Decomposition of λK_v into kites and 4-cycles

Selda Küçükçifçi *
Department of Mathematics
Koç University
Istanbul
Turkey
skucukcifci@ku.edu.tr

Salvatore Milici †
Dipartimento di Matematica e Informatica
Università di Catania
Catania
Italia
milici@dmi.unict.it

Abstract

Given a collection of graphs $\mathcal{H}$, an $\mathcal{H}$*−decomposition* of λK_v is a decomposition of the edges of λK_v into isomorphic copies of graphs in $\mathcal{H}$. A *kite* is a triangle with a tail consisting of a single edge. In this paper we investigate the decomposition problem when $\mathcal{H}$ is the set containing a kite and a 4-cycle, that is; this paper gives a complete solution to the problem of decomposing λK_v into r kites and s 4-cycles for every admissible values of v, λ, r and s.

AMS classification: 05B05.
Keywords: Graph decomposition; G-design; kite; 4-cycle.

1 Introduction and Definitions

Let G be a finite and simple graph. A G-design of order v and index λ is a pair (V, C) where V is the vertex set of λK_v (λ copies of the undirected complete graph on v vertices) and C is a collection of isomorphic copies of the graph G, called *blocks*, which partition the edges of λK_v. A *kite* is a triangle with a tail consisting of a single edge. A *kite system of order* v *and index* λ is a G-design of order v and index λ, where G is a kite. In what follows we will denote the kite, having vertices $\{a_1, a_2, a_3, a_4\}$ and edges $\{\{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2\}, \{a_3, a_4\}\}$ by $(a_1, a_2, a_3) - a_4$. It is well-known [12] and [16] that a kite system of order v and index λ exists if

*Research supported by Scientific and Technological Research Council of Turkey Grant Number: 110T692

†Research supported by MIUR and by C. N. R. (G. N. S. A. G. A.), Italy

and only if $\lambda v(v-1) \equiv 0 \pmod 8$. A *4-cycle system of order v and index λ* is a G-design of order v and index λ, where G is a 4-cycle. It is well-known [15] that a 4-cycle system of order v and index λ exists if and only if $\lambda(v-1)$ is even and $\lambda v(v-1) \equiv 0 \pmod 8$. In what follows we will denote the 4-cycle C_4 having vertices $\{a_1, a_2, a_3, a_4\}$ and edges $\{\{a_1, a_2\}, \{a_2, a_3\}, \{\{a_3, a_4\}, \{a_1, a_4\}\}$ by (a_1, a_2, a_3, a_4).

Given a collection of graphs $\mathcal{H}$, an *$\mathcal{H}$−decomposition* of λK_v is a decomposition of the edges of λK_v into isomorphic copies of graphs in $\mathcal{H}$. The copies of $H \in \mathcal{H}$ in the decomposition are called *blocks*. The case when $\mathcal{H}$ is a set of cycles of length at most v is the well-known Alspach's conjecture [4]. For $\lambda = 1$ there are several results for a decomposition of K_v into cycles of more than one length (see [3], [5], [6], [8], [9], [11] and [13]). The Alspach's conjecture has been completely settled in the case when all components of $\mathcal{H}$ are cycles of the same length (see [6] and [17]); a solution has been shown to exist in all such cases. The case when all components of $\mathcal{H}$ are stars, has been studied by Lin and Shyu [14] for $\lambda = 1$. Recently many authors have settled decompositions of λK_v into stars S_k and cycles C_h in the case $\lambda \geq 1$ and $h = k - 1 = 3, 4, 5$ and in the case $\lambda = 1$ and $h = k = 4$ (see [1], [2], [19]).

Let $J(v, \lambda)$ denote the set of all pairs (r, s) such that there exists a decomposition of λK_v into r copies of kites and s copies of 4-cycles. Given $v \geq 4$ and $\lambda \geq 1$, let $s_v = \frac{\lambda v(v-1)}{8}$ and define $I(v, \lambda)$ according to the following table

Table 1		
$\lambda \pmod 4$	v	$I(v,\lambda)$
$\lambda = 1$	$1 \pmod 8$	$\{(s_v - x, x), x = 0, 1, ..., s_v - 3, s_v\}$
1,3	$0 \pmod 8$	$\{(s_v - x, x), x = 0, 1, ..., s_v - \frac{v}{2}\}$
$1, 3,\ \lambda > 1$	$1 \pmod 8$	$\{(s_v - x, x), x = 0, 1, ..., s_v - 2, s_v\}$
2	$0, 1 \pmod 4$	$\{(s_v - x, x), x = 0, 1, ..., s_v - 2, s_v\}$
$0,\ \lambda \geq 4$	$v \geq 4$	$\{(s_v - x, x), x = 0, 1, ..., s_v - 2, s_v\}$

In this paper we investigate the decomposition of λK_v into kites and 4-cycles. In particular, we will prove the following result:

Main Theorem. *Let $v \geq 4$ and $\lambda \geq 1$ be positive integers satisfying the condition given in Table* 1. *Then $J(v, \lambda) = I(v, \lambda)$.*

2 Preliminaries and necessary conditions

In this section we will give necessary conditions for the existence of a decomposition of λK_v into kites and 4-cycles.

Lemma 2.1. *Let v and λ be positive integers. If $(r,s) \in J(v,\lambda)$, then*
(1) *when* $\lambda \equiv 1,3 \pmod 4$, $v \equiv 0,1 \pmod 8$;
(2) *when* $\lambda \equiv 2 \pmod 4$, $v \equiv 0,1 \pmod 4$;
(3) *when* $\lambda \equiv 0 \pmod 4$, $v \geq 4$.

Proof. Let $(r,s) \in J(v,\lambda)$. Since r and s are non negative integers such that $4r + 4s = \lambda\frac{v(v-1)}{2}$, we easily obtain the necessary conditions. □

Lemma 2.2. *Let $v \equiv 1 \pmod 8$. If $(r,s) \in J(v,1)$ then $(r,s) \in I(v,1)$.*

Proof. Let D be a decomposition of K_v into r copies of kites and s copies of 4-cycles. Assume $r = 1$. Let $K^1 = (x_1,x_2,x_3) - x_4$ be the unique kite in D. Since x_3 has even degree in K_v and incident with 3 edges in K^1, $K_v - E(K^1)$ can not be decomposed into 4-cycles. Now assume $r = 2$. Let $K^1 = (x_1,x_2,x_3) - x_4$ and $K^2 = (y_1,y_2,y_3) - y_4$ denote those two copies of kites in D. Since $x_i, i = 3,4$ has even degree in K_v and is incident with odd number of edges in K^1, we obtain $x_3 = y_3$ and $x_4 = y_4$ or $x_3 = y_4$ and $x_4 = y_3$. But then the edge (x_3,x_4) appears twice in K_v, which is impossible. □

Lemma 2.3. *Let $v \equiv 1 \pmod 8$ and λ be an odd integer with $\lambda > 1$. If $(r,s) \in J(v,\lambda)$ then $(r,s) \in I(v,\lambda)$.*

Proof. Let D be a decomposition of λK_v into r copies of kites and s copies of 4-cycles. Assume $r = 1$. Let $K^1 = (x_1,x_2,x_3) - x_4$ be the unique kite in D. Since x_3 has even degree in λK_v and is incident with 3 edges in K^1, $\lambda K_v - K^1$ can not be decomposed into 4-cycles. □

Lemma 2.4. *Let $v \equiv 0 \pmod 8$ and λ be an odd integer. If $(r,s) \in J(v,\lambda)$ then $(r,s) \in I(v,\lambda)$.*

Proof. Let D be a decomposition of λK_v into r copies of kites and s copies of 4-cycles. Let $K^1, K^2, ..., K^r$ denote those r copies of kites. Since λ and $v-1$ are odd, each vertex of λK_v must occur in some K^i as a vertex with odd degree. Then necessarily we have $r \geq \frac{v}{2}$. □

Lemma 2.5. *Let $v \equiv 0,1 \pmod 4$ and $\lambda \equiv 2 \pmod 4$ or $v \geq 4$ and $\lambda \equiv 0 \pmod 4$. If $(r,s) \in J(v,\lambda)$ then $(r,s) \in I(v,\lambda)$.*

Proof. Let D be a decomposition of λK_v into r copies of kites and s copies of 4-cycles. Assume $r = 1$. Let $K^1 = (x_1,x_2,x_3) - x_4$ be the unique kite in D. Since x_3 has even degree in λK_v and is incident with 3 edges in K^1, $\lambda K_v - K^1$ can not be decomposed into 4-cycles. □

Combining Lemmas 2.1, 2.2, 2.3, 2.4 and 2.5 we obtain the following result.

