On the basis number of the wreath product of paths with wheels and some related problems

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Abstract

In this paper, we investigate the basis number for the wreath product of wheels with paths. Also, as a related problem, we construct a minimum cycle basis of the same.

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1 Introduction

In the article *the basis number of Cartesian product of some graphs,* Ali and Marougi [1] gave an upper bound of the basis number of the Cartesian product of two graphs in terms of the factors. Also in the article *the basis number of the powers of the complete graph,* Alsardary and Wojciechowski [3] proved that the basis number of the d times Cartesian product of the complete graph is bounded above by 9. Jaradat [14] and Jaradat and Alzoubi [15] treated the problem of finding the basis number for the lexicographic product by presenting an upper bound in terms of the second factor. In [10] and [11] upper bounds for the basis numbers in terms of

*This paper has been done during my sabbatical leave from Yarmouk University the factors are obtained for the direct product of two bipartite graphs and strong product of a graph with a bipartite graph. In a related problem, Imrich and Stadler [9] constructed minimum cycle bases for Cartesian and strong products of graphs, in terms of minimum cycle bases of the factors. Berger [4] and Jaradat [12] solved the same problem for the lexicographical product. Hammack [7] constructed a minimum cycle basis for the direct product of two bipartite graphs. In [6, 8] minimum cycle bases for the direct product of complete graphs is constructed. Recently, Bradshwa and Hammack, constructed a minimum cycle bases of the direct product of cycle with a graph. The problem of finding the basis number and constructing a minimum cycle basis for the wreath product appears to be more complicated. The problem has been solved for some special cases see [2,13]. In this paper, we mainly prove that the basis number of the wreath product of wheels with paths is bounded above by 4 and we construct a minimum cycle basis for the same. Our proof is the first step towards giving a general upper bound of the basis number of a graph with a path and constructing a minimum cycle basis of the same.

For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. The set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X\backslash Y) \cup (Y\backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that the dimension of the cycle space is the *cyclomatic number* or the *first Betti number*

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r \tag{1}$$

where r is the number of components.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a d-fold if each edge of G occurs in at most d of the cycles in $\mathcal{B}$. *The basis number*, $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a *d-fold* basis. The *length*, $|C|$, of the element C of the cycle space $\mathcal{C}(G)$ is the number of its edges. The *length* $l(\mathcal{B})$ *of a cycles basis* $\mathcal{B}$ is the sum of the lengths of its elements: $l(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. A *minimum cycle basis* (MCB) is a cycle basis with minimum length. Further, $\lambda(G)$ is defined to be the length of the longest element in a minimum cycle basis of G.

The first important result concerning the basis number is the lemma of MacLane [17] when he proved that a graph G is planar if and only if the basis number of G is less than or equal to 2. Later on Schemeichel [18] utilized the ideas of MacLane and defined the basis number of a graph in its recent form. In fact, Schemeichel proved that for any integer r there is a graph with basis number greater than or equal to r. Moreover, he investigated the basis number of certain important classes of non-planar graphs, specifically, complete graphs and complete bipartite graphs.

The cycle space is a weighted matroid where each element C has weight $|C|$. Hence the Greedy Algorithm [19] can always be used to extract an MCB. For completeness, we give the following two definitions: Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The Cartesian product $G \Box H$ has the vertex set $V(G \Box H) = V(G) \times V(H)$ and the edge set $E(G \Box H) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$. The wreath product $G\rho H$ has the vertex set $V(G\rho H) = V(G) \times V(H)$ and the edge set $E(G\rho H) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and there is $\alpha \in \mathrm{Aut}(H)$ such that $\alpha(v_1) = v_2\}$.

One can note that $G \Box H$ is a spanning subgraph of $G\rho H$. Also

$$|E(G \Box H)| = |V(G)||E(H)| + |V(H)||E(G)|.$$

In the rest of this paper, $f_B(e)$ stand for the number of elements of B containing the edge e and $E(B) = \cup_{C \in B} E(C)$ where $B \subseteq \mathcal{C}(G)$.

2 The Basis Number of $\mathbf{W_n \rho P_m}$

In this section, we investigate the basis number of the wreath product of paths with wheels. Let $\{v_1, v_2, \ldots, v_m\}$ be a set of vertices and $P_m = v_1v_2 \ldots v_m$. Then the automorphism group of P_m consists of two elements the identity, I, and the automorphism α which is defined as in the following:

$$\alpha(v_j) = v_{m-j+1}; j = 1, 2, \ldots, m.$$

Let $\{u_1, u_2, ..., u_n\}$ be the vertex set of the wheel W_n with $d_{W_n}(u_1) = n - 1$ and $W_n - u_1 = C_{n-1}$ where $C_{n-1} = u_2u_3...u_nu_2$. Then $W_n\rho P_m$ is decomposable into

$$(W_n \Box P_m) \cup \left(\cup_{i=1}^{n-1} M_{u_i u_{i+1}}\right) \cup \left(\cup_{i=3}^{n} M_{u_1 u_i}\right) \cup \left(M_{u_2 u_n}\right)$$

where M_{xy} is the graph with the edge set

$$\{(x, v_j)(y, v_{m-j+1}), (x, v_{m-j+1})(y, v_j) | j = 1, 2, \ldots, \lfloor m/2 \rfloor\}.$$

Thus

$$\begin{aligned}|E(W_n\rho P_m)| &= |E(W_n\Box P_m)| + \sum_{i=1}^{n-1} 2\left\lfloor\frac{m}{2}\right\rfloor + \sum_{i=3}^{n} 2\left\lfloor\frac{m}{2}\right\rfloor + 2\left\lfloor\frac{m}{2}\right\rfloor \\ &= (2n-1)m + n(m-1) + 2(n-1)\left\lfloor\frac{m}{2}\right\rfloor + 2(n-2)\left\lfloor\frac{m}{2}\right\rfloor \\ &\quad +2\left\lfloor\frac{m}{2}\right\rfloor \\ &= 3mn - 2m - n + 4(n-1)\left\lfloor\frac{m}{2}\right\rfloor.\end{aligned}$$

Hence,

$$\dim \mathcal{C}(W_n\rho P_m) = 2mn - 2m - n + 4(n-1)\left\lfloor\frac{m}{2}\right\rfloor + 1.$$

