On the complexity of recognizing tenacious graphs

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Abstract

We consider the relationship between the minimum degree $\delta(G)$ of a graph and the complexity of recognizing if a graph is T-tenacious. Let $T \geq 1$ be a rational number. We first show that if $\delta(G) \geq \frac{Tn}{T+1}$, then G is T-tenacious. On the other hand, for any fixed $\epsilon > 0$, we show that it is NP-hard to determine if G is T-tenacious, even for the class of graphs with $\delta(G) \geq (\frac{T}{T+1} - \epsilon)n$.

Keywords: NP-complete problem, tenacity, tenacious, NP-hard.

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1. Introduction

We consider only graphs without loops or multiple edges. Our terminology will be standard except as indicated; a good reference for any undefined terms is [2]. We use $V(G)$, $\alpha(G)$, and $\omega(G)$ to denote the vertex set, independence number and number of components in a graph G, respectively. We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\mid V(G) \mid$ the set of vertices, the set of edges and the order of G, respectively. The concept of tenacity of a graph G was introduced in [4,5], as a useful measure of the "vulnerability" of G. In [5] Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs but they didn't show the complete proof of the third case. In [18] we showed a new and complete proof for

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case three of the Harary Graphs. In [12], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [3 - 27], the authors studied more about this new invariant. The tenacity of a graph G, $T(G)$, is defined by $T(G) = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$. A connected graph G is called T-tenacious if $\mid S \mid +\tau(G-S) \geq T\omega(G-S)$ holds for any subset S of vertices of G with $\omega(G-S) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+\tau(G-S)}{\omega(G-S)}$.

The Mix-tenacity $T_m(G)$ of a graph G is defined as

$$T_m(G) = \min_{A \subset E(G)} \{\frac{\mid A \mid +\tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of vertices) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. Cozzens et al. in [4], called this parameter Edge-tenacity, but Moazzami changed the name of this parameter to Mix-tenacity in [16]. It seems Mix-tenacity is a better name for this parameter. $T(G)$ and $T_m(G)$ turn out to have interesting properties.

After the pioneering work of Cozzens, Moazzami, and Stueckle in [4,5], several groups of researchers have investigated tenacity, and its related problems. In [20] and [21] Piazza et al. used the $T_m(G)$ as Edge-tenacity. But this parameter is a combination of cutset $A \subset E(G)$ and the number of vertices of a largest component, $\tau(G-A)$. It may be observed that in the definition of $T_m(G)$, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. Also this parameter didn't seem very satisfactory for Edge-tenacity. Thus Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity, $T_e(G)$, in [16]. The Edge-tenacity $T_e(G)$ of a graph G is defined as

$$T_e = \min_{A \subset E(G)} \{\frac{\mid A \mid +\tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of edges) of a largest com-

ponent of $G - A$ and $\omega(G - A)$ is the number of components of $G - A$. This new measure of vulnerability involves edges only and thus is called the Edge-tenacity. Since 1992 there were several interesting questions. But the question " How difficult is it to recognize T-tenacious graphs? " has remained an interesting open problem for some time. The question was first raised by Moazzami in [11]. Our purpose in [19] was to show that for any fixed positive rational number T, it is NP-hard to recognize T-tenacious graphs. To prove this we showed that it is NP-hard to recognize T-tenacious graphs by reducing a well-known NP-complete variant of INDEPENDENT SET.
Any undefined terms can be found in the standard references on graph theory, including Bondy and Murty [2].

2. Main Results

We begin by considering the following problem. Let $T \geq 1$ be any rational number.

Not T-TENACIOUS

INSTANCE: An undirected graph G.

QUESTION: Does there exist $X \subseteq V(G)$ with $\omega(G - X) > 1$ such that $T\omega(G - X) > |X| + m(G - X)$

Theorem 1. *Not T-TENACIOUS is NP-complete.*

To prove this, we will reduce the following problem, which is known [1] to be NP-complete for any fixed β, $0 < \beta < 1$.

INDEPENDENT β-MAJORITY

INSTANCE: An undirected graph G on n vertices.

QUESTION: Is $\alpha(G) \geq \beta n$?

Proof of theorem 1. We reduce INDEPENDENT β-MAJORITY to Not T-TENACIOUS. Let $T = \frac{a}{b} \geq 1$ for positive integers a and b, and fix β where $0 < \beta < 1$. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and let $k = \lceil \beta n \rceil$. Construct G' from G as follow. First we add a set A includes n complete graphs $A_1, \ldots, A_n$ with

$$|V(A_i)| = h = \lceil Tn \rceil - n + k, \ i = 1 \ldots n,$$

to G and join v_i to any vertex in $A_i, 1 \leq i \leq n$. Then add another set C of br independent vertices to G, where $r > 2$ is an integer. Now add a set