Lemma 2.6. *Let v and λ be positive integers such that there exists a decomposition of λK_v into kites and 4-cycles. Then $J(v,\lambda) \subseteq I(v,\lambda)$.*

3 Constructions and related structures

In this section we will introduce some useful definitions and results and discuss some constructions we will use in proving the main theorem. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [10] and its online updates.

A 3−GDD is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a finite set of vertices, $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is a partition of V into subsets, called *groups* and $\mathcal{B}$ is a collection of isomorphic copies of K_3, called *blocks*, which partitions the edges of $K_{g_1,g_2,\ldots,g_n}$ on V, where $|G_i| = g_i$. If for $i = 1, 2, \ldots, t$, there are u_i groups of size g_i, we say that the 3−GDD is of type $g_1^{u_1} g_2^{u_2} \ldots g_t^{u_t}$.

We recall the existence of some 3−GDDs we will need in the paper.

Lemma 3.1. *[10] There exists a 3-GDD of type*

- 2^t, 4^t *for each* $t \equiv 0, 1 \pmod 3$, $t \geq 3$ *(see page 255, Theorem 4.1 in [10]);*
- $4^1 2^{3t}$, $8^1 4^{3t}$ *for each* $t \geq 1$ *(see page 255, Theorem 4.2 in [10]).*

The following result will be used to partition the edges of $\lambda K_{v,u}$ into 4-cycles.

Theorem 3.2. *[7] There exists a decomposition of $\lambda K_{v,u}$ into 4-cycles if and only if*

(1) $v, u \geq 2$; (2) $\lambda vu \equiv 0 \pmod 4$; (3) $\lambda v \equiv \lambda u \equiv 0 \pmod 2$.

We need the following definition. Let (s_1, t_1) and (s_2, t_2) be two pairs of non-negative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If X and Y are two sets of pairs of non negative integers, then $X + Y$ denotes the set $\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}$. If X is a set of pairs of non negative integers and h is a positive integer, then $h * X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

Lemma 3.3. *There exists a decomposition of $\lambda K_{2,2,2}$ into r kites and s 4-cycles for all $(r, s) \in \{(3(\lambda - x), 3x), x = 0, 1, \ldots, \lambda\}$.*

Proof. Let A be a decomposition of $K_{2,2,2}$ into $3 - 3x$ kites and $3x$ 4-cycles for $x = 0, 1$ (see step 1 in Appendix). Repeat A λ-times. The result is a decomposition of $\lambda K_{2,2,2}$ into r kites and s 4-cycles for each $(r, s) \in \lambda * \{(3, 0), (0, 3)\}$. □

The next lemma is an immediate consequence of the definition of $I(v, \lambda)$.

Lemma 3.4. *Let v and λ be positive integers. Then*
(1) *when $\lambda = 1 + 2t$, $t \geq 1$, $I(v, \lambda) = I(v, 1) + t * I(v, 2)$;*
(2) *when $v \equiv 0, 1 \pmod 4$, $I(v, 2t) = t * I(v, 2)$;*
(3) *when $v \geq 4$, $I(v, 4t) = t * I(v, 4)$.*

Lemma 3.5. *Let v and λ be positive integers. If $I(v, i) = J(v, i)$ for each $i = 1, 2, 3, 4$, then $I(v, \lambda) \subseteq J(v, \lambda)$ for each $\lambda > 4$.*

Proof. Suppose at first $\lambda = 1 + 2t$, $t \geq 0$. Let $(r, s) \in I(v, \lambda)$. Thus r and s are nonnegative integers with $r + s = \lambda\frac{v(v-1)}{8}$. By Lemma 3.4, we have $I(v, \lambda) = I(v, 1) + t * I(v, 2)$. Then there exist two pairs of non negative integers $(r_1, s_1) \in I(v, 1)$ and $(r_2, s_2) \in t * I(v, 2)$ such that $(r, s) = (r_1, s_1) + (r_2, s_2)$. Since $I(v, i) = J(v, i)$, for $i = 1, 2$, we have $(r_1, s_1) \in J(v, 1)$ and $(r_2, s_2) \in t * J(v, 2)$. Then there exists a decomposition of λK_v into $r = r_1 + r_2$ kites and $s = s_1 + s_2$ 4-cycles. Hence $(r, s) \in J(v, \lambda)$. For $\lambda \equiv 0, 2 \pmod 4$ we proceed as above. This completes the proof. □

Lemma 3.6. *Let v, u, w and λ be nonnegative integers. If*
(1) *$(r_1, s_1) \in J(u + w, \lambda)$;*
(2) *there exists a decomposition of $\lambda K_{v+w} - \lambda K_w$ into r_2 copies of kites and s_2 copies of 4-cycles and*
(3) *there exists a decomposition of $\lambda K_{v,u}$ into $\lambda\frac{vu}{4}$ 4-cycles,*
then $(r_1 + r_2, s_1 + s_2 + \lambda\frac{vu}{4}) \in J(v + u + w, \lambda)$.

Proof. Construct a decomposition of λK_{u+w} into r_1 copies of kites and s_1 copies of 4-cycles. Construct a decomposition of $\lambda K_{v+w} - \lambda K_w$ into r_2 copies of kites and s_2 copies of 4-cycles and finally construct a decomposition of $\lambda K_{v,u}$ into $\lambda\frac{vu}{4}$ 4-cycles. Then the result is a decomposition of λK_{v+u+w} into $r_1 + r_2$ kites and $s_1 + s_2 + \lambda\frac{vu}{4}$ 4-cycles. □

Let V and W be two sets with $|V| = v$, $|W| = w$ and $W \subseteq V$. Let $J(v, w, \lambda)$ denote the set of all pairs (r, s) such that there exists a decomposition of $\lambda K_v - \lambda K_w$ into r kites and s 4-cycles.

Lemma 3.7. *Let v, g, t, u and w be non-negative integers such that $v = 2gt + 2u + w$. If*
(1) *there exists a $3-GDD$ of type u^1g^t;*
(2) *a decomposition of λK_{2u+w} into r_1 kites and s_1 4-cycles;*
(3) *a decomposition of $\lambda K_{2g+w} - \lambda K_w$ into r_2 kites and s_2 4-cycles;*
(4) *a decomposition of $\lambda K_{2,2,2}$ into r_3 kites and s_3 4-cycles with $(r_3, s_3) \in \lambda * \{(3, 0), (0, 3)\}$,*
then there exists a decomposition of $\lambda K_{2gt+2u+w}$ into r kites and s 4-cycles such that $(r, s) \in J(2u + w, \lambda) + t * J(2g + w, w, \lambda) + h * \{(3(\lambda - i), 3i), i =$

$0, 1, ..., \lambda\}$, *where* $h = \frac{g^2t(t-1)}{6} + \frac{gtu}{3}$ *is the number of blocks of the* 3*−GDD of type* u^1g^t.