Now, for any two edges la, ab and a path lab of order 3 and for any j, we recall the following cycles of [12]

$$\mathcal{R}_{ab}^{(j)} = (a, v_j)(a, v_{j+1})(b, v_{m-j})(b, v_{m-j+1})(a, v_j),$$

$$\mathcal{N}_{ab}^{(j)} = (a, v_j)(b, v_{m-j+1})(a, v_{m-j+1})(b, v_j)(a, v_j),$$

$$\mathcal{K}_{ab}^{(j)} = (a, v_j)(b, v_j)(b, v_{j+1})(a, v_{j+1})(a, v_j),$$

and

$$\mathcal{Z}_{ab} = (a, v_{\lfloor\frac{m}{2}\rfloor})(a, v_{\lfloor\frac{m}{2}\rfloor+1})(b, v_{\lfloor\frac{m}{2}\rfloor+1})(a, v_{\lfloor\frac{m}{2}\rfloor}).$$

Also, we introduce the following cycles:

$$\mathcal{U}_{lab}^{(j)} = (l, v_j)(a, v_j)(b, v_j)(l, v_j),$$

and

$$\mathcal{C}_{lab}^{(j)} = (l, v_j)(a, v_{m-j+1})(b, v_{m-j+1})(l, v_j).$$

Let

$$\mathcal{R}_{ab}^{+} = \cup_{j=2 \text{ and } j \text{ is even}}^{s} \mathcal{R}_{ab}^{(j)}, \qquad \mathcal{R}_{ab}^{-} = \cup_{j=1 \text{ and } j \text{ is odd}}^{s} \mathcal{R}_{ab}^{(j)},$$

$$\mathcal{K}_{ab}^{+} = \cup_{j=2 \text{ and } j \text{ is even}}^{m-1} \mathcal{K}_{ab}^{(j)}, \qquad \mathcal{K}_{ab}^{-} = \cup_{j=1 \text{ and } j \text{ is odd}}^{m-1} \mathcal{K}_{ab}^{(j)},$$

$$\mathcal{C}_{lab} = \begin{cases} \cup_{i=1}^{m} \mathcal{C}_{lab}^{(j)}, & \text{if } m \text{ is even}, \\ \cup_{i=1 \text{ and } j\neq\lfloor\frac{m}{2}\rfloor+1}^{m} \mathcal{C}_{lab}^{(j)}, & \text{if } m \text{ is odd}, \end{cases},$$

where

$$s = \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor, & \text{if } m \text{ is odd,} \\ \left\lfloor \frac{m}{2} \right\rfloor - 1, & \text{if } m \text{ is even,} \end{cases}$$

and

$$\mathcal{N}_{ab} = \cup_{j=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{N}_{ab}^{(j)}.$$

Also, let

$$\mathcal{R}_{ab} = \mathcal{R}_{ab}^{+} \cup \mathcal{R}_{ab}^{-}, \mathcal{K}_{ab} = \mathcal{K}_{ab}^{+} \cup \mathcal{K}_{ab}^{-} \text{ and } C_{lab}^{*} = C_{lab} \cup C_{alb} \cup C_{bal}.$$

Lemma 2.1. *The set C_{lab}^{*} is linearly independent. Moreover, any linear combinations of cycles of C_{lab}^{*} contains at least one edge incident with a vertex of the form (l, v_j) where $1 \leq j \leq m$.*

Proof: Note that each of C_{lab}, C_{alb} and C_{bal} consists of pairwise edge disjoint cycles. Thus, each of which is linearly independent. To show that C_{lab}^{*} is linearly independent, it suffices to show that any non trivial linear combination of cycles of C_{lab}^{*} is non empty. Let C be the linear combination of cycles of $Z = \{z_1, z_2, \ldots, z_k\} \subseteq C_{lab}^{*}$. Then we consider the following three cases:

Case 1. The set Z contains at least one cycle from C_{bal}, say $z_1 = C_{bal}^{(j)}$. Note that $(l, v_j)(a, v_j) \in E(z_1)$ and it does not appear in any other cycle of C_{lab}^{*}. Thus, $(l, v_j)(a, v_j) \in E(\bigoplus_{i=1}^{k} z_i) = E(C)$.

Case 2. The set Z does not contain any cycle from C_{bal} but contains at least one cycle of C_{alb}, say $z_1 = C_{alb}^{(j)}$. Then as in Case 1, $(l, v_j)(b, v_j) \in E(C)$.

Case 3. The set Z consists only of cycles from C_{lab}. Since C_{lab} is linearly independent, as a result $E(C) \neq \varnothing$. Since $E(C_{lab}) - \{(l, v_j)(b, v_{m-j+1}) | 1 \leq j \leq m\}$ is an edge set of a forest and since the linear combination of a linearly independent set of cycles is a cycle or a union of edge disjoint cycles, as a result C must contain at least one edge of the form $(l, v_j)(b, v_{m-j+1})$. From Cases 1, 2 and 3, C must contains an edge adjacent with a vertex of $\{(l, v_j) : 1 \leq j \leq m\}$. ■

The following proposition will be needed in the forthcoming result:

Proposition 2.2 (Jaradat et al. [16]) *Let A and B be two linearly independent sets of cycles such that $E(A) \cap E(B)$ is an edge set of a forest. Then $A \cup B$ is linearly independent.*

Note that

$$(W_n - u_n u_2)\, \rho P_m = W_n \rho P_m - \left(E(M_{u_2 u_n}) \cup E(u_n u_2 \square V(P_m))\right).$$

Thus,

$$\begin{aligned}\dim \mathcal{C}((W_n - u_n u_2)\,\rho P_m) &= \dim \mathcal{C}(W_n \rho P_m) - 2\left\lfloor \frac{m}{2} \right\rfloor - m \\ &= 2nm - 3m - n + 4(n-2)\lfloor \frac{m}{2} \rfloor + 2\lfloor \frac{m}{2} \rfloor + 1\end{aligned}$$

To address the problem of finding the basis number of $W_n \rho P_m$, we first find a basis for $(W_n - u_2 u_n)\rho P_m$, then we extend it to a basis for $W_n \rho P_m$. Let

$$\begin{aligned}\mathcal{B}_4 &= \mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{C}^*_{u_1u_3u_4} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4} \\ &\cup \mathcal{K}_{u_1u_2} \cup \mathcal{K}_{u_1u_3} \cup \mathcal{K}_{u_3u_4},\end{aligned}$$

if m is odd, and

$$\begin{aligned}\mathcal{B}_4 &= \mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{C}^*_{u_1u_3u_4} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{Z}_{u_2u_3} \cup \mathcal{K}_{u_1u_2} \cup \\ &\mathcal{K}_{u_1u_3} \cup \mathcal{K}_{u_3u_4},\end{aligned}$$

if m is even. Then we have the following result.

Proposition 2.3: The set $\mathcal{B}_4$ as described above is a 4-fold basis for $\mathcal{C}(W_4 - u_2u_4)\rho P_m$.

