b-coloring of Cartesian product of odd graphs

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Abstract

A b-coloring of a graph G with k colors is a proper coloring of G using k colors in which each color class contains a color dominating vertex, that is, a vertex which has a neighbor in each of the other color classes. The largest positive integer k for which G has a b-coloring using k colors is the b-chromatic number $b(G)$ of G. The b-spectrum $S_b(G)$ of a graph G is the set of positive integers k, $\chi(G) \leq k \leq b(G)$, for which G has a b-coloring using k colors. A graph G is b-continuous if $S_b(G)= \{\chi(G), \ldots, b(G)\}$. It is known that for any two graphs G and H, $b(G\Box H) \geq \max\{b(G), b(H)\}$, where $\Box$ stands for the Cartesian product. In this paper, we determine some families of graphs G and H for which $b(G\Box H) \geq b(G) + b(H) - 1$. Further we show that if O_{k_i}, $i = 1, 2, \ldots, n$ are odd graphs with $k_i \geq 4$ for each i, then $O_{k_1}\Box O_{k_2}\Box \ldots \Box O_{k_n}$ is b-continuous and $b(O_{k_1}\Box O_{k_2}\Box \ldots \Box O_{k_n}) = 1 + \sum_{i=1}^{n} k_i$.

Key Words: b-coloring, b-continuity, Cartesian product, odd graphs.
AMS Subject Classification: 05C15

1 Introduction

All graphs considered in this paper are finite, simple and undirected. A b-coloring of a graph G is a proper coloring of G in which each color class has a color dominating vertex (c.d.v.), that is, a vertex that has a neighbor in each of the other color classes. The b-chromatic number $b(G)$ of G is the largest k such that G has a b-coloring using k colors. For a given b-coloring of a graph, a set of c.d.v.'s, one from each class is known as a color dominating system (c.d.s.) of that b-coloring. Recently, there has been an increasing interest in the study of b-coloring. See, for instance, [3], [6], [10–15]. The concept of b-coloring was introduced by Irving and Manlove [8] in analogy to the achromatic number of a graph G (which gives the maximum number of color classes in a complete coloring of G [7]). They have shown that the determination of $b(G)$ is NP-hard for general graphs, but polynomial for trees. From the very definition of $b(G)$, the chromatic number $\chi(G)$ of G is the least k for which G admits a b-coloring using k colors. Thus $\chi(G) \leq b(G) \leq 1+\Delta(G)$, where $\Delta(G)$ is the maximum degree of G.

While considering the hypercube Q_3, it is easy to note that Q_3 has a b-coloring using 2 colors and 4 colors but none with 3 colors. Thus a statement similar to the interpolation theorem for complete coloring [7] is not true for b-coloring. Graphs G for which there exists a b-coloring using k colors for every $k \in \{\chi(G), \dots, b(G)\}$ are known as b-continuous graphs. Recently, there had been several papers on b-continuity of graphs ([2], [4], [5], [9]). Some of the known families of graphs which are b-continuous are chordal graphs, cographs and P_4-sparse graphs ([2],[4]). The b-spectrum of a graph G, denoted by $S_b(G)$, is defined by:

$$S_b(G) = \{k \ : \ G \text{ has a } b\text{-coloring using } k \text{ colors}\}.$$

Clearly $S_b(G) \subseteq \{\chi(G), \dots, b(G)\}$ and G is b-continuous iff $S_b(G) = \{\chi(G), \dots, b(G)\}$.

The Cartesian product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, denoted by $G \Box H$, has vertex set $V_1 \times V_2$, and two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \Box H$ iff either $x_1 = x_2$ and y_1 is adjacent to y_2 in H, or $y_1 = y_2$ and x_1 is adjacent to x_2 in G.

Let n and k be positive integers, $m = 2n+k$. We denote by $[m]$ the set $\{1, 2, \ldots, m\}$ and by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. The Kneser graph $K(m,n)$ [16] has vertex set $\binom{[m]}{n}$ in which two vertices are adjacent iff the corresponding n-subsets are disjoint. When $k = 1$, we have the odd graphs. The famous Petersen graph is the odd graph $K(5,2)$.

This paper deals with the b-chromatic number of Cartesian products of Odd graphs. The study of the b-chromatic number of Cartesian product of graphs was initiated by Kouider and Mahéo in [13] wherein they have proved the following results.