Proof. Let $(X, \{G, G_1, \ldots, G_t\}, \mathcal{B})$ be a 3−GDD of type u^1g^t where G is the group of size u and $G_1, \ldots, G_t$ are the t groups of size g. Let W be a set of size w such that $X \cap W = \varnothing$. On each block $B \in \mathcal{B}$ place a decomposition of $\lambda K_{2,2,2}$ into r_3 kites and s_3 4-cycles with $(r_3, s_3) \in \lambda * \{(3,0),(0,3)\}$. Place a decomposition of λK_{2u+w} into r_1 kites and s_1 4-cycles on $(G \times \{0,1\}) \cup W$. For each $i = 1, 2, ..., t$, place a decomposition of $\lambda K_{2g+w} - \lambda K_w$ into r_2 kites and s_2 4-cycles on $G_i \times \{0,1\}$. The result is a decomposition of $\lambda K_{2gt+2u+w}$ on $V = (X \times \{0,1\}) \cup W$ into r kites and s 4-cycles with $(r, s) \in J(u+w, \lambda) + t * J(2g+w, w, \lambda) + h * \{(3(\lambda - i), 3i), i = 0, 1, ..., \lambda\}$, where $h = \frac{g^2t(t-1)}{6} + \frac{gtu}{3}$ is the number of blocks of the 3−GDD of type u^1g^t.

□

Lemma 3.8. *Let* v, g, t *and* w *be non-negative integers such that* $v = 2gt + w$. *If*

(1) *there exists a* 3*−GDD of type* g^t;

(2) *a decomposition of* λK_{2g+w} *into* r_1 *kites and* s_1 *4-cycles;*

(3) *a decomposition of* $\lambda K_{2g+w} - \lambda K_w$ *into* r_2 *kites and* s_2 *4-cycles;*

(4) *a decomposition of* $\lambda K_{2,2,2}$ *into* r_3 *kites and* s_3 *4-cycles with* $(r_3, s_3) \in \lambda * \{(3,0),(0,3)\}$,

then there exists a decomposition of λK_{2gt+w} *into* r *kites and* s *4-cycles such that* $(r, s) \in J(2g+w, \lambda) + (t-1) * J(2g+w, w, \lambda) + h * \{(3(\lambda - i), 3i), i = 0, 1, ..., \lambda\}$, *where* $h = \frac{g^2t(t-1)}{6}$ *is the number of blocks of the* 3*−GDD of type* g^t.

Proof. Let $(X, \{G_1, G_2, \ldots, G_t\}, \mathcal{B})$ be a 3−GDD of type g^t, where G_is are the groups of size g. Let W be a set of size w such that $X \cap W = \varnothing$. On each block $B \in \mathcal{B}$ place a decomposition of $\lambda K_{2,2,2}$ into r_3 kites and s_3 4-cycles with $(r_3, s_3) \in \lambda * \{(3,0),(0,3)\}$. Place a decomposition of λK_{2g+w} into r_1 kites and s_1 4-cycles on $(G_1 \times \{0,1\}) \cup W$. For each $i = 2, ..., t$ place a decomposition of $\lambda K_{2g+w} - \lambda K_w$ into r_2 kites and s_2 4-cycles on $G_i \times \{0,1\}$. The result is a decomposition of λK_{2gt+w} into r kites and s 4-cycles on $V = (X \times \{0,1\}) \cup W$, where $(r, s) \in J(2g+w, \lambda) + (t-1) * J(2g+w, w, \lambda) + h * \{(3(\lambda - i), 3i), i = 0, 1, ..., \lambda\}$ where $h = \frac{g^2t(t-1)}{6}$ is the number of blocks of the 3−GDD of type g^t.

□

4 Proof of Main Theorem

By Lemma 2.6 $J(v,\lambda) \subseteq I(v,\lambda)$, $\lambda \geq 1$. We need to show that $I(v,\lambda) \subseteq J(v,\lambda)$. Thanks to the result in Lemma 3.5, it suffices to prove that $I(v,i) \subseteq J(v,i)$ for each $i = 1,2,3,4$. We consider each of these cases in the subsections below.

4.1 $\lambda = 1$

Lemma 4.1. *For any $v \equiv 0,1,8,9 \pmod{24}$, $I(v,1) = J(v,1)$.*

Proof. For $v = 8$ and 9, see steps 10 and 11 of the Appendix. Let $v = 8t+w$, $t \equiv 0,1 \pmod 3$, $t \geq 3$ and $w \in \{0,1\}$. Start with a 3–GDD of type 4^t. Give weight 2 to every point of the 3–GDD. Applying Lemma 3.8 with $g = 4$ and $w \in \{0,1\}$, we obtain a decomposition of K_v into r kites and s 4-cycles with $(r,s) \in h*\{(3,0),(0,3)\}+t*J(8+w,1)$, where $h = \frac{8t(t-1)}{3}$ is the number of blocks of the 3-GDD of type 4^t. Since $|t*J(8+w,1)| \geq 3$ and $h*\{(3,0),(0,3)\}$=$\{(8t(t-1)-3i,3i), i=0,1,..., \frac{8t(t-1)}{3}\}$, it is easy to see that $I(8t+w,1) = h*\{(3,0),(0,3)\}+t*J(8+w,1)$. Since $h*\{(3,0),(0,3)\}+t*J(8+w,1) \subseteq J(8t+w,1)$, it follows that $I(v,1) \subseteq J(v,1)$. □

Lemma 4.2. *For any $v \equiv 16,17 \pmod{24}$, then $I(v,1) = J(v,1)$.*

Proof. For $v = 16$ and 17, see steps 12 and 13 of the Appendix. Let $v = 8t+w$, $t \equiv 2 \pmod 3$, $t \geq 5$ and $w \in \{0,1\}$. Start with a 3–GDD of type $8^1 4^{t-2}$. Give weight 2 to every point of the 3–GDD. Applying Lemma 3.7 with $g = 4$, $u = 8$ and $w \in \{0,1\}$, we obtain a decomposition of K_v into r kites and s 4-cycles with $(r,s) \in (t-2)*J(8+w,1)+J(16+w,1)+h*\{(3,0),(0,3)\}$, where $h = \frac{8(t-2)(t+1)}{3}$ is the number of blocks of the 3–GDD of type $8^1 4^{t-2}$. Since $|(t-2)*J(8+w,1)+J(16+w,1)| \geq 3$ and $h*\{(3,0),(0,3)\}$=$\{(8(t-2)(t+1)-3i,3i), i=0,1,..., \frac{8(t-2)(t+1)}{3}\}$, it is easy to see that $I(8t+w,1) = (t-2)*J(8+w,1)+J(16+w,1)+h*\{(3,0),(0,3)\}$. Since $(t-2)*J(8+w,1)+J(16+w,1)+h*\{(3,0),(0,3)\} \subseteq J(8t+w,1)$, it follows that $I(v,1) \subseteq J(v,1)$. □

4.2 $\lambda = 2$

Lemma 4.3. *For any $v \equiv 0,1,4,5 \pmod{12}$, $I(v,2) = J(v,2)$.*

Proof. For $v = 4$ and 5, see steps 15 and 17 of the Appendix. Let $v = 4t+w$, $t \equiv 0,1 \pmod 3$, $t \geq 3$ and $w \in \{0,1\}$. Start with a 3–GDD of type 2^t. Give weight 2 to every point of the 3–GDD. Applying Lemma 3.8 with $g = 2$ and $w \in \{0,1\}$, we obtain a decomposition of $2K_v$ into r kites and s 4-cycles with $(r,s) \in h*\{(6,0),(3,3),(0,6)\}+t*J(4+w,2)$, where $h = \frac{2t(t-1)}{3}$

is the number of blocks of the 3−GDD of type 2^t. Since $|t * J(4+w,2)| \geq 3$ and $h * \{(6,0),\ (3,3),\ (0,6)\} = \{(4t(t-1)-3i, 3i),\ i = 0,1,\ldots,\frac{4t(t-1)}{3}\}$ it is easy to see that $I(4t+w,2) = h * \{(6,0),(3,3),(0,6)\} + t * J(4+w,2)$. Since $h * \{(6,0),\ (3,3),\ (0,6)\} + t * J(4+w,2) \subseteq J(4t+w,2)$, it follows that $I(v,2) \subseteq J(v,2)$. □