Proof. First, suppose m is odd, Note that each set of $\mathcal{R}^+_{u_2u_3}, \mathcal{R}^-_{u_3u_2}$ and $\mathcal{N}_{u_2u_3}$ is linearly independent because each of which consists only of pairwise edge disjoint cycles. Since $E(\mathcal{R}^+_{u_2u_3}) \cap E(\mathcal{R}^-_{u_3u_2}) = \varnothing$, $\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2}$ is linearly independent by Proposition 2.2. It is clear that any linear combination of cycles of $\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2}$ contains an edge of $\{(u_2, v_j)(u_2, v_{j+1}) | 1 \leq j \leq m-1\}$ which does not occur in any cycle of $\mathcal{N}_{u_2u_3}$. Hence, $\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3}$ is linearly independent. Now, by Lemma 2.1, any linear combinations of cycles of $\mathcal{C}^*_{u_1u_2u_3}$ contains an edge incident with a vertex of the form (u_1, v_j) for $1 \leq j \leq m$ which is not a vertex of any cycle of $\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3}$. Thus, $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3}$ is linearly independent. Since the cycle $\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3}$ contains $(u_1, v_{\lfloor \frac{m}{2} \rfloor+1})(u_2, v_{\lfloor \frac{m}{2} \rfloor+1})$ which does not occur in any cycle of $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3}$, $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3}$ is linearly independent. Observe that each cycle of $\mathcal{K}^-_{u_1u_2} \cup \mathcal{K}^+_{u_1u_3}$ contains an edge of the form $(u_1, v_j)(u_1, v_{j+1})$ for $1 \leq j \leq m-1$ and no other cycle of $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3} \cup \mathcal{K}^-_{u_1u_2} \cup \mathcal{K}^+_{u1u_3}$ have such edge. Hence $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3} \cup \mathcal{K}^-_{u_1u_2} \cup \mathcal{K}^+_{u_1u_3}$ is a linearly independent set. Similarly, each cycle of $\mathcal{K}^+_{u_1u_2} \cup \mathcal{K}^-_{u_1u_3}$ contains an edge of the form $(u_2, v_j)(u_2, v_{j+1})$

for even $1 \leq j \leq m-1$, or an edge of the form $(u_3, v_j)(u_3, v_{j+1})$ for odd $1 \leq j \leq m-1$. Since non of the previous two edges appears in any other cycles of $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3} \cup \mathcal{K}_{u_1u_2} \cup \mathcal{K}_{u_1u_3}$, we conclude that $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3} \cup \mathcal{K}_{u_1u_2} \cup \mathcal{K}_{u_1u_3}$ is linearly independent. Note that $\mathcal{C}^*_{u_1u_3u_4}$ and $\mathcal{C}^*_{u_4u_3u_1}$ are symmetric in roles. Then, any linear combination of cycles of $\mathcal{C}^*_{u_1u_3u_4}$ contains an edge incident with a vertex of the form (u_4, v_j) for $j \leq m$, by lemma 2.1. Since no cycle of $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3} \cup \mathcal{K}_{u_1u_2} \cup \mathcal{K}_{u_1u_3}$ has an edge incident with such vertex, we have that $\mathcal{C}^*_{u_1u_2u_3} \cup \mathcal{C}^*_{u_1u_3u_4} \cup \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_3u_2} \cup \mathcal{N}_{u_2u_3} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3} \cup \mathcal{K}_{u_1u_2} \cup \mathcal{K}_{u_1u_3}$ is linearly independent. Now, clearly that any linear combination of cycle of $\mathcal{K}_{u_3u_4}$ contains an edge of the form $(u_4, v_j)(u_4, v_{j+1})$ for $1 \leq j \leq m-1$ which is not an edge of any cycle of $\mathcal{B}_4 - \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4} \cup \mathcal{K}_{u_3u_4}$. Hence, $\mathcal{B}_4 - \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}$ is linearly independent. Finally, the cycle $\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}$ contains the edge $(u_1, v_{\lfloor \frac{m}{2} \rfloor+1})(u_4, v_{\lfloor \frac{m}{2} \rfloor+1})$ and no cycle of $\mathcal{B}_4 - \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}$ contains such edge. Therefore, $\mathcal{B}_4$ is linearly independent. Now,

$$\begin{aligned}
|\mathcal{B}_4| &= |\mathcal{C}^*_{u_1u_2u_3}| + |\mathcal{C}^*_{u_1u_3u_4}| + \sum_{j=1 \text{ and } j \text{ is even}}^{\lfloor \frac{m}{2} \rfloor} |\mathcal{R}^{(j)}_{u_2u_3}| + \\
&\quad \sum_{j=1 \text{ and } j \text{ is odd}}^{\lfloor \frac{m}{2} \rfloor} |\mathcal{R}^{(j)}_{u_3u_2}| + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} |\mathcal{N}^{(j)}_{u_3u_2}| + |\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3}| + \\
&\quad |\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}| + |\mathcal{K}_{u_1u_2}| + |\mathcal{K}_{u_1u_3}| + |\mathcal{K}_{u_3u_4}| \\
&= 3(m-1) + 3(m-1) + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + 1 + 1 + 3(m-1) \\
&= 9m + 2\left\lfloor \frac{m}{2} \right\rfloor - 7 \\
&= 10m - 8 \\
&= \dim \mathcal{C}((W_4 - u_2u_4)\rho P_m).
\end{aligned}$$

Then, $\mathcal{B}_4$ is a basis for $\mathcal{C}(W_4 - u_2u_4)\rho P_m$. To complete the proof, we have to show that for any edge $e \in E((W_4 - u_2u_4)\rho P_m)$, $f_{\mathcal{B}_4}(e) \leq 4$. 1) If $e \in M_{u_1u_2}$, then e appears only in $\mathcal{C}^*_{u_1u_2u_3}$. And so, $f_{\mathcal{B}_4}(e) = f_{\mathcal{C}^*_{u_1u_2u_3}}(e) = 2$. 2) If $e \in M_{u_2u_3}$, then e appears only in $\mathcal{C}^*_{u_1u_2u_3}, \mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_2u_3}$ and $\mathcal{N}_{u_3u_2}$. And so, $f_{\mathcal{B}_4}(e) = f_{\mathcal{C}^*_{u_1u_2u_3}}(e) + f_{\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_2u_3}}(e) + f_{\mathcal{N}_{u_3u_2}}(e) = 2+1+1$. 3) If $e \in M_{u_3u_4}$, then e appears only in $\mathcal{C}^*_{u_1u_3u_4}$. And so, $f_{\mathcal{B}_4}(e) = f_{\mathcal{C}^*_{u_1u_3u_4}}(e) = 2$. 4) If $e \in M_{u_1u_3}$, then e appears only in $\mathcal{C}^*_{u_1u_2u_3}$ and $\mathcal{C}^*_{u_1u_3u_4}$. And so, $f_{\mathcal{B}_4}(e) = f_{\mathcal{C}^*_{u_1u_2u_3}}(e) + f_{\mathcal{C}^*_{u_1u_3u_4}}(e) = 2+2$. 5) If $e \in M_{u_1u_4}$, then e appears only in