Theorem 1.1 (M. Kouider and M. Mahéo [13])
For any two graphs G and H, $b(G \square H) \geq \max\{b(G), b(H)\}$.

Theorem 1.2 (M. Kouider and M. Mahéo [13])
Let G and H be two graphs such that G has a $b(G)$-stable (that is, independent) dominating system, and H has a $b(H)$-stable dominating system. Then $b(G \square H) \geq b(G) + b(H) - 1$, and the graph $G \square H$ has a $(b(G) + b(H) - 1)$-stable dominating system.

The above result can be generalized as follows (with the same proof).

Observation 1.3
Let G and H be two graphs such that G has a k-stable (that is, independent) dominating system, and H has an l-stable dominating system. Then $G \square H$ has a $(k+l-1)$-stable dominating system.

One of the main problems concerning b-colorings is to completely characterize those graphs G and H for which $b(G \square H) = \max\{b(G), b(H)\}$. Equivalently, one has to characterize those graphs G and H for which $b(G \square H) > \max\{b(G), b(H)\}$. Theorem 1.2 gives one such family. In this paper, we find a few more classes of graphs G and H for which $b(G \square H) \geq b(G) + b(H) - 1$. These include odd graphs. In particular, we prove that for odd graphs O_{k_i}, $1 \le i \le n$ and $k_i \ge 4$ for each i, $O_{k_1}\Box O_{k_2}\Box\ldots\Box O_{k_n}$ is b-continuous and $b(O_{k_1}\Box O_{k_2}\Box\ldots\Box O_{k_n}) = 1 + \sum_{i=1}^{n} k_i$.

2 b-coloring of Cartesian product of odd graphs

We start this section with the following observation.

Observation 2.1

(i) If G has a b-coloring using k colors and H has a b-coloring using l colors with $k \le l$, then $G\Box H$ has a b-coloring using l colors (and hence $b(G\Box H) \ge l$).

(ii) If G and H are b-continuous graphs, then $S_b(G\Box H) \supseteq \{\chi(G\Box H) = \max\{\chi(G), \chi(H)\}, \ldots, \max\{b(G), b(H)\}\}$. In particular, if G and H are b-continuous and $b(G\Box H) = \max\{b(G), b(H)\}$, then $G\Box H$ is b-continuous.

Proof. (i) Let G be a graph having a b-coloring using k colors and let the colors used be $0, 1, \ldots, k-1$. Also let H be a graph having a b-coloring using l ($\ge k$) colors and let the colors used be $0, 1, \ldots, l-1$. Now color the vertex (x, y) of $G\Box H$ with $(i + j)$ $(mod\ l)$ if the color of x is i and y is j. Choose one layer, corresponding to some x with color 0, we get a copy of H where the set of vertices $\{(x, y) : y$ is a c.d.v. in $H\}$ forms a c.d.s. in $G\Box H$. Moreover this is proper, as the end vertices of any edge in $G\Box H$ have distinct colors: consider an edge $((x, y_1), (x, y_2))$. As $(y_1, y_2) \in E(H)$, the colors of y_1 and y_2 are different. The reasoning is similar for an edge $((x_1, y), (x_2, y))$. Thus $G\Box H$ has a b-coloring using l colors and hence $b(G\Box H) \ge l$.

(ii) Proof follows immediately from (i). ∎

We now define a family $\mathscr{F}$ of graphs.

Definition 2.2

Let $\mathscr{F}$ be the family of graphs H such that for every $l \in S_b(H)$, there exists

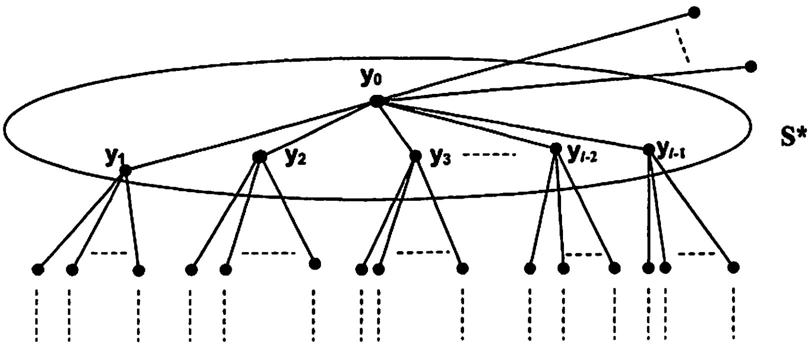

Figure 1: A graph H of the family $\mathscr{F}$

a b-coloring using l colors for H with a c.d.s. $S^* = \{y_0, y_1, \ldots, y_{l-1}\}$ such that