Lemma 4.4. *For any $v \equiv 8,9 \pmod{12}$, then $I(v,2) = J(v,2)$.*

Proof. For $v = 8$ and 9 see steps 18 and 19 of the Appendix. Let $v = 4t+w$, $t \equiv 2 \pmod 3$, $t \geq 5$ and $w \in \{0,1\}$. Start with a 3−GDD of type $4^1 2^{t-2}$. Give weight 2 to every point of the 3−GDD. Applying Lemma 3.7 with $g = 2$, $u = 4$ and $w \in \{0,1\}$, we obtain a decomposition of $2K_v$ into r kites and s 4-cycles with $(r,s) \in (t-2) * J(4+w,2) + J(8+w,2) + h * \{(6,0),(3,3),(0,6)\}$, where $h = \frac{2(t-2)(t+1)}{3}$ is the number of blocks of the 3−GDD of type $4^1 2^{t-2}$. Since $|(t-2) * J(4+w,2) + J(8+w,2)| \geq 3$ and $h * \{(6,0),(3,3),(0,6)\} = \{(2(t-2)(t+1)-3i, 3i), i = 0,1,\ldots,\frac{2(t-2)(t+1)}{3}\}$ it is easy to see that $I(4t+w,2) = (t-2) * J(4+w,2) + J(8+w,2) + h * \{(6,0),(3,3),(0,6)\}$. Since $(t-2) * J(4+w,2) + J(8+w,2) + h * \{(6,0),(3,3),(0,6)\} \subseteq J(4t+w,2)$, it follows that $I(v,2) \subseteq J(v,2)$. □

4.3 $\lambda = 3$

Lemma 4.5. *For any $v \equiv 0,1 \pmod 8$, $I(v,3) = J(v,3)$.*

Proof. Lemma 3.4 and the results in Sections 4.1 and 4.2 give $I(v,3) = I(v,1) + I(v,2) = J(v,1) + J(v,2)$. Since it is easy to see that $J(v,1) + J(v,2) \subseteq J(v,3)$, it follows that $I(v,3) \subseteq J(v,3)$. □

4.4 $\lambda = 4$

Lemma 4.6. *For any $v \equiv 0,1 \pmod 4$, $I(v,4) = J(v,4)$.*

Proof. Lemma 3.4 and the results of Section 4.2 gives $I(v,4) = 2 * I(v,2) = 2 * J(v,2)$. Since it is easy to see that $2 * J(v,2) \subseteq J(v,4)$, we have $I(v,4) \subseteq J(v,4)$. □

Lemma 4.7. *For any $v \equiv 2,3,6,7 \pmod{12}$, $I(v,4) = J(v,4)$.*

Proof. For $v = 6$ and 7, see steps 20 and 22 of the Appendix. Let $v = 4t+w$, $t \equiv 0,1 \pmod 3$, $t \geq 3$ and $w \in \{2,3\}$. Start with a 3−GDD of type 2^t. Give weight 2 to every point of the 3−GDD. Applying Lemma 3.8 with $g = 2$ and $w \in \{2,3\}$, we obtain a decomposition of $4K_v$ into r kites and s 4-cycles with $(r,s) \in h * \{(12,0),(9,3),(6,6),(3,9),(0,12)\} + J(4+w,4) + (t-1) * J(4+w,w,4)$, where $h = \frac{2t(t-1)}{3}$ is the number of blocks of the 3−GDD of type 2^t (see step 14 of the Appendix for $J(6,2,4)$). Since $|J(4+w,4) + (t-$

$1) * J(4+w,w,4)| \geq 3$ and $h * \{(12,0),(9,3),(6,6),(3,9),(0,12)\} = \{(8t(t-1)-3i,3i), i = 0,1,..., \frac{2t(t-1)}{3}\}$ it is easy to see that $I(4t+w,4) = h*\{(12,0), (9,3), (6,6), (3,9), (0,12)\} + J(4+w,4) + (t-1) * J(4+w,w,4)$. Since $h*\{(12,0), (9,3), (6,6), (3,9), (0,12)\} + J(4+w,4) + (t-1) * J(4+w,w,4) \subseteq J(v,4)$, it follows that $I(v,4) \subseteq J(v,4)$. □

Lemma 4.8. *For any* $v \equiv 10, 11 \pmod{12}$, *then* $I(v,4) = J(v,4)$.

Proof. For $v = 10$ and 11, see steps 23 and 24 of the Appendix. Let $v = 4t+w$, $t \equiv 2 \pmod 3$, $t \geq 5$ and $w \in \{2,3\}$. Start with an 3-GDD of type $4^1 2^{t-2}$. Give weight 2 to every point of the 3-GDD. Applying Lemma 3.7 with $g = 2$, $u = 4$ and $w \in \{2,3\}$, we obtain a decomposition of $4K_v$ into r kites and s 4-cycles with $(r,s) \in (t-2) * J(4+w,w,4) + J(8+w,4) + h * \{(12,0),(9,3),(6,6),(3,9),(0,12)\}$, where $h = \frac{2(t-2)(t+1)}{3}$ is the number of blocks of the 3-GDD of type $4^1 2^{t-2}$ (see step 16 of the Appendix for $J(7,3,4)$). Now because $|(t-2) * J(4+w,w,4) + J(8+w,4)| \geq 3$ and $h * \{(12,0), (9,3), (6,6), (3,9), (0,12)\} = \{(2(t-2)(t+1) - 3i, 3i), i = 0,1,..., \frac{2(t-2)(t+1)}{3}\}$ it is easy to verify that $I(4t+w,4) = (t-2) * J(4+w,w,4) + J(8+w,4) + h*\{(12,0),(9,3),(6,6),(3,9),(0,12)\}$. Since $(t-2)* J(4+w,w,4) + J(8+w,4) + h*\{(12,0),(9,3),(6,6),(3,9),(0,12)\} \subseteq J(v,4)$, it follows that $I(v,4) \subseteq J(v,4)$.

□

5 Appendix

1. A decomposition of $K_{2,2,2}$ into $3-x$ kites and x 4-cycles, $x = 0, 3$ having the vertex set $V(K_{2,2,2}) = \{a,b\} \cup \{1,2\} \cup \{x,y\}$.

   - $(0,3)$:
     $\{(1,x,2,y),(a,x,b,y),(a,1,b,2)\}$.
   - $(3,0)$:
     $\{(y,2,a)-1,(b,2,x)-a,(b,y,1)-x\}$.

2. A decomposition of $K_{10} - K_2$ into 11 kites having the vertex set $\{0,1,...,7\} \cup \{a,b\}$ with hole on $\{a,b\}$.

   - $(11,0) \in J(10,2,1)$:
     $\{(a,3,0)-5,\ (a,5,2)-4,\ (a,7,1)-4,\ (a,6,4)-3,\ (b,4,0)-6,\ (b,5,1)-6,\ (b,2,6)-3,\ (b,7,3)-5,\ (2,3,1)-0,\ (6,7,5)-4,\ (2,0,7)-4\}$.

3. A decomposition of $K_{11} - K_3$ into 13 kites having the vertex set $\{0,1,...,7\} \cup \{a,b,\infty\}$ with hole on $\{a,b,\infty\}$.

   - $(13,0) \in J(11,3,1)$:
     $\{(a,3,0)-5,\ (a,5,2)-\infty,\ (a,7,1)-4,\ (a,6,4)-\infty,\ (b,4,0)-\infty,\ (b,5,1)-\infty,\ (b,2,6)-\infty,\ (b,3,7)-\infty,\ (2,1,3)-\infty,\ (6,7,5)-\infty,\ (2,0,7)-4,\ (1,0,6)-3,\ (3,5,4)-2\}$.

4. A decomposition of $K_{12} - K_4$ into 15 kites having the vertex set $\{0,1,...,7\} \cup \{a,b,c,d\}$ with hole on $\{a,b,c,d\}$.