$C^*_{u_1u_3u_4}$. And so, $f_{\mathcal{B}_4}(e) = f_{C^*_{u_1u_3u_4}}(e) = 2$. 6) If $e \in u_1\Box P_m$, then e appears in $\mathcal{K}_{u_1u_2}$ and $\mathcal{K}_{u_1u_3}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{K}_{u_1u_2}}(e) + f_{\mathcal{K}_{u_1u_3}}(e) = 1 + 1$. 7) If $e \in u_2\Box P_m$, then e appears in $\mathcal{K}_{u_1u_2}$ and $\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_2u_3}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{K}_{u_1u_2}}(e) + f_{\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_2u_3}}(e) = 1 + 1$. 8) If $e \in u_3\Box P_m$, then e appears in $\mathcal{K}_{u_1u_3}, \mathcal{K}_{u_3u_4}$ and $\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_2u_3}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{K}_{u_1u_2}}(e) + f_{\mathcal{K}_{u_3u_4}}(e) + f_{\mathcal{R}^+_{u_2u_3} \cup \mathcal{R}^-_{u_2u_3}}(e) = 1+1+1$. 9) If $e \in u_4\Box P_m$, then e appears in $\mathcal{K}_{u_3u_4}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{K}_{u_3u_4}}(e) = 1$. 10) If $e \in u_1u_2\Box V(P_m - v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears in $\mathcal{K}_{u_1u_2}$ and $C^*_{u_1u_2u_3}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{K}_{u_1u_2}}(e) + f_{C^*_{u_1u_2u_3}}(e) \leq 2+1$. 11) If $e \in u_2u_3\Box V(P_m - v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears in $C^*_{u_1u_2u_3}$. And so $f_{\mathcal{B}_4}(e) = f_{C^*_{u_1u_2u_3}}(e) = 1$. 12) If $e \in u_3u_4\Box V(P_m - v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears in $\mathcal{K}_{u_3u_4}$ and $C^*_{u_1u_3u_4}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{K}_{u_1u_2}}(e) + f_{C^*_{u_1u_2u_3}}(e) \leq 2 + 1$. 13) If $e \in u_1u_3\Box V(P_m - v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears in $\mathcal{K}_{u_1u_3}$ and $C^*_{u_1u_2u_3}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{K}_{u_1u_3}}(e) + f_{C^*_{u_1u_2u_3}}(e) \leq 2 + 1$. 14) If $e \in u_1u_4\Box V(P_m - v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears in $C^*_{u_1u_3u_4}$. And so $f_{\mathcal{B}_4}(e) = f_{C^*_{u_1u_3u_4}}(e) = 1$. 15) If $e = (u_1, v_{\lfloor \frac{m}{2} \rfloor+1})(u_2, v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears in $\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3}, \mathcal{K}_{u_1u_2}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}}(e) + f_{\mathcal{K}_{u_1u_2}}(e) \leq 1 + 2$. 16) If $e = (u_2, v_{\lfloor \frac{m}{2} \rfloor+1})(u_3, v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears only in $\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3}}(e) = 1$. 17) If $e = (u_3, v_{\lfloor \frac{m}{2} \rfloor+1})(u_4, v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears in $\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}, \mathcal{K}_{u_3u_4}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}}(e) + f_{\mathcal{K}_{u_3u_4}}(e) \leq 1 + 2$. 18) If $e = (u_1, v_{\lfloor \frac{m}{2} \rfloor+1})(u_3, v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears in $\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3}, \mathcal{K}_{u_1u_3}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_2u_3}}(e) + f_{\mathcal{K}_{u_1u_3}}(e) = 1+2$. 19) If $e = (u_1, v_{\lfloor \frac{m}{2} \rfloor+1})(u_4, v_{\lfloor \frac{m}{2} \rfloor+1})$, then e appears only in $\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}$. And so $f_{\mathcal{B}_4}(e) = f_{\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_3u_4}}(e) = 1$. The argument is the same for m is even. ■

We will define a cycle basis for $(W_k - u_2u_n)\rho P_m$ inductively, beginning with the case $(W_4 - u_2u_4)\rho P_m$ addressed in Proposition 2.3. In preparation for that for each $k \geq 5$ we let

$$\mathcal{B}_k = \begin{cases} C^*_{u_1u_{k-1}u_k} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor+1)}_{u_1u_{k-1}u_n} \cup \mathcal{K}_{u_{k-1}u_k}, & \text{if } m \text{ is odd.} \\ C^*_{u_1u_{k-1}u_k} \cup \mathcal{K}_{u_{k-1}u_k}, & \text{if } m \text{ is even.} \end{cases} \quad (2)$$

By using a similar argument to those in Proposition 2.1 and 2.3 and by counting the number of cycles containing e where $e \in E(\mathcal{B}_k)$, one can get the following remark:

Remark 1: *By our construction of $\mathcal{B}_k$, we have the following:*

1) $\mathcal{B}_k$ is linearly independent.

2) Any linear combination of cycles of elements of $\mathcal{B}_n$ contains an edge incident with (u_n, v_j) for some j.

3) If $e \in E(M_{u_1u_{k-1}})$ or $e \in E(M_{u_{k-1}u_k})$ or $e \in E(M_{u_1u_k})$, then $f_{\mathcal{B}_k}(e) = 2$.

4) If $e \in E(u_{k-1}u_1u_k\Box V(P_m))$ or $e \in E(\{u_{k-1}, u_k\}\Box P_m)$, then $f_{\mathcal{B}_k}(e) = 1$.

5) If $e \in E(u_{k-1}u_k\Box V(P_m))$, then $f_{\mathcal{B}_k}(e) \leq 3$.