(i) $\{y_1, \ldots, y_{l-1}\} \subseteq N_H(y_0)$ and $N_H(y_i) \cap N_H(y_j) = \{y_0\}$, $1 \leq i \neq j \leq l-1$,
(ii) the sets $\{y_1, \ldots, y_{l-1}\}$ and $\bigcup_{i=1}^{l-1} N_H(y_i)$ are independent sets in H. (See Figure 1).

Note that the girth of all odd graphs other than the Petersen graph is 6. Moreover it is shown in [1] that the family of odd graphs with the exception of the Petersen graph belongs to $\mathscr{F}$.

The importance of the family $\mathscr{F}$ is seen from the next theorem.

Theorem 2.3
Let G be any graph and $H \in \mathscr{F}$. If G has a b-coloring using k colors and H has a b-coloring using l colors and if $3 \leq k < l$, then $G \square H$ has a b-coloring using $k + l - 1$ colors.

Proof. Let g be a b-coloring of G using k colors with a c.d.s. $S = \{x_0, \ldots, x_{k-1}\}$. As $H \in \mathscr{F}$ and $l \in S_b(H)$, there exists a b-coloring using l colors, say h, for H with a c.d.s. $S^* = \{y_0, \ldots, y_{l-1}\}$ satisfying the conditions (i) and (ii) of Definition 2.2. Let U_i denote the color class con-

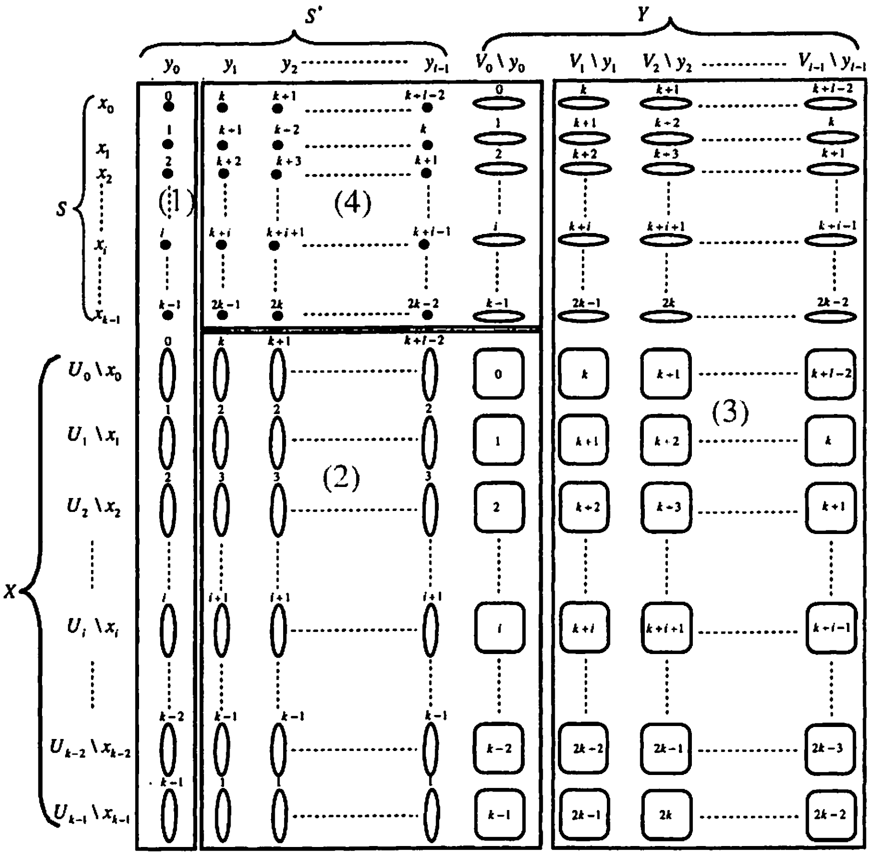

Figure 2: Coloring c given in the proof of Theorem 2.3

taining x_i, $0 \le i \le k-1$ in G, and V_j denote the color class containing y_j, $0 \le j \le l-1$ in H. Let $X = V(G) \backslash S$ and $Y = V(H) \backslash S^*$. We produce a b-coloring c for $G \square H$ using $k+l-1$ colors by means of g and h as follows:

(1) For $x \in U_i$, $i = 0, 1, \ldots, k-1$, set

$$c(x, y_0) = i. \text{ (See box (1) in Figure 2).}$$

(2) Consider the vertices in $X \times ((S^* \cup V_0) - \{y_0\})$ (See box (2) in Figure 2).

(i) For $x \in U_0 - \{x_0\}$, $y \in ((S^* \cup V_0) - \{y_0\})$, set

$$c(x,y) = \begin{cases} k+j-1 & \text{if } y = y_j,\ j = 1,2,\ldots,l-1, \\ 0 & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

(ii) For $x \in X \backslash U_0$, $y \in ((S^* \cup V_0) - \{y_0\})$, set

$$c(x,y) = \begin{cases} 1 + [i\ (mod\ (k-1))] & \text{if } x \in U_i,\ 1 \le i \le k-1,\ y \in S^* - \{y_0\}, \\ c(x, y_0) & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

(3) Next consider the vertices in $V(G) \times (Y \backslash V_0)$ (See box (3) in Figure 2). For $x \in U_i$, $0 \le i \le k-1$ and $y \in V_j - \{y_j\}$, $1 \le j \le l-1$, set

$$c(x,y) = k + [(i+j-1)\ (mod\ (l-1))].$$

(4) Finally for the vertices in $S \times ((S^* \cup V_0) - \{y_0\})$ (See box (4) in Figure 2), set

$$c(x,y) = \begin{cases} k + [(i+j-1)\ (mod\ (l-1))] & \text{if } x = x_i,\ y = y_j,\ i \ge 0,\ j > 0, \\ c(x, y_0) & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

We have to show that c is a b-coloring. Clearly c uses $k+l-1$ colors. Recall that two vertices (x,y) and (x',y') are adjacent in $G \square H$ if and only if $x = x'$ and y is adjacent to y' in H or $y = y'$ and x is adjacent to x' in G. As $k < l$ and the subgraph induced by the Cartesian product of two independent sets is independent, c is proper. Actually, what we have done is to give a circular rotation of colors whenever there arises a conflict in coloring due to presence of edges.

Consider the set of vertices $(S \times \{y_0\}) \cup (\{x_0\} \times S^*)$ whose cardinality is $k+l-1$. We shall show that the vertices of this set are c.d.v.'s of distinct color classes. By definition of c, one can see that they belong to distinct color classes. As $H \in \mathscr{F}$, $y_1, y_2, \ldots, y_{l-1}$ are all neighbors of y_0 in H and therefore the vertices of $S \times \{y_0\}$ are c.d.v.'s of their corresponding color classes. Next, consider the set $\{x_0\} \times (S^* \backslash y_0)$. If x_0 has a neighbor in each of the class $U_i \backslash \{x_i\}$, $i \ne 0$, then the vertices of $\{x_0\} \times (S^* \backslash y_0)$ are c.d.v's for the color classes $k, k+1, \ldots, k+l-2$. Otherwise, there exists at least one i in $\{1, 2, \ldots, k-1\}$ for which the above condition fails. Without loss of generality, assume that x_0 has no neighbor in $U_1 \backslash \{x_1\}, \ldots, U_r \backslash \{x_r\}, 1 \leq r \leq k-1$. In this case no vertex of $\{x_0\} \times S^* \backslash \{y_0\}$ has a neighbor with its color in $\{2, 3, \ldots, 1 + (r \bmod k-1)\}$ (see box 2 of Figure 2).