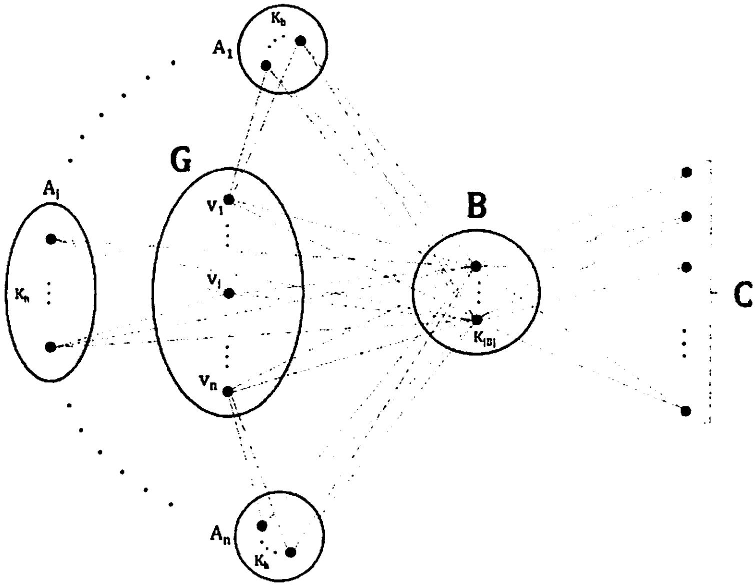

Figure 1: Construct Graph G' from G

B of $ar-2$ vertices which induces a complete graph, and join each vertex of B to every vertex of $V(G)\cup A\cup C$. It suffices to show that $\alpha(G)\geq k$ if and only if G' is not T-tenacious.

First suppose that G contains an independent set I with $|I|=k$. Define $X'\subseteq V(G')$ by $X'=(V(G)-I)\cup B$. Then

$$\omega(G'-X') = n+|C| = n+br$$
$$|X'| = n-k+|B| = n-k+ar-2$$
$$m(G'-X') = h+1 = \lceil Tn\rceil - n+k+1$$
$$\downarrow$$

$$\begin{aligned}T\omega(G'-X') = Tn+ar &> (\lceil Tn\rceil -1)+ar\\ &= (\lceil Tn\rceil - n+k+1)+(n-k+ar-2)\\ &= m(G'-X')+|X'|\end{aligned}$$

Therefore G' is not T-tenacious.

Conversely, suppose G' is not T-tenacious. Then exists $X' \subseteq V(G')$ with $\omega(G'-X') > 1$ such that $T\omega(G'-X') > |X'| + m(G'-X')$. Clearly $B \subseteq X'$.

Claim 1. $|X'|+m(G'-X') \geq |X'-(A\cup C)|+m(G'-(X'-(A\cup C)))$.

Proof. Suppose $X'' = X' - (A \cup C)$ and $M(G'-X'')$ is a largest component of $G'-X''$. Then $M(G'-X'')-(X'-X'')$ is a component of $G'-X'$ and

$$\begin{aligned} m(G'-X') &\geq |M(G'-X'')-(X'-X'')| \\ &\geq |M(G'-X'')|-|X'-X''| \\ &= m(G'-X'')-|X'|+|X''| \end{aligned}$$

$$\rightarrow |X'|+m(G'-X') \geq |X''|+m(G'-X'')$$

□

We may also assume $X' \cap (A \cup C) = \phi$; otherwise

$$\begin{aligned} T\omega(G'-(X'-(A\cup C))) &\geq \omega(G'-X') \\ &> |X'|+m(G'-X') \\ &\geq |X'-(A\cup C)|+m(G'-(X'-(A\cup C))) \end{aligned}$$

And we could use $X'-(A\cup C)$ instead of X'.

Let

$$X = X' \cap V(G), \quad x = |X|, \quad x' = |X'|$$

$$m' = m(G'-X'), \quad w = \omega(G-X), \quad w' = \omega(G'-X')$$

Then

$$\begin{aligned} x' &= x+|B| = x+ar-2 \\ w' &= w+x+|C| = w+x+br \\ m' &\geq h+1 \end{aligned}$$

Claim 2. $\lceil Tn \rceil - x - m' + 1 \geq 0$.

Proof.

$$w' \leq n+|C| = n+br$$

$$x'+m' < Tw' \leq T(n+br) = Tn+ar$$

$$\begin{aligned}
&\rightarrow \quad Tn + ar - x' - m' > 0 \\
&\rightarrow \quad \lceil Tn + ar - x' - m' \rceil \geq 1 \\
&\rightarrow \quad \lceil Tn \rceil + ar - x' - m' - 1 \geq 0 \\
&\rightarrow \quad \lceil Tn \rceil - x - m' + 1 \geq 0
\end{aligned}$$

□

$$Tw' > x' + m'$$

$$\rightarrow \quad Tw + Tx + ar > x + ar - 2 + m'$$

$$\downarrow$$

$$\begin{aligned}
Tw &> x - Tx + m' - 2 \\
&= (T-1)(\lceil Tn \rceil - x - m' + 1) - (T-1)(\lceil Tn \rceil - m' + 1) + m' - 2 \\
&\geq -(T-1)(\lceil Tn \rceil - m' + 1) + m' - 2 \\
&= Tm' - (T-1)\lceil Tn \rceil - T - 1 \\
&\geq T(h+1) - (T-1)\lceil Tn \rceil - T - 1 \\
&= T(\lceil Tn \rceil - n + k + 1) - (T-1)\lceil Tn \rceil - T - 1 \\
&= \lceil Tn \rceil - Tn + Tk - 1 \\
&\geq Tk - 1
\end{aligned}$$

$$\rightarrow$$

$$\begin{aligned}
w &> k - \frac{1}{T} \\
w &\geq k
\end{aligned}$$

Since it is possible to form an independent set in G by choosing one vertex from each component of $G - X$, we conclude $\alpha(G) \geq k$.