6) $|\mathcal{B}_k| = \begin{cases} 4k-3, & \text{if } m \text{ is odd,} \\ 4k-1, & \text{if } m \text{ is even.} \end{cases}$

Proposition 2.4. *Let $\mathcal{B}_4$ be as described ahead of Proposition 2.3, and for $n \geq 5$ let $\mathcal{B}_n$ be as in (2). Then $\mathcal{B}((W_n - u_2u_n)\rho P_m) = \mathcal{B}_n \cup \mathcal{B}_{n-1} \cup \ldots \cup \mathcal{B}_5 \cup \mathcal{B}_4$ is a 4-fold basis for $\mathcal{C}((W_n - u_2u_n)\rho P_m)$.*

Proof. As in Proposition 2.3, we prove the case where m is odd and similarly we can prove the case m is even. We use induction on n to show that $\mathcal{B}((W_n - u_2u_n)\rho P_m)$ is a basis for $\mathcal{C}((W_n - u_2u_n)\rho P_m)$ when $n \geq 4$. By Proposition 2.3, $\mathcal{B}_4$ is a 4-fold basis for $\mathcal{C}((W_4 - u_2u_4)\rho P_m)$. Let $n > 4$ and suppose $\mathcal{B}^* = \mathcal{B}_{n-1} \cup \mathcal{B}_{n-2} \cup \ldots \cup \mathcal{B}_5 \cup \mathcal{B}_4$ is a 4-fold basis for $\mathcal{C}((W_{n-1} - u_2u_{n-1})\rho P_m)$. We now show $\mathcal{B}_n \cup \mathcal{B}^*$ is a 4-fold basis for $\mathcal{C}((W_n - u_2u_n)\rho P_m)$. This is equivalent of showing that $|\mathcal{B}_n \cup \mathcal{B}^*| = \dim(\mathcal{C}((W_n - u_2u_n)\rho P_m))$ and that $\mathcal{B}^* \cup \mathcal{B}_n$ is a 4-fold linearly independent set. Observe that $\mathcal{B}_n$ and $\mathcal{B}^*$ are disjoint because each elements of $\mathcal{B}_n$ contains an edge incident with (u_n, v_j) for some j and no elements of $\mathcal{B}^*$ have such edges. So $|\mathcal{B}_n \cup \mathcal{B}^*| = |\mathcal{B}_n| + |\mathcal{B}^*|$. By the inductive hypothesis, $|\mathcal{B}^*| = \dim(\mathcal{C}((W_{n-1} - u_2u_{n-1})\rho P_m)) = 2nm - 6m - n + 4(n-2)\lfloor \frac{m}{2} \rfloor + 3$. By (6) of Remark 1, it follows that $|\mathcal{B}_n| = 4n - 3$, so $|\mathcal{B}_n \cup \mathcal{B}^*| = 2nm - 6m - n + 4(n-2)\lfloor \frac{m}{2} \rfloor + 3 + 4m - 3 = 2nm - 2m - n + 4(n-2)\lfloor \frac{m}{2} \rfloor$ which is equal to $\dim(\mathcal{C}((W_n - u_2u_n)\rho P_m))$. Next we show $\mathcal{B}_n \cup \mathcal{B}^*$ is linearly independent. The set $\mathcal{B}^*$ is linearly independent by the inductive hypothesis. As we indicated above $\mathcal{B}_n$ is linearly independent. We must thus only show that $\text{Span}(\mathcal{B}^*) \cap \text{Span}(\mathcal{B}_n) = \{0\}$. To see this is true, suppose $O \in \text{Span}(\mathcal{B}_n) \cap \text{Span}(\mathcal{B}^*)$. Since $O \in \text{Span}(\mathcal{B}_n)$, either $O = 0$ or O contains an edge incident with a vertex of $V(u_n\Box\{v_1, v_2, \ldots, v_m\})$. But since $O \in \text{Span}(\mathcal{B}^*)$, O can have no such edges, so $O = 0$. To complete the proof, we show that $\mathcal{B}((W_n - u_2u_n)\rho P_m)$ has fold 4. Note that $E(\mathcal{B}^*) \cap E(\mathcal{B}_n) = E(M_{u_1u_{n-1}}) \cup (u_1u_{n-1}\Box V(P_m)) \cup (u_{n-1}\Box P_m)$. Thus, if $e \in E(\mathcal{B}^*) \cap E(\mathcal{B}_n)$, then e appears only in cycles of $\mathcal{B}_{n-1}$ and $\mathcal{B}_n$. Hence, by Proposition 2.3 and Remark 1, $f_{\mathcal{B}((W_n - u_2u_n)\rho P_m)}(e) = f_{\mathcal{B}_{n-1}}(e) + f_{\mathcal{B}_n}(e) \leq 2 + 2$. Also, if $e \in E(\mathcal{B}_n) - E(\mathcal{B}^*) \cap E(\mathcal{B}_n)$, then by Remark 1 $f_{\mathcal{B}((W_n - u_2u_n)\rho P_m)}(e) \leq 3$. Finally, if $e \in E(\mathcal{B}^*) - E(\mathcal{B}^*) \cap E(\mathcal{B}_n)$, then by the inductive step $f_{\mathcal{B}((W_n - u_2u_n)\rho P_m)}(e) = f_{\mathcal{B}^*}(e) \leq 4$. ■

Let

$$\mathcal{B}_{u_2u_n} = \mathcal{C}_{u_2u_nu_1} \cup \mathcal{U}_{u_1u_2u_n}. \tag{3}$$

Then, one can remark the following:

Remark 2: *By our construction of $\mathcal{B}_{u_2u_n}$, we have the following:*

1) If $e \in E(M_{u_1u_2})$ or $e \in E(M_{u_2u_n})$ or $e \in E(M_{u_1u_n})$ or $e \in E(u_1u_2\Box V(P_m))$ or $e \in E(u_2u_n\Box V(P_m))$, then $f_{\mathcal{B}_{u_2u_n}}(e) = 1$.

2) If $e \in E(u_1u_n\Box V(P_m))$, then $f_{\mathcal{B}_{u_2u_n}}(e) = 2$.

3) $|\mathcal{B}_{u_2u_n}| = \begin{cases} 2m-1, & \text{if } m \text{ is odd,} \\ 2m, & \text{if } m \text{ is even.} \end{cases}$

Theorem 2.5: *The set $\mathcal{B}(W_n\rho P_m) = \mathcal{B}((W_n - u_2u_n)\rho P_m) \cup \mathcal{B}_{u_2u_n}$ is a 4-fold basis of $\mathcal{C}(W_n\rho P_m)$* where $\mathcal{B}_{u_2u_n}$ is as in (3).