In order to overcome this case we shall recolor some of the vertices in $\{x_0\} \times Y$ by using the fact that these colors are also present in box (4) of Figure 2. Recall that S^* is a star having center y_0 and with $y_1, \ldots, y_{l-1}$ forming an independent set in H. As the y_j's are c.d.v.'s in H, each y_j should have a neighbor in $V_s \backslash \{y_s\}$, for each $s = 1, \ldots, j-1, j+1, \ldots, l-1$. Call such a neighbor in $V_s \backslash \{y_s\}$ as y_{j_s}. As x_0 is adjacent to $x_1, \ldots, x_r$, the vertex (x_0, y_j) is adjacent to the vertices $(x_1, y_j), \ldots, (x_r, y_j)$ receiving the colors $k + [j \ (mod \ (l-1))], \ldots, k + [(j+r-1) \ (mod \ (l-1))]$, respectively. Also since the vertices (x_0, y_{j_s}) for $1 \leq j \leq l-1$ and $s = 1 + [j \ (mod \ (l-1))], \ldots, 1 + [(j+r) \ (mod \ (l-1))]$ form an independent set (by (ii) of Definition 2.2), by arbitrarily coloring these vertices by distinct colors from $\{2, 3, \ldots, 1 + [r \ (mod \ (k-1))]\}$, it is seen that the set of vertices $\{(x_0, y_j) : 1 \leq j \leq l-1\}$ forms c.d.v.'s of their corresponding color classes. ∎

Corollary 2.4
If $H \in \mathscr{F}$ and $b(G) < b(H)$, then $b(G \square H) \geq b(G) + b(H) - 1$. ∎

Let us now consider the odd graphs O_k, $k \geq 4$. In [1] and [9], it was shown that $b(O_k) = k+1$, O_k is b-continuous and that $O_k \in \mathscr{F}$, $k \geq 4$. We also know that $\chi(O_k) = 3$ [17]. We now show that the Cartesian products of odd graphs are b-continuous.

Theorem 2.5
If $G = O_{k_1}$ and $H = O_{k_2}$ are odd graphs, where $k_1, k_2 \geq 4$, then $b(O_{k_1} \square O_{k_2}) = k_1 + k_2 + 1$ and $O_{k_1} \square O_{k_2}$ is b-continuous.

Proof. As mentioned already, odd graphs belong to $\mathscr{F}$. Thus by Theorem 2.3, if O_{k_1} has a b-coloring using k colors and O_{k_2} has a b-coloring using l colors and if $3 \leq k < l$, then $G \square H$ has a b-coloring using $k + l - 1$ colors. We now consider the case when $k \geq l \geq 4$.

Claim. If O_{k_1} has a b-coloring using k colors and O_{k_2} has a b-coloring using l colors (where $4 \leq l \leq k$) colors, then $G \square H$ has a b-coloring using $k + l - 1$ colors.

Assume for the moment that the claim is true. Then if O_{k_1} has a b-coloring using k colors and O_{k_2} has a b-coloring using l colors and if $3 \leq k < l$ or if $k \geq l \geq 4$, then $O_{k_1} \square O_{k_2}$ has a b-coloring using $k + l - 1$ colors.

We know that $b(O_k) = k + 1$. Thus $O_{k_1} \square O_{k_2}$ has a b-coloring using $(k_1 + 1) + (k_2 + 1) - 1 = k_1 + k_2 + 1$ colors and hence $k_1 + k_2 + 1 \leq b(O_{k_1} \square O_{k_2}) \leq \Delta(O_{k_1} \square O_{k_2}) + 1 = \Delta(O_{k_1}) + \Delta(O_{k_2}) + 1 = k_1 + k_2 + 1$. Therefore $b(O_{k_1} \square O_{k_2}) = k_1 + k_2 + 1$.

Next let us prove that $O_{k_1} \square O_{k_2}$ is b-continuous. As odd graphs are b-continuous, $S_b(O_{k_1}) = \{3, 4, \ldots, k_1 + 1\}$, and $S_b(O_{k_2}) = \{3, 4, \ldots, k_2 + 1\}$ and therefore we see that $\{6, 7, \ldots, k_1 + k_2 + 1\} \subseteq S_b(O_{k_1} \square O_{k_2})$. We know that $\chi(O_{k_1}) = \chi(O_{k_2}) = 3$, $b(O_{k_1}) = k_1 + 1 \geq 4 + 1 = 5$ and $b(O_{k_2}) = k_2 + 1 \geq 4 + 1 = 5$. Hence, by Observation 2.1, $\{3, 4, 5\} \subseteq S_b(O_{k_1} \square O_{k_2})$. This proves that $O_{k_1} \square O_{k_2}$ is b-continuous.