   - $(15,0) \in J(12,4,1)$:
     $\{(a,1,0)-7,\ (a,3,2)-1,\ (a,5,4)-3,\ (a,6,7)-3,\ (b,4,6)-3,\ (b,3,1)-4,\ (b,2,0)-3,\ (b,7,5)-2,\ (c,6,0)-4,\ (c,7,1)-5,\ (c,4,2)-6,\ (c,3,5)-6,\ (5,0,d)-2,\ (1,6,d)-3,\ (d,4,7)-2\}$.

5. A decomposition of $K_{13} - K_5$ into 17 kites having the vertex set $\{0,1,...,7\} \cup \{a,b,c,d,\infty\}$ with hole on $\{a,b,c,d,\infty\}$.

   - $(17,0) \in J(13,5,1)$:
     $\{(a,0,1)-\infty,\ (a,3,2)-\infty,\ (a,5,4)-3,\ (a,7,6)-\infty,\ (b,4,6)-3,\ (b,1,3)-\infty,\ (b,2,0)-\infty,\ (b,7,5)-\infty,\ (c,6,0)-4,\ (c,1,7)-\infty,\ (c,4,2)-6,\ (c,3,5)-6,\ (5,0,d)-2,\ (1,6,d)-3,\ (d,7,4)-\infty,\ (0,3,7)-2,\ (2,5,1)-4\}$.

6. A decomposition of $K_{14} - K_6$ into 19 kites having the vertex set $\{0,1,...,7\} \cup \{a,b,c,d,e,f\}$ with hole on $\{a,b,c,d,e,f\}$.

- $(19,0) \in J(14,6,1)$:
  $\{(a,1,0)-e,\ (a,3,2)-d,\ (a,5,4)-e,\ (a,7,6)-d,\ (b,4,3)-6,$
  $(b,0,7)-4,\ (b,2,1)-6,\ (b,6,5)-0,\ (c,2,0)-3,\ (c,3,1)-7,$
  $(c,6,4)-2,\ (c,7,5)-3,\ (4,0,d)-3,\ (1,5,d)-7,\ (2,6,e)-1,$
  $(3,7,e)-5,\ (0,6,f)-3,\ (1,4,f)-7,\ (f,5,2)-7\}.$

7. A decomposition of $K_{15}-K_7$ into 21 kites having the vertex set $\{0,1,...,7\} \cup \{a,b,c,d,e,f,\infty\}$ with hole on $\{a,b,c,d,e,f,\infty\}$.

   - $(19,0) \in J(15,7,1)$:
     $\{(a,1,0)-e,\ (a,3,2)-d,\ (a,5,4)-e,\ (a,7,6)-d,\ (b,4,3)-\infty,$
     $(b,0,7)-\infty,\ (b,2,1)-6,\ (b,5,6)-\infty,\ (c,2,0)-\infty,\ (c,3,1)-\infty,$
     $(c,6,4)-\infty,\ (c,7,5)-\infty,\ (4,0,d)-3,\ (1,5,d)-7,\ (2,6,e)-1,$
     $(3,7,e)-5,\ (0,6,f)-3,\ (1,4,f)-7,\ (f,5,2)-\infty,\ (4,2,7)-1,$
     $(5,0,3)-6\}.$

8. A decomposition of $K_{16}-K_8$ into 23 kites having the vertex set $\{0,1,...,7\} \cup \{a,b,c,d,e,f,g,h\}$ with hole on $\{a,b,c,d,e,f,g,h\}$.

   - $(23,0) \in J(16,8,1)$:
     $\{(a,1,0)-g,\ (a,7,5)-h,\ (a,3,6)-c,\ (b,1,2)-a,\ (b,6,0)-e,$
     $(b,4,7)-e,\ (c,3,2)-g,\ (c,4,0)-5,\ (c,1,7)-d,\ (d,4,3)-b,$
     $(d,2,0)-f,\ (d,1,5)-2,\ (e,4,5)-c,\ (e,3,1)-f,\ (6,e,2)-7,$
     $(6,f,5)-b,\ (2,f,4)-a,\ (f,7,3)-0,\ (g,7,6)-d,\ (g,5,3)-h,$
     $(g,4,1)-6,\ (7,0,h)-1,\ (4,6,h)-2\}.$

9. A decomposition of $K_{17}-K_9$ into 25 kites having the vertex set $\{0,1,...,7\}\cup\{a,b,c,d,e,f,g,h,\infty\}$ with hole on $\{a,b,c,d,e,f,g,h,\infty\}$.

   - $(23,0) \in J(17,9,1)$:
     $\{(a,1,0)-g,\ (a,7,5)-h,\ (a,3,6)-c,\ (b,1,2)-a,\ (b,6,0)-e,$
     $(b,4,7)-e,\ (c,3,2)-g,\ (c,0,4)-\infty,\ (c,1,7)-d,\ (d,4,3)-b,$
     $(d,2,0)-f,\ (d,5,1)-\infty,\ (e,4,5)-c,\ (e,3,1)-f,\ (e,2,6)-\infty,$
     $(6,f,5)-b,\ (2,f,4)-a,\ (f,3,7)-\infty,\ (g,7,6)-d,\ (g,5,3)-h,$
     $(g,4,1)-6,\ (7,0,h)-1,\ (4,6,h)-2,\ (5,\infty,2)-7,\ (3,\infty,0)-5\}.$

10. $J(8,1)=I(8,1)$.
    Let $\{1,...,7\} \cup \{\infty\}$ be the vertex set. Then

    - $(7,0) \in J(8,1)$:
      Develop the base block $(1,3,0)-\infty$ in (mod 7).
    - $(6,1) \in J(8,1)$:
      $\{(6,1,3)-7,\ (2,6,7)-\infty,\ (3,\infty,2)-1,\ (1,\infty,4)-7,\ (1,7,5)-\infty,$
      $(5,4,6)-\infty,\ (2,4,3,5)\}.$

- $(5,2) \in J(8,1)$:
  $\{(5,\infty,6)-4,\ (5,4,2)-\infty,\ (6,7,3)-\infty,\ (4,\infty,7)-5,\ (2,3,1)-\infty,\ (2,7,1,6),\ (3,5,1,4)\}$.
- $(4,3) \in J(8,1)$:
  $\{(3,2,5)-1,\ (4,3,6)-2,\ (4,1,7)-3,\ (1,2,\infty)-4,\ (5,6,7,\infty),\ (2,4,5,7),\ (1,3,\infty,6)\}$.

11. $J(9,1)$=$I(9,1)$.
    Let $\{0,1,...,8\}$ be the vertex set. Then

    - $(9,0) \in J(9,1)$:
      Develop the base block $(1,3,0)-4$ in (mod 9).
    - $(8,1) \in J(9,1)$:
      $\{(6,0,5)-4,\ (1,7,6)-3,\ (0,8,7)-2,\ (2,5,8)-1,\ (3,0,1)-5,\ (4,0,2)-6,\ (7,5,3)-8,\ (8,6,4)-7,\ (1,2,3,4)\}$.
    - $(7,2) \in J(9,1)$:
      $\{(7,8,2)-1,\ (3,4,0)-2,\ (8,0,1)-7,\ (4,5,7)-0,\ (6,0,5)-8,\ (4,6,8)-3,\ (7,6,3)-5,\ (1,6,2,5),\ (4,2,3,1)\}$.
    - $(6,3) \in J(9,1)$:
      $\{(7,4,2)-1,\ (5,8,3)-2,\ (0,6,1)-3,\ (4,6,8)-7,\ (4,5,0)-8,\ (5,6,7)-0,\ (1,5,2,8),\ (1,4,3,7),\ (2,6,3,0)\}$.
    - $(5,4) \in J(9,1)$:
      $\{(7,5,2)-0,\ (6,7,3)-1,\ (7,0,4)-2,\ (5,8,0)-3,\ (6,0,1)-4,\ (1,2,8,7),\ (1,5,4,8),\ (2,3,4,6),\ (3,5,6,8)\}$.
    - $(4,5) \in J(9,1)$:
      $\{(5,0,2)-1,\ (6,0,3)-2,\ (7,0,4)-3,\ (8,0,1)-4,\ (5,6,7,8),\ (1,7,5,3),\ (1,5,4,6),\ (2,6,8,4),\ (3,7,2,8)\}$.
    - $(3,6) \in J(9,1)$.
      $\{(4,7,2)-1,\ (5,8,3)-2,\ (6,0,1)-3,\ (5,6,8,7),\ (1,4,3,7),\ (1,5,2,8),\ (2,0,3,6),\ (4,5,0,8),\ (4,6,7,0)\}$.
    - $(0,9) \in J(9,1)$:
      Develop the base block $(0,2,5,4)$ in (mod 9).