Proof. We show the case where m is odd. The cycle $\mathcal{U}^{(j)}_{u_1u_2u_n}$ contains $(u_2,v_j)(u_n,v_j)$ for each j which does not appears in any other cycle of $\mathcal{B}((W_n - u_2u_n)\rho P_m) \cup \mathcal{U}_{u_1u_2u_n}$. Hence $\mathcal{B}((W_n - u_2u_n)\rho P_m) \cup \mathcal{U}_{u_1u_2u_n}$ is linearly independent. Similarly, for each j the cycle $\mathcal{C}^{(j)}_{u_2u_nu_1}$ contains $(u_2,v_j)(u_n,v_{m-j+1})$ and no other cycle of $\mathcal{B}(W_n\rho P_m)$ has such edge. Thus, $\mathcal{B}(W_n\rho P_m)$ is linearly independent. Now,

$$\begin{aligned} |\mathcal{B}(W_n\rho P_m)| &= |\mathcal{B}((W_n - u_2u_n)\rho P_m)| + |\mathcal{B}_{u_2u_n}| \\ &= 2nm - 2m - n + 4(n-2)\lfloor\frac{m}{2}\rfloor + 2m - 1 \\ &= 2nm - 2m - n + 4(n-1)\lfloor\frac{m}{2}\rfloor + 1 \\ &= \dim \mathcal{C}(W_n\rho P_m). \end{aligned}$$

Hence, $\mathcal{B}(W_n\rho P_m)$ is a basis for $\mathcal{C}(W_n\rho P_m)$. Now, we show that $\mathcal{B}(W_n\rho P_m)$ is a 4-fold basis. Note that $E(\mathcal{B}_{u_2u_n}) \cap E(\mathcal{B}((W_n - u_2u_n)\rho P_m)) = E(M_{u_1u_2}) \cup E(u_1u_2\Box V(P_m)) \cup E(u_1u_n\Box V(P_m))$. Now, let $e \in E(W_n\rho P_m)$. Then we consider the following cases:

Case 1: $e \in E(\mathcal{B}_{u_2u_n}) \cap E(\mathcal{B}((W_n - u_2u_n)\rho P_m))$. Then we have the following:

1) If $e \in E(M_{u_1u_2})$, then e appears only in $\mathcal{B}_{u_2u_n}$ and $\mathcal{B}_4$, and so by Remark 2 and (1) of proposition 2.3 $f_{\mathcal{B}(W_n\rho P_m)}(e) = f_{\mathcal{B}_{u_2u_n}}(e) + f_{\mathcal{B}_4}(e) \leq 1+2$.

2) If $e \in E(u_1u_2\Box V(P_m))$, then e appears only in $\mathcal{B}_{u_2u_n}$ and $\mathcal{B}_4$ and so by Remark 2 and (9) and (14) of proposition 2.3 $f_{\mathcal{B}(W_n\rho P_m)}(e) = f_{\mathcal{B}_{u_2u_n}}(e) + f_{\mathcal{B}_4}(e) \leq 3+1$.

3) If $e \in E(u_1u_n\Box V(P_m))$, then e appears only in $\mathcal{B}_{u_2u_n}$ and $\mathcal{B}_n$ and so by Remark 1 and Remark 2 $f_{\mathcal{B}(W_n\rho P_m)}(e) = f_{\mathcal{B}_{u_2u_n}}(e) + f_{\mathcal{B}_n}(e) \leq 1+2$.

Case 2: $e \notin E(\mathcal{B}_{u_2u_n}) \cap E(\mathcal{B}((W_n - u_2u_n)\rho P_m))$. Then by Proposition 2.4 and Remark 2, $f_{\mathcal{B}(W_n\rho P_m)}(e) \leq 4$. ■

Theorem 2.6. For any $n \geq 4$ and $m \geq 2$, $3 \leq b(W_n \rho P_m) \leq 4$.
Proof. By Proposition 2.5, it suffices to show that $b(W_n \rho P_m) \geq 3$. Suppose that $b(W_n \rho P_m) \leq 2$. Then $\mathcal{C}(W_n \rho P_m)$ has a 2-fold basis, say $\mathcal{B}$. Since the girth of $W_n \rho P_m$ is 3,

$$\begin{aligned}
2|E(W_n \rho P_m)| &\geq 3|\mathcal{B}| \\
2(3nm - 2m - n + 4(n-1)\lfloor \frac{m}{2} \rfloor + 1) &\geq 3(2nm - 2m - n + \\
& \quad 4(n-1)\lfloor \frac{m}{2} \rfloor + 1) \\
2m + n &\geq 4(n-1)\lfloor \frac{m}{2} \rfloor + 1 \\
2m + n &\geq 2(n-1)(m+1) \\
4m &\geq 2nm + n - 2.
\end{aligned}$$

But $n \geq 4$, thus

$$4m \geq 8m + 2,$$

which is a contradiction. ■

3 Minimum cycle basis of $W_n \rho P_m$

A related problem to the basis number is the construction of a minimum cycle basis, to address such problem for $W_n \rho P_m$, we construct it for $(W_n - u_2 u_n)\rho P_m$ then we extended it for $W_n \rho P_m$ as we did in the previous section. Let

$$\begin{aligned}
\mathcal{B}_4^* &= \mathcal{C}^*_{u_1 u_2 u_3} \cup \mathcal{C}^*_{u_1 u_3 u_4} \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor + 1} \mathcal{U}^{(j)}_{u_1 u_2 u_3} \right) \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor + 1)}_{u_1 u_3 u_4} \cup \\
& \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{R}^{(j)}_{u_1 u_2} \right) \cup \mathcal{K}_{u_1 u_2} \cup \mathcal{K}_{u_2 u_3} \cup \mathcal{K}_{u_3 u_4}
\end{aligned}$$

if m is odd and

$$\begin{aligned}
\mathcal{B}_4^* &= \mathcal{C}^*_{u_1 u_2 u_3} \cup \mathcal{C}^*_{u_1 u_3 u_4} \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{U}^{(j)}_{u_1 u_2 u_3} \right) \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor - 1} \mathcal{R}^{(j)}_{u_1 u_2} \right) \cup \\
& (\cup_{j=1}^{3} \mathcal{Z}_{u_i u_{i+1}}) \cup \mathcal{Z}_{u_2 u_1} \cup (\cup_{j=1}^{3} \mathcal{K}_{u_i u_{i+1}}) - \left(\cup_{j=1}^{3} \mathcal{K}^{\lfloor \frac{m}{2} \rfloor}_{u_i u_{i+1}} \right)
\end{aligned}$$

if m is even.