Proof of the Claim.

Since O_{k_1} satisfies the conditions (i) and (ii) of Definition 2.2, there exists a c.d.s. $S = \{x_0, x_1, \ldots, x_{k-1}\}$ such that $\langle S \rangle$ is a star with center at x_1. Since $\chi(O_{k_1}) = 3$, $V(O_{k_1})$ is $S_0 \cup S_1 \cup S_2$ (union of the three color classes), where we may suppose that $x_0 \in S_0$ and $x_1 \in S_1$. We now assume the notations given in the proof of Theorem 2.3 and give the following coloring:

(1) For $x \in U_i$, $i = 0, 1, \ldots, k - 1$, set

$$c(x, y_0) = i. \text{ (See box (1) in Figure 3)}$$

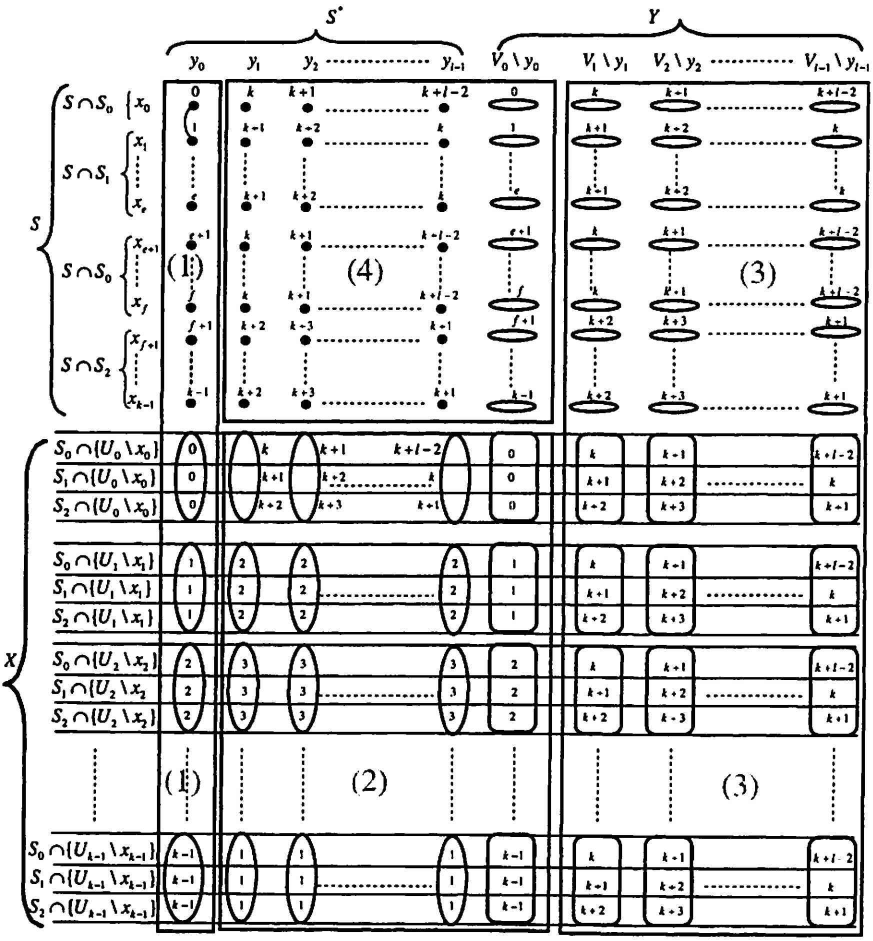

Figure 3: Coloring c given in the proof of Theorem 2.5

(2) Consider the vertices in $X \times (S^* \cup V_0 - \{y_0\})$. (See box (2) in Figure 3)

(i) For $x \in U_0 - \{x_0\}$ and $y \in (S^* \cup V_0 - \{y_0\})$, set

$$c(x,y) = \begin{cases} k + [(i+j-1)\ (mod\ (l-1))] & \text{if } x \in U_0 \cap S_i - \{x_0\},\ 0 \le i \le 2, \\ & y = y_j,\ 1 \le j \le l-1, \\ 0 & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