□

Define $\Omega(r)$ to be the class of all graphs with $\delta(G) \geq rn$, where $n = |V(G)|$. We prove the following two results for any rational number $T \geq 1$.

Theorem 2. *Let G be a graph in $\Omega\left(\frac{T}{T+1}\right)$. Then G is T-tenacious.*

Theorem 3. *For any fixed $\varepsilon > 0$ it is NP-hard to recognize T-tenacious graphs in $\Omega\left(\frac{T}{T+1} - \varepsilon\right)$.*

Proof of theorem 2. Let $X \subseteq V(G)$ such that $\omega(G-X) > 1$ and $Z \subseteq V(G)$ be the vertex set of a component of $G-X$ having the fewest number of vertices.
Let

$$n = |V(G)|, \quad x = |X|, \quad z = |Z|, \quad w = \omega(G-X)$$

Then

$$\frac{T}{T+1}n \leq \delta \leq n-1, \quad z \leq \frac{n-x}{w} \leq \frac{n-x}{2}$$

Hence if $w \in Z$, $d(w) \leq x + z - 1$, Thus

$$\delta \leq x + z - 1$$

Therefore

$$\begin{aligned} \delta &\leq x + \frac{n-x}{w} - 1 \\ \delta &\leq x + \frac{n-x}{2} - 1 = \frac{n+x-2}{2} \end{aligned}$$

Claim 3. $T + 1 \leq x$.

Proof.

$$\frac{T}{T+1}n \leq \delta \leq n-1 \quad \rightarrow \quad T+1 \leq n$$

$$\frac{T}{T+1}n \leq \delta \leq \frac{n+x-2}{2} \quad \rightarrow \quad \frac{T-1}{T+1}n + 2 \leq x$$

$$\downarrow$$

$$T+1 = \frac{T-1}{T+1}(T+1) + 2 \leq \frac{T-1}{T+1}n + 2 \leq x$$

□

We must show that

$$Tw \leq x + m(G-X)$$

we instead show that

$$Tw \leq x$$

We consider the following two conditions:

1) $nT < x(T+1)$

$$\rightarrow \quad T(n-x) \leq x$$

$$\xrightarrow{w \leq n-x} \quad Tw \leq T(n-x) \leq x$$

2) $x(T+1) \leq nT$

$$\begin{aligned}
\textit{Claim 3} \quad &\rightarrow \quad x(T+1)(x-(T+1)) \leq nT(x-(T+1)) \\
&\rightarrow \quad x-(T+1) \leq nT\left(\frac{1}{T+1}-\frac{1}{x}\right) \\
&\rightarrow \quad x+\frac{nT}{x}-T-1 \leq \frac{T}{T+1}n \\
&\rightarrow \quad x+\frac{T}{x}(n-x)-1 \leq \frac{T}{T+1}n \leq \delta \leq x+\frac{n-x}{w}-1 \\
&\rightarrow \quad \frac{T}{x} \leq \frac{1}{w} \\
&\rightarrow \quad Tw \leq x
\end{aligned}$$

□

Proof of theorem 3. Given $\varepsilon > 0$ and $T = \frac{a}{b} \geq 1$, choose β such that $0 < \beta < 1$, and then choose r sufficiently large such that

$$\frac{ar-2}{(a+b)r+n(Tn-n+\beta n+3)} > \frac{T}{T+1} - \varepsilon \tag{1}$$

$$\left(\varepsilon(a+b)\cdot r > \left(\frac{T}{T+1}-\varepsilon\right) \times n(Tn-n+\beta n+3)+2\right)$$

The reduction described in the proof of Theorem 1 yields a graph G' with

$$\begin{aligned}
|V(G')| &= n(h+1)+|B|+|C| \\
&= (a+b)r-2+n(\lceil Tn\rceil - n + \lceil \beta n\rceil + 1) \\
&< (a+b)r+n(Tn-n+\beta n+3)
\end{aligned}$$

and

$$\delta(G') = |B| = ar-2$$

By (1) it follows that

$$\begin{aligned}
\delta(G') &= ar-2 \\
&> \left(\frac{T}{T+1}-\varepsilon\right)((a+b)r+n(Tn-n+\beta n+3)) \\
&> \left(\frac{T}{T+1}-\varepsilon\right)|V(G')|
\end{aligned}$$

$$\rightarrow \quad G' \in \Omega\left(\frac{T}{T+1}-\varepsilon\right)$$

This establishes that it is NP-hard to recognize T-tenacious graphs in this class.

□

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