Proposition 3.1: The set $\mathcal{B}_4^*$ which described above is a basis of

$$\mathcal{C}\left((W_4 - u_2u_4)\rho P_m\right).$$

Proof. We prove the theorem for the case where m is odd and similarly we can do it for the case where m is even. To show that $\mathcal{C}^*_{u_1u_2u_3} \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{U}^{(j)}_{u_1u_2u_3}\right)$ is linearly independent, it suffices to show that each cycle of $\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor+1} \mathcal{U}^{(j)}_{u_1u_2u_3}$ is independent of the cycles of $\mathcal{C}^*_{u_1u_2u_3}$. For some $j \leq \lfloor \frac{m}{2} \rfloor$, suppose that $\mathcal{U}^{(j)}_{u_1u_2u_3}$ is the linear combinations of the cycles of $Z = \{z_1, z_2, \ldots, z_r\}$. Since $\mathcal{U}^{(j)}_{u_1u_2u_3}$ contains edges $e_1 = (u_1, v_j)(u_2, v_j), e_2 = (u_2, v_j)(u_3, v_j)$ and $e_3 = (u_1, v_j)(u_3, v_j)$ which appear only in $C^{(m-j+1)}_{u_3u_2u_1}$, $C^{(m-j+1)}_{u_1u_2u_3}$ and $C^{(m-j+1)}_{u_2u_1u_3}$, respectively, as a result $C^{(m-j+1)}_{u_3u_2u_1}$, $C^{(m-j+1)}_{u_1u_2u_3}$ and $C^{(m-j+1)}_{u_2u_1u_3} \in Z$, say $z_1 = C^{(m-j+1)}_{u_3u_1u_2}, z_2 = C^{(m-j+1)}_{u_1u_2u_3}$, and $z_3 = C^{(m-j+1)}_{u_1u_3u_2}$. Now, since $e_4 = (u_1, v_j)(u_3, v_{m-j+1}) \in E(z_1 \oplus z_2 \oplus z_3)$ and $e_4 \notin E(\mathcal{U}^{(j)}_{u_1u_2u_3})$ and since e_4 belongs only to z_1 and $C^{(j)}_{u_1u_2u_3}$, as a result $C^{(j)}_{u_1u_2u_3} \in Z$, say $z_4 = C^{(j)}_{u_1u_2u_3}$. Similarly, since $e_5 = (u_3, v_j)(u_1, v_{m-j+1}) \in E(z_1 \oplus z_2 \oplus z_3 \oplus z_4)$ and $e_5 \notin E(\mathcal{U}^{(j)}_{u_1u_2u_3})$ and since e_5 belongs only to z_2 and $C^{(j)}_{u_3u_2u_1}$, we get $C^{(j)}_{u_3u_2u_1} \in Z$, say $z_5 = C^{(j)}_{u_3u_2u_1}$. Finally, since $e_6 = (u_1, v_{m-j+1})(u_2, v_j) \in E(z_1 \oplus z_2 \oplus z_3 \oplus z_4 \oplus z_5)$ and $e_6 \notin E(\mathcal{U}^{(j)}_{u_1u_2u_3})$ and since e_6 belongs only to z_2 and $C^{(j)}_{u_2u_1u_3}$, we have $C^{(j)}_{u_2u_1u_3} \in Z$, say $z_6 = C^{(j)}_{u_2u_1u_3}$. To this end, $E(u_1u_2u_3u_1 \Box v_{m-j+1}) \subseteq E(z_1 \oplus z_2 \oplus z_3 \oplus z_4 \oplus z_5 \oplus z_6)$ and the edges $u_1u_2 \Box v_{m-j+1}, u_2u_3 \Box v_{m-j+1}$ and $u_1u_3 \Box v_{m-j+1}$ belong only to $C^{(j)}_{u_3u_2u_1}$, $C^{(j)}_{u_1u_2u_3}$ and $C^{(j)}_{u_2u_1u_3}$, respectively. Therefore,

$$E(u_1u_2u_3u_1 \Box v_{m-j+1}) \subseteq E(\bigoplus_{i=1}^{r} z_i) = O.$$

This is a contradiction. By continuing employing the same arguments as in the proof of Proposition 3.1, we can show that $\mathcal{B}_4^*$ is linearly independent.

Now,

$$
\begin{aligned}
|\mathcal{B}_4^*| &= |\mathcal{C}^*_{u_1u_2u_3}| + |\mathcal{C}^*_{u_1u_3u_4}| + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor +1} |\mathcal{U}^{(j)}_{u_1u_2u_3}| + |\mathcal{U}^{(\lfloor \frac{m}{2} \rfloor +1)}_{u_1u_3u_4}| \\
&\quad + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} |\mathcal{R}^{(j)}_{u_1u_2}| + |\mathcal{K}_{u_1u_2}| + |\mathcal{K}_{u_2u_3}| + |\mathcal{K}_{u_3u_4}| \\
&= 3(m-1) + 3(m-1) + \left\lfloor \frac{m}{2} \right\rfloor + 1 + 1 + \left\lfloor \frac{m}{2} \right\rfloor + 3(m-1) \\
&= 9m + 2\left\lfloor \frac{m}{2} \right\rfloor - 7 \\
&= 10m - 8 \\
&= \dim \mathcal{C}((W_4 - u_2u_4)\rho P_m).
\end{aligned}
$$

Thus, $\mathcal{B}_4$ is a basis for $\mathcal{C}(W_4 - u_2u_4)\rho P_m$. ■

Now for each $k \geq 5$ we let

$$
\mathcal{B}_k^* = \begin{cases} \mathcal{C}^*_{u_1u_{k-1}u_k} \cup \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor +1)}_{u_1u_{k-1}u_n} \cup \mathcal{K}_{u_{k-1}u_k} & \text{if } m \text{ is odd} \\ \mathcal{C}^*_{u_1u_{k-1}u_k} \cup \mathcal{Z}_{u_ku_{k-1}} \cup \mathcal{K}_{u_{k-1}u_k} - \{\mathcal{K}^{\lfloor \frac{m}{2} \rfloor}_{u_{k-1}u_k}\} & \text{if } m \text{ is even.} \end{cases} \tag{4}
$$

Using the same argument as in Proposition 2.4, one can easily prove the following result:

Proposition 3.3. *Let $\mathcal{B}_4^*$ be as described in Proposition 3.2, and for $n \geq 5$ let $\mathcal{B}_n^*$ be as described in (4). Then $\mathcal{B}^*((W_n - u_2u_n)\rho P_m) = \mathcal{B}_n^* \cup \mathcal{B}_{n-1}^* \cup \ldots \cup \mathcal{B}_5^* \cup \mathcal{B}_4^*$ is a basis for $\mathcal{C}((W_n - u_2u_n)\rho P_m)$.* ■

Theorem 3.4. *Let $\mathcal{B}^*((W_n - u_2u_n)\rho P_m)$ and $\mathcal{B}_{u_2u_n}$ be as described in Proposition 3.3 and in (3), respectively. Then the set $\mathcal{B}^*(W_n\rho P_m) = \mathcal{B}^*((W_n - u_2u_n)\rho P_m) \cup \mathcal{B}_{u_2u_n}$ is a minimum cycle basis of $\mathcal{C}(W_n\rho P_m)$.*

