\gamma+\xi$ is odd, there are two subcases depending on their parities.

Case 1.1. Suppose that all of β, γ and ξ are odd. Let $S = \{w_1^k, w_3^k, \ldots, w_\beta^k : 1 \leq k \leq 2\} \cup \{w_1^k, w_3^k, \ldots, w_\gamma^k : 3 \leq k \leq 4\} \cup \{w_1^k, w_3^k, \ldots, w_\xi^k : 5 \leq k \leq 6\}$, then $|S| = (\lceil \frac{\beta}{2} \rceil + \lceil \frac{\gamma}{2} \rceil + \lceil \frac{\xi}{2} \rceil) \times 2 = n+1$, that is, $\alpha(G) \geq n+1$.

Case 1.2. Suppose that there is exactly one of β, γ and ξ which is odd. Without loss of generality, let $\xi \geq 1$ be odd, and both β and γ are even. Let $S = \{v_2, v_3\} \cup \{w_2^k, w_4^k, \ldots, w_\beta^k : 1 \leq k \leq 2\} \cup \{w_2^k, w_4^k, \ldots, w_\gamma^k : 3 \leq k \leq 4\} \cup \{w_2^5, w_4^5, \ldots, w_{\xi-1}^5\} \cup \{w_1^6, w_3^6, \ldots, w_\xi^6\}$, then $|S| = 2+\beta+\gamma+\lfloor \frac{\xi}{2} \rfloor + \lceil \frac{\xi}{2} \rceil = n$, that is, $\alpha(G) \geq n$.

Case 2. Suppose that $G \in \mathcal{F}_2$, and thus $\Delta(G) = 4$. Note that $j = \frac{n-1}{2}$, see Figure 1(b). For even j, let $S = \{v_1\} \cup \{w_2^k, w_4^k, \ldots, w_j^k : 1 \leq k \leq 4\}$, then $|S| = 2j+1 = n$. For odd j, let $S = \{w_1^k, w_3^k, \ldots, w_j^k : 1 \leq k \leq 4\}$, then $|S| = 4 \times \lceil \frac{j}{2} \rceil = n+1$. Hence, $\alpha(G) \geq n$.

Cases 1 and 2 imply that $\alpha(G) \geq n$, a contradiction with G being a $(C_{\leq n}, K_n; 2n)$-graph. Hence $R(C_{\leq n}, K_n) \leq 2n$ for odd $n \geq 5$, and the theorem holds. □

4. Proof of Theorem 3

Let $G'(0, 2, \gamma, \gamma, \xi, \xi)$ be a graph of order $2n+2$ and girth at least $n+1$, which is similar to the structure of $G \in \mathcal{F}_1$, and it is defined on vertices $\{v_i, w_j^2, w_k^3, w_k^4, w_l^5, w_l^6 : 1 \leq i \leq 4, 1 \leq j \leq 2, 1 \leq k \leq \gamma, 1 \leq l \leq \xi, \gamma \leq \xi, \gamma+\xi = n-2\}$ as in Figure 5.

By the same argument as in the proof of Lemma 3, we can obtain the following lemma.

Lemma 4. *If γ and ξ are even, then $\alpha(G(0, 2, \gamma, \gamma, \xi, \xi)) = n$.*

Proof of Theorem 3. (1) $R(C_{\leq n}, K_{n+1}) = 2n+3$ for odd $n \geq 5$.

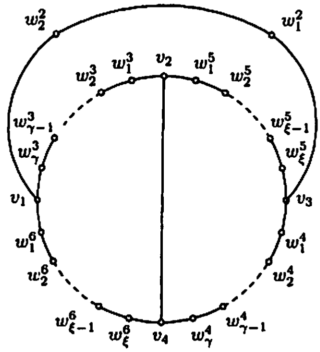


Figure 5: Structure of $G'(0, 2, \beta, \beta, \xi, \xi)$

For odd $\frac{n+1}{2}$, let $H \cong G(0, \frac{n+1}{2} - 1, \frac{n+1}{2} - 1)$ with order $2n+2$, then $\alpha(H) = n$ by Lemma 3. For even $\frac{n+1}{2}$, let $H \cong G(2, \frac{n+1}{2} - 2, \frac{n+1}{2} - 2)$ with order $2n+2$, then $\alpha(H) = n$ by Lemma 3. Hence $R(C_{\leq n}, K_{n+1}) \geq 2n+3$ for odd $n \geq 5$. We will prove that $R(C_{\leq n}, K_{n+1}) \leq 2n+3$. Assume that there exists a $(C_{\leq n}, K_{n+1}; 2n+3)$-graph G. If $\delta(G) \geq 3$, then $|E(G)| > 2n+8$, a contradiction with Corollary 1(d). If $\delta(G) \leq 2$, let v be a vertex of degree $\delta(G)$ and $H = G[V(G) \setminus N[v]]$, then $|V(H)| \geq 2n$. Since $R(C_{\leq n}, K_n) = 2n$ for odd $n \geq 5$ by Theorem 2, H contains an independent set of n vertices. These n vertices together with v would be an independent set of $n+1$ vertices in G, a contradiction. Hence $R(C_{\leq n}, K_{n+1}) \leq 2n+3$ for odd $n \geq 5$, and the theorem holds.

(2) $R(C_{\leq n}, K_{n+1}) = 2n+3$ for even $n \geq 16$.

For odd $\frac{n}{2}$, let $H \cong G'(0, 2, \frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}-1)$, then $\alpha(H) = n$ by Lemma 4. For even $\frac{n}{2}$, let $H \cong G'(0, 2, \frac{n}{2}-2, \frac{n}{2}-2, \frac{n}{2}, \frac{n}{2})$, then $\alpha(H) = n$ by Lemma 4. Hence $R(C_{\leq n}, K_{n+1}) \geq 2n+3$ for even $n \geq 6$. We will prove that $R(C_{\leq n}, K_{n+1}) \leq 2n+3$ for even $n \geq 16$. Assume that there exists a $(C_{\leq n}, K_{n+1}; 2n+3)$-graph G. If $\delta(G) \geq 3$, then $|E(G)| > 2n+5$, a contradiction with Corollary 1(d), and thus $\delta(G) \leq 2$. Let v be a vertex of degree $\delta(G)$, and $H = G[V(G) - N[v]]$. If $\delta(G) \leq 1$, then $|V(H)| \geq 2n+1$. Since $R(C_{\leq n}, K_n) = 2n+1$ for even $n \geq 6$ by Theorem 2, the appropriate n vertices of $V(H)$ together with v would be an independent set of $n+1$ vertices in G, a contradiction. Hence $\delta(G) = 2$, and H is a $(C_{\leq n}, K_n; 2n)$-graph. By Lemma 2 and n being even, we have $H \in \mathcal{F}_1$. Notice that $V(G) = 2n+3$ and $n \geq 16$, so G is a planar graph by Lemma 1(b). Hence $N[v]$ has to lie in one of four $(n+1)$-faces of H. Therefore, since $\delta(G) = 2$, each vertex of $N(v)$ is adjacent to at least one vertex which is incident with this face. In any case, we have $g(G) < n+1$, a contradiction. Hence

$R(C_{\leq n}, K_{n+1}) \leq 2n+3$ for even $n \geq 16$, and the theorem holds. □

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