12. $J(16,1)$=$I(16,1)$.
    By Lemma 2.6 $J(16,1) \subseteq I(16,1)$. We need to show that $I(16,1) \subseteq J(16,1)$. Consider Lemma 3.6 with $v = u = 8$, $w = 0$. Using the results of $J(8,1)$ in 10 of the Appendix and a decomposition of $K_{8,8}$ into 16 4-cycles (see [18]) we can get $\{(14,16),(13,17),....,(8,22)\} \subseteq J(16,1)$. The rest follows by applying Lemma 3.6 with $v = 8$, $w = x$, $u = 8-x$, $x \in \{2,4,6,8\}$ and using results in 2, 4, 6, 8 and 10 of the Appendix, respectively.

13. $J(17,1)$=$I(17,1)$.
By Lemma 2.6 $J(17,1) \subseteq I(17,1)$. We need to show that $I(17,1) \subseteq J(17,1)$. Consider Lemma 3.6 with $v = u = 8$, $w = 1$. Using the results in 11 of the Appendix and a decomposition of $K_{8,8}$ into 16 4-cycles (see [18]) we can get $\{(18,16),(17,17),....,(3,31),(0,34)\} \subseteq J(17,1)$. The rest follows by applying Lemma 3.6 with $v = 8$, $w = x+1$, $u = 8-x$, $x \in \{2,4,6,8\}$ and using results in 3, 5, 7, 9 and 11 of the Appendix, respectively.

14. A decomposition of $2K_6 - 2K_2$ into $7-x$ kites and x 4 cycles for $x = 0,1,2,3,4,5,7$ having the vertex set $\{0,1,2,3\} \cup \{a,b\}$ with hole on $\{a,b\}$. $J(6,2,2)$=$\{(7,0),(6,1),...(2,5),(0,7)\}$.

   - $(7,0) \in J(6,2,2)$:
     $\{(1,3,a)-0,\ (2,3,b)-1,\ (2,0,a)-1,\ (1,b,0)-3,\ (1,0,2)-a,\ (1,3,2)-b,\ (b,0,3)-a\}$.
   - $(6,1) \in J(6,2,2)$:
     $\{(1,3,a)-0,\ (b,3,2)-a,\ (2,0,b)-1,\ (1,b,0)-2,\ (2,a,1)-3,\ (0,a,3)-b,\ (0,1,2,3)\}$.
   - $(5,2) \in J(6,2,2)$:
     $\{(1,3,b)-0,\ (a,3,2)-b,\ (2,3,a)-1,\ (1,b,0)-a,\ (0,3,1)-2,\ (b,2,0,3),\ (0,2,1,a)\}$.
   - $(4,3) \in J(6,2,2)$:
     $\{(a,3,1)-0,\ (b,1,3)-2,\ (0,3,2)-1,\ (1,2,0)-3,\ (a,0,b,2),\ (a,2,b,0),\ (3,b,1,a)\}$.
   - $(3,4) \in J(6,2,2)$:
     $\{(2,3,1)-0,\ (0,3,2)-1,\ (1,3,0)-2,\ (a,0,b,1),\ (a,1,b,2),\ (a,2,b,3),\ (a,3,b,0)\}$.
   - $(2,5) \in J(6,2,2)$:
     $\{(2,3,1){-}0, (2,3,0){-}1, (0,2,1,3), (a,0,b,1), (a,1,b,2), (a,2,b,3), (a,3,b,0)\}$.
   - $(0,7) \in J(6,2,2)$:
     $\{(2,3,1,0), (2,3,0,1), (0,2,1,3), (a,0,b,1), (a,1,b,2), (a,2,b,3), (a,3,b,0)\}$

15. $J(4,2)$=$I(4,2)$.
Let $\{0,1,2,3\}$ be the vertex set.

   - $(3,0) \in J(4,2)$:
     $\{(3,0,2)-1,(0,1,3)-2,(0,2,1)-3\}$.
   - $(2,1) \in J(4,2)$:
     $\{(3,2,0)-1,(2,3,1)-0,(0,2,1,3)\}$.

- $(0,3) \in J(4,2)$:
  $\{(0,1,2,3),(0,2,3,1),(0,2,1,3)\}$.

16. A decomposition of $2K_7 - 2K_3$ into $9-x$ kites and x 4-cycles for $x = 0,1,2,3,4,5,6,7,9$ having the vertex set $\{0,1,2,3\} \cup \{a,b,c\}$ with hole on $\{a,b,c\}$. $J(7,3,2)=\{(9,0),(8,1),...,(2,7),(0,9)\}$.

    - $(9,0) \in J(7,3,2)$:
      $\{(3,b,0)-a,\ (3,c,1)-b,\ (3,a,2)-c,\ (1,2,a)-0,\ (0,2,b)-1,\ (1,0,c)-2,\ (a,3,1)-0,\ (3,b,2)-1,\ (3,c,0)-2\}$.
    - $(8,1) \in J(7,3,2)$:
      $\{(1,3,a)-0,\ (0,2,a)-1,\ (1,3,c)-0,\ (0,2,c)-1,\ (b,2,1)-0,\ (b,3,2)-1,\ (b,0,3)-2,\ (b,1,0)-3,\ (a,2,c,3)\}$.
    - $(7,2) \in J(7,3,2)$:
      $\{(0,3,b)-1,\ (a,3,1)-b,\ (2,a,3)-c,\ (2,b,0)-a,\ (0,1,c)-2,\ (1,2,a)-0,\ (0,c,2)-3,\ (b,2,1,3),\ (c,1,0,3)\}$.
    - $(6,3) \in J(7,3,2)$:
      $\{(2,b,0)-a,\ (3,1,a)-0,\ (0,c,1)-b,\ (3,2,b)-1,\ (1,a,2)-c,\ (3,0,c)-2,\ (a,2,1,3),\ (b,0,2,3),\ (c,1,0,3)\}$.
    - $(5,4) \in J(7,3,2)$:
      $\{(2,3,a)-0,\ (0,3,b)-1,\ (1,c,0)-a,\ (c,0,2)-b,\ (c,3,1)-2,\ (a,1,b,3),\ (b,2,1,0),\ (c,2,0,3),\ (a,1,3,2)\}$.
    - $(4,5) \in J(7,3,2)$:
      $\{(0,1,2)-b,\ (1,2,3)-b,\ (3,0,2)-c,\ (1,0,3)-c,\ (b,0,c,1),\ (a,0,b,1),\ (a,2,b,3),\ (a,1,c,2),\ (a,3,c,0)\}$.
    - $\{(3,6),(2,7),(0,9)\} \subseteq J(7,3,2)$:
      Theorem 3.2 with $u=4$, $v=3$ and $\lambda=2$ together with the step 15 of the Appendix give $\{(3,6),(2,7),(0,9)\} \subseteq J(7,3,2)$.

17. $J(5,2)=I(5,2)$.
    Let $\{0,1,2,3,4\}$ be the vertex set.