Proof. We start where m is odd: By employing the same arguments as in the proof of Proposition 2.6, one can easily prove that $\mathcal{B}^*(W_n\rho P_m)$ is a basis for $\mathcal{C}(W_n\rho P_m)$. Now we show that $\mathcal{B}^*(W_n\rho P_m)$ is minimal. Let $\mathcal{N} = W_n\rho P_m - V(P_n)\square P_m$. Note that, from the construction of $\mathcal{B}_4^*$ and $\mathcal{B}_k^*$, for each $k \geq 5$, $\mathcal{B}^*(W_n\rho P_m) = \left(\cup_{i=2}^{n} \mathcal{C}^*_{u_1u_iu_{i+1}}\right) \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor +1} \mathcal{U}^{(j)}_{u_1u_2u_3}\right) \cup \left(\cup_{i=3}^{n} \mathcal{U}^{(\lfloor \frac{m}{2} \rfloor +1)}_{u_1u_iu_{i+1}}\right) \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{R}^{(j)}_{u_1u_2}\right) \cup \left(\cup_{i=1}^{n} \mathcal{K}_{u_iu_{i+1}}\right) \cup \mathcal{B}_{u_2u_n}$. Also, each cycle of $\mathcal{B}(\mathcal{N}) = \mathcal{B}^*(W_n\rho P_m) - \left(\cup_{i=1}^{n} \mathcal{K}_{u_iu_{i+1}}\right) \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{R}^{(j)}_{u_1u_2}\right)$ is of length 3.

Moreover, one can easily see that $\dim C(\mathcal{N}) = |\mathcal{B}(\mathcal{N})|$ and so $\mathcal{B}^*(W_n\rho P_m) - \left(\cup_{i=1}^{n}\mathcal{K}_{u_iu_{i+1}}\right) \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor}\mathcal{R}_{u_1u_2}^{(j)}\right)$ is a basis of the graph $\mathcal{N}$. Observe that any cycle containing any edge of $V(P_n)\Box P_m$ is of length at least 4. Therefore, $\mathcal{B}(\mathcal{N})$ is a maximum linearly independent set of $C(W_n\rho P_m)$ consisting of 3-cycles. Since $\left(\cup_{i=1}^{n}\mathcal{K}_{u_iu_{i+1}}\right) \cup \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor}\mathcal{R}_{u_1u_2}^{(j)}\right)$ is a set of 4-cycles and the cycle space is a matroid, as a result $\mathcal{B}(W_n\rho P_m)$ is minimum.

To treat the even m we follow, world by word, the proof of the odd case taking into the account that

$$\mathcal{B}^*(W_n\rho P_m) - \left(\cup_{j=1}^{\lfloor \frac{m}{2} \rfloor - 1}\mathcal{R}_{u_1u_2}^{(j)}\right) \cup \left(\cup_{i=1}^{n}\left(\mathcal{K}_{u_iu_{i+1}} - \mathcal{K}_{u_iu_{i+1}}^{\lfloor \frac{m}{2} \rfloor}\right)\right)$$

is a cycle basis for $W_n\rho P_m - V(P_n)\Box\left(P_m - u_{\lfloor \frac{m}{2} \rfloor}u_{\lfloor \frac{m}{2} \rfloor+1}\right)$ and it is a maximum linearly independent set of $C(W_n\rho P_m)$ consisting of three cycles. ■

Corollary 3.5. $l(W_n\rho P_m) = \begin{cases} 13nm - 16n - 25\lfloor \frac{m}{2} \rfloor, & \text{if } m \text{ is odd,} \\ 13nm - 14n - 25\lfloor \frac{m}{2} \rfloor + 4, & \text{if } m \text{ is even,} \end{cases}$ and $\lambda(W_n\rho P_m) = 4$. ■

References

[1] A.A. Ali and G.T. Marougi, The basis number of Cartesian product of some graphs, *J. of the Indian Math. Soc.* 58, 123-134 (1992).

[2] M.K. Al-Qeyyam and M.M.M. Jaradat, On the basis number and the minimum cycle bases of the wreath product of some graphs II. *J. Combin. Math. Combin. Comput.* 72, 65-72 (2010).

[3] A.S. Alsardary and J. Wojciechowski, The basis number of the powers of the complete graph, *Discrete Math. 188*, no. 1-3, 13-25 (1998).

[4] Berger, F., Minimum cycle bases in graphs", PHD thesis, Munich 2004.

[5] Z. Bradshaw and R. Hammack, Minimum cycle bases of the direct products of graphs with cycles, Ars Mathematica Contemporanea 2, 101-119 (2009).

[6] Z. Bradshaw and M.M.M. Jaradat, Minimum cycle bases for direct products of K_2 with complete graphs. *Australasian Journal of Combinatorics* 43, 127–131, (2009).

[7] R. Hammack, Minimum cycle bases of direct products of bipartite graphs, *Australasian Journal of Combinatorics* 36, 213–221, (2006).

[8] R. Hammack, Minimum cycle bases of direct products of complete graphs, *Information Processing Letters*, 102, 214-218, (2007).

[9] W. Imrich and P. Stadler, Minimum cycle bases of product graphs, *Australasian Journal of Combinatorics* 26, 233-244 (2002) .

[10] M.M.M. Jaradat, On the basis number of the direct product of graphs, *Australasian Journal of Combinatorics* 27, 293-306 (2003).

[11] M.M.M. Jaradat, An upper bound of the basis number of the strong product of graphs, *Discussion Mathematicea Graph Theory* 25, no. 3, 391-406 (2005).

[12] M.M.M. Jaradat, Minimal cycle bases of the lexicographic product of graphs, *Discussions Mathematicae Graph Theory* 28(2) (2008) 229-247

[13] M.M.M. Jaradat, On the basis number and the minimum cycle bases of the wreath product of some graphs I, *Discussiones Mathematicae Graph Theory* 26, 113-134 (2006).

[14] M.M.M. Jaradat, A new upper bound of the basis number of the lexicographic product of graphs, *Ars Combinatoria* 97, 423-442 (2010).

[15] M.M.M. Jaradat and M.Y Alzoubi, An upper bound of the basis number of the lexicographic product of graphs, *Australasian Journal of Combinatorics* 32, 305-312 (2005).

[16] M.M.M. Jaradat, M.Y. Alzoubi and E.A. Rawashdeh, The basis number of the Lexicographic product of different ladders, *SUT Journal of Mathematics* 40, no. 2, 91-101 (2004).

[17] S. MacLane, A combinatorial condition for planar graphs, *Fundamenta Math.* 28, 22-32 (1937).

[18] E.F. Schmeichel, The basis number of a graph, *J. Combin. Theory Ser. B* 30, no. 2, 123-129 (1981).

[19] D.J.A. Welsh, Kruskal's theorem for matroids, *Proc. Cambridge Phil, Soc.,* 64, 3-4 (1968).