(ii) For $x \in X \backslash U_0,\ y \in (S^* \cup V_0 - \{y_0\})$, set

$$c(x,y) = \begin{cases} 1 + [i\ (mod\ (k-1))] & \text{if } x \in U_i, \text{ and } y \in S^* - \{y_0\}, \\ c(x, y_0) & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

(3) Next consider the vertices in $V(G) \times (Y \backslash V_0)$. (See box (3) in Figure 3)

For $x \in S_i,\ 0 \le i \le 2$ and $y \in V_j - \{y_j\},\ 1 \le j \le l-1$, set

$$c(x,y) = k + [(i+j-1)\ (mod\ (l-1))].$$

(4) Finally consider the vertices in $S \times (S^* \cup V_0) - \{y_0\})$ (See box (4) in Figure 3), set

$$c(x,y) = \begin{cases} k + [(i+j-1)\ (mod\ (l-1))] & \text{if } x \in S \cap S_i,\ 0 \le i \le 2,\ y = y_j, \\ & 1 \le j \le l-1 \\ c(x, y_0) & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

Checking that the coloring is proper is similar to what was given in the proof of Theorem 2.3. We note that x_0 is adjacent to x_1 in $\langle S \rangle$. If x_0 has a neighbor in $U_1 - \{x_1\}$, then we are done. Otherwise the vertex (x_0, y_j), for $1 \le j \le l-1$, has no neighbors in the color class 2. In order to overcome this, we recolor its neighbors in $\{x_0\} \times V(O_{k_2})$. This can be done as in the proof of Theorem 2.3. This gives the desired b-coloring for $O_{k_1} \square O_{k_2}$ using $k + l - 1$ colors. ∎

Note that the c.d.s. $\{\{x_0\} \times S^*\} \cup \{S \times \{y_0\}\}$ of $O_{k_1} \square O_{k_2}$ obtained in Theorem 2.5, contains a vertex of degree one in the induced subgraph of $\{\{x_0\} \times S^*\} \cup \{S \times \{y_0\}\}$.

Corollary 2.6

If O_{k_i}, $i = 1, 2, \ldots, n$ are odd graphs with $k_i \geq 4$ for each i, then $O_{k_1} \square O_{k_2} \square \ldots \square O_{k_n}$ is b-continuous and $b(O_{k_1} \square O_{k_2} \square \ldots \square O_{k_n}) = 1 + \sum\limits_{i=1}^{n} k_i$.

Proof. Proof is by induction on n. For $n = 2$, the result is true by Theorem 2.5. So assume that the result is true for all $j \leq n-1$ where $n \geq 3$. We now prove the result for n. Consider $O_{k_1} \square O_{k_2} \square \ldots \square O_{k_n} = (O_{k_1} \square O_{k_2} \square \ldots \square O_{k_{n-1}}) \square O_{k_n}$. By induction hypothesis $b(O_{k_1} \square O_{k_2} \square \ldots \square O_{k_{n-1}}) = 1 + \sum\limits_{i=1}^{n-1} k_i$ and $O_{k_1} \square O_{k_2} \square \ldots \square O_{k_{n-1}}$ is b-continuous. Note that by applying the technique used in Theorem 2.5 step by step to $O_{k_1} \square O_{k_2} \square \ldots \square O_{k_{n-1}}$, we can find a b-coloring using k colors (where $3 = \max\{\chi(O_{k_1}) \square \chi(O_{k_2}) \square \ldots \square \chi(O_{k_{n-1}})\} \leq k \leq b(O_{k_1} \square O_{k_2} \square \ldots \square O_{k_{n-1}})$) for which there is a c.d.s. S of $O_{k_1} \square O_{k_2} \square \ldots \square O_{k_{n-1}}$ which has a vertex of degree one in $\langle S \rangle$. Also as mentioned above $3 = \max\{\chi(O_{k_1}), \chi(O_{k_2}), \ldots, \chi(O_{k_{n-1}})\}$. Thus arguments similar to Theorem 2.5 can be used to prove that $(O_{k_1} \square O_{k_2} \square \ldots \square O_{k_{n-1}}) \square O_{k_n}$ is b-continuous and $b(O_{k_1} \square O_{k_2} \square \ldots \square O_{k_n}) = 1 + \sum\limits_{i=1}^{n} k_i$. ∎

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