    - $(5,0) \in J(5,2)$:
      Develop the base block $\{(1,2,4)-0\}$ in (mod 5).
    - $(0,5) \in J(5,2)$:
      Develop the base block $\{(0,1,3,2)\}$ in (mod 5)
    - $(4,1) \in J(5,2)$:
      $\{(4,2,1)-0,(3,4,0)-1,(1,4,3)-2,(0,4,2)-3,(0,2,1,3)\}$.
    - $(3,2) \in J(5,2)$:
      $\{(3,4,2)-1,(1,4,0)-2,(4,3,1)-0,(0,4,2,3),(0,2,1,3)\}$.

- $(2,3) \in J(5,2)$:
  $\{(2,3,0)-1,(2,4,1)-0,(4,0,3,1),(4,0,2,3),(4,3,1,2)\}$.

18. $J(8,2)$=$I(8,2)$.

  - By step 10 of the Appendix $\{(14,0),(13,1),....,(8,6)\} \subseteq J(8,2)$. Lemma 3.6 with $u=v=4$ and $w=0$ gives $\{(6,8),(5,9),....,(2,12),(0,14)\} \subseteq J(8,2)$.
  - $(7,7) \in J(8,2)$:
    Let K_8 be $\{0,...,6\} \cup \{\infty\}$ be the vertex set. Develop the base blocks $\{(1,2,3)-0,(\infty,0,2,5)\}$ in (mod 7).

19. $J(9,2)$=$I(9,2)$.
  Let $\{0,1,2,3\}$ be the vertex set.

  - By step 11 of the Appendix $\{(18,0),(17,1),....,(3,15),(0,18)\} \subseteq J(9,2)$. Lemma 3.6 with $u=v=4$ and $w=1$ gives $(2,16) \in J(8,2)$.

20. $J(6,4)$=$I(6,4)$.
  Let $\{0,1,2,3,4\} \cup \{\infty\}$ be the vertex set. Then

  - $(15,0) \in J(6,4)$:
    Develop the base blocks $\{(1,3,0)-\infty,(1,2,\infty)-0,(1,2,3)-0\}$ in (mod 5).
  - $(14,1) \in J(6,4)$:
    Develop the base blocks $\{(1,\infty,0)-3,(1,2,0)-3\}$ in (mod 5). Then add the blocks $\{(3,\infty,1)-0,\ (4,\infty,2)-1,\ (4,\infty,3)-2,\ (2,\infty,0)-3,\ (\infty,0,4,1)\}$.
  - $(13,2) \in J(6,4)$:
    Develop the base blocks $\{(0,\infty,2)-1,(0,\infty,2)-1\}$ in (mod 5). Then add the blocks $\{(3,4,1)-0,\ (3,4,2)-1,\ (1,4,0)-2,\ (0,3,1,2),\ (0,3,2,4)\}$.
  - $(12,3) \in J(6,4)$:
    Develop the base blocks $\{(0,\infty,2)-1,(0,\infty,2)-1\}$ in (mod 5). Then add the blocks $\{(3,4,1)-0,\ (4,2,0)-1,\ (1,2,3,4),\ (0,3,1,2),\ (0,3,2,4)\}$.
  - $(11,4) \in J(6,4)$:
    $\{(\infty,3,1)-0,\ (2,3,0)-\infty,\ (4,\infty,2)-1,\ (0,\infty,3)-2,\ (1,2,\infty)-4,\ (1,0,4)-3,\ (\infty,3,2)-0,\ (\infty,2,4)-1,\ (\infty,4,0)-2,\ (\infty,1,3)-4,\ (\infty,0,1)-3,\ (0,4,2,3),\ (1,0,3,4),\ (1,4,0,2),\ (4,3,1,2)\}$.

- $(10,5) \in J(6,4)$:
  Develop the base blocks $\{(1,3,0)-\infty, (1,2,\infty)-0, (0,1,3,2)\}$ in (mod 5).
- $(9,6) \in J(6,4)$:
  Develop the base blocks $\{(1,\infty,0)-3, (0,4,2,3)\}$ in (mod 5). Then add the blocks $\{(3,\infty,1)-0,\ (4,\infty,2)-1,\ (4,\infty,3)-2,\ (2,\infty,0)-3,\ (\infty,0,4,1)\}$.
- $(8,7) \in J(6,4)$:
  Develop the base blocks $\{(1,2,0,\infty), (0,\infty,2)-1)\}$ in (mod 5). Then add the blocks $\{(3,4,1)-0,\ (3,4,2)-1,\ (1,4,0)-2,\ (0,3,1,2),\ (0,3,2,4)\}$.
- $(7,8) \in J(6,4)$:
  Develop the base blocks $\{(1,2,0,\infty), (1,\infty,2)-0\}$ in (mod 5). Then add the blocks $\{(3,4,1)-0,\ (4,2,0)-1,\ (1,2,3,4),\ (0,3,1,2),\ (0,3,2,4)\}$.
- $(6,9) \in J(6,4)$:
  $\{(\infty,3,1)-0,\ (2,3,0)-\infty,\ (4,\infty,2)-1,\ (0,\infty,3)-2,\ (1,2,\infty)-4,\ (1,0,4)-3,\ (\infty,0,4,3),\ (\infty,0,2,4),\ (\infty,2,0,1),\ (\infty,4,1,3),\ (\infty,2,3,1),\ (0,4,2,3),\ (1,0,3,4),\ (1,4,0,2),\ (4,3,1,2)\}$.
- $(5,10) \in J(6,4)$:
  Develop the base blocks $\{(3,1,0,\infty), (\infty,1,2)-0, (0,1,3,5)\}$ in (mod 5).
- $(4,11) \in J(6,4)$:
  Develop the base blocks $\{(1,0,3,\infty), (0,4,2,3)\}$ in (mod 5) and add the blocks $\{(3,\infty,1)-0,\ (4,\infty,2)-1,\ (4,\infty,3)-2,\ (2,\infty,0)-3,\ (\infty,0,4,1)\}$.
- $(3,12) \in J(6,4)$:
  Develop the base blocks $\{(1,2,0,\infty), (1,2,0,\infty)\}$ in (mod 5) and add the blocks $\{(3,4,1)-0, (3,4,2)-1, (1,4,0)-2, (0,3,1,2),\ (0,3,2,4)\}$.
- $(2,13) \in J(6,4)$:
  Develop the base blocks $\{(1,0,3,\infty), (0,4,2,3)\}$ in (mod 5) and add the blocks $\{(1,2,\infty)-0, (3,4,\infty)-0, (0,1,3,2), (\infty,3,0,4),\ (\infty,1,4,2)\}$.
- $(0,15) \in J(6,4)$:
  Develop the base blocks $\{(1,0,3,\infty), (1,0,3,\infty), (0,4,2,3)\}$ in (mod 5).

21. A decomposition of $4K_5 - 4K_2$ into $9-x$ kites and x 4-cycles for $x = 0,3,9$ having the vertex set $\{0,1,2\} \cup \{a,b\}$ with hole on $\{a,b\}$.

- $(9,0) \in J(5,2,4)$:
$\{(1,2,a)-0,(2,0,b)-1,(2,0,a)-1,(1,0,b)-2,(2,0,1)-b,$
$(0,1,a)-2,(b,0,2)-1,(0,1,a)-2,(1,2,b)-0\}$.
- $(6,3) \in J(5,2,4)$:
$\{(0,1,2)-a,(2,0,1)-a,(1,2,0)-a,(0,1,b)-2,(1,2,b)-0,$
$(2,0,a)-1,(0,a,1,b),(0,a,2,b),(1,a,2,b)\}$.
- $(0,9) \in J(5,2,4)$:
$\{(0,1,2,a), (0,1,2,b), (1,2,0,a), (1,2,0,b), (2,0,1,b), (2,0,1,a),$
$(0,a,1,b), (0,a,2,b), (1,a,2,b)\}$.

22. $J(7,4)$=$I(7,4)$.
Let $\{0,1,2,3,4,5,6\}$ be the vertex set.

- Lemma 3.6 with $v=3$, $u=2$ and $w=2$, Lemma 4.6 and step 21 of the Appendix give $\{(15,6),(14,7),...,(2,19),(0,21)\} \subseteq J(7,4)$.
- $(7,14) \in J(7,4)$:
Develop the base blocks $\{(0,1,2)-4,\ (0,3,5,1),\ (0,3,5,1)\}$ in (mod 7).
- $(21,0) \in J(7,4)$:
Develop the base blocks $\{(3,4,1)-0,(1,4,2)-0,(1,4,2)-5\}$ in (mod 7).
- $(20,1) \in J(7,4)$:
Develop the base blocks $\{(1,4,2)-0,(1,4,2)-5\}$ in (mod 7). Then add the blocks $\{(2,4,3)-1,\ (4,6,5)-0,\ (0,4,1)-5,$ $(3,6,2)-1,\ (0,6,1)-2,\ (4,5,3)-0,\ (0,2,5,6)\}$.
- $(19,2) \in J(7,4)$:
Develop the base blocks $\{(1,4,2)-0,(1,4,2)-5\}$ in (mod 7). Then add the blocks $\{(2,4,3)-1,\ (4,6,5)-3,\ (0,4,1)-5,$ $(3,6,2)-1,\ (0,6,1)-2,\ (0,3,4,5),\ (0,2,5,6)\}$.
- $(18,3) \in J(7,4)$.
Develop the base blocks $\{(1,4,2)-0,(1,4,2)-5\}$ in (mod 7). Then add the blocks $\{(0,5,2)-1,\ (0,4,1)-2,\ (3,6,5)-4,$ $(2,3,4)-5,\ (0,6,5,1),\ (1,3,4,6),\ (3,2,6,0)\}$.
- $(17,4) \in J(7,4)$.
Develop the base blocks $\{(1,4,2)-0,(1,4,2)-5\}$ in (mod 7). Then add the blocks $\{(2,4,3)-1,\ (0,4,1)-5,\ (4,6,5)-3,$ $(1,2,0,6),\ (1,2,5,0),\ (3,4,5,6),\ (3,2,6,0)\}$.
- $(16,5) \in J(7,4)$.
Develop the base blocks $\{(1,4,2)-0,(1,4,2)-5\}$ in (mod 7). Then add the blocks $\{(0,5,2)-1,\ (0,4,1)-2,\ (0,6,5,1),\ (1,3,4,6),$ $(2,4,5,3),\ (3,4,5,6),\ (3,2,6,0)\}$.

23. $J(10,4)$=$I(10,4)$

- Lemma 4.6 with $v = 3$, $u = 5$ and $w = 2$ and steps 21, 22 of the Appendix give $\{(30,15),(29,16),....,(2,43),(0,45)\} \subseteq J(10,4)$.
- $\{(31,14),...,(35,10)\} \subseteq J(10,4)$:
Let $X = \{a_i, i \in Z_5\}$ and $Y = \{b_i, i \in Z_5\}$ be two disjoint sets. Let $V = X \cup Y$ be the vertex set. Form the following 25 kites $\{(a_{i+1}, a_{i+3}, b_i) - a_i, (b_{i+3}, b_{i+4}, a_{i+2}) - b_i, (a_{i+1}, a_{i+3}, b_i) - a_i, (b_{i+3}, b_{i+4}, a_{i+2}) - b_i, (b_i, b_{i+2}, a_{i+1}) - a_i, i \in Z_5\}$ and 10 4-cycles $\{(a_i, a_{i+1}, b_{i+1}, b_{i+4}), (a_i, b_{i+1}, a_{i+3}, b_{i+3}), i \in Z_5\}$ on V. Take a decomposition of $2K_5$ with $(r_1, s_1) \in J(5,2)$ on X and a decomposition of $2K_5$ with $(r_2, s_2) \in J(5,2)$ on Y. Thus we obtain a decomposition of $4K_{10}$ into $25 + r_1 + r_2$ kites and $10 + s_1 + s_2$ 4-cycles.
- $\{(36,9),...,(40,5)\} \subseteq J(10,4)$:
Let $X = \{a_i, i \in Z_5\}$ and $Y = \{b_i, i \in Z_5\}$ be two disjoint sets. Let $V = X \cup Y$ be the vertex set. Form the following 30 kites $\{(a_{i+1}, a_{i+3}, b_i) - a_i, (b_{i+3}, b_{i+4}, a_{i+2}) - b_i, (a_{i+1}, a_{i+3}, b_i) - a_i, (b_{i+3}, b_{i+4}, a_{i+2}) - b_i, (b_{i+1}, b_{i+3}, a_i) - b_i, (a_{i+1}, a_{i+2}, b_i) - a_i, i \in Z_5\}$ and 5 4-cycles $\{(a_i, a_{i+1}, b_{i+2}, b_{i+4}), i \in Z_5\}$ on V. Take a decomposition of $2K_5$ with $(r_1, s_1) \in J(5,2)$ on X and a decomposition of $2K_5$ with $(r_2, s_2) \in J(5,2)$ on Y. Hence we obtain a decomposition of $4K_{10}$ into $30 + r_1 + r_2$ kites and $5 + s_1 + s_2$ 4-cycles.
- $\{(41,4),...,(45,0)\} \subseteq J(10,4)$:
Let $X = \{a_i, i \in Z_5\}$ and $Y = \{b_i, i \in Z_5\}$ be two disjoint sets. Let $V = X \cup Y$ be the vertex set. Form the following 35 kites $\{(a_{i+1}, a_{i+3}, b_i) - a_i, (b_{i+3}, b_{i+4}, a_{i+2}) - b_i, (a_{i+1}, a_{i+3}, b_i) - a_i, (b_{i+3}, b_{i+4}, a_{i+2}) - b_i, (b_{i+1}, b_{i+3}, a_i) - b_i, (a_{i+1}, a_{i+2}, b_i) - a_i, (b_i, b_{i+2}, a_{i+1}) - a_i, i \in Z_5\}$ on V. Take a decomposition of $2K_5$ with $(r_1, s_1) \in J(5,2)$ on X and a decomposition of $2K_5$ with $(r_2, s_2) \in J(5,2)$ on Y. Hence we obtain a decomposition of $4K_{10}$ into $35 + r_1 + r_2$ kites and $s_1 + s_2$ 4-cycles.

24. $J(11,4)$=$I(11,4)$.

- Lemma 4.6 with $v = 3$, $u = 6$ and $w = 2$, Lemma 4.6 together with step 21 of the Appendix give $\{(37,18),(36,19),....,(2,53),(0,55)\} \subseteq J(11,4)$.
- $\{(38,17),...,(45,10)\} \subseteq J(11,4)$:
Let $X = \{a_i, i \in Z_5\}$ and $Y = \{b_i, i \in Z_5\}$ be two disjoint sets. Let $V = X \cup Y$ be the vertex set. Form the following 35 kites

$\{(a_{i+1},\infty,b_i)-a_i,(b_{i+4},\infty,a_{i+2})-b_i,(b_{i+3},b_{i+4},a_{i+2})-a_i,$ $(a_{i+1},\infty,b_i)-a_i,(b_{i+4},\infty,a_{i+2})-b_i,(b_{i+3},b_{i+4},a_{i+2})-a_i,$ $(b_i,b_{i+2},a_{i+1})-a_i, i\in Z_5\}$ and 10 4-cycles $\{(a_i,a_{i+1},b_{i+1},b_{i+4}),$ $(a_i,b_{i+1},a_{i+3},b_{i+3}), i\in Z_5\}$ on V. Take a decomposition of $2K_5$ with $(r_1,s_1)\in J(5,2)$ on X and a decomposition of $2K_5$ with $(r_2,s_2)\in J(5,2)$ on Y.Thus we obtain a decomposition of $4K_{10}$ into $35+r_1+r_2$ kites and $10+s_1+s_2$ 4-cycles.

- $\{(46,9),...,(50,5)\}\subseteq J(11,4)$:
  Let $X=\{a_i,i\in Z_5\}$ and $Y=\{b_i,i\in Z_5\}$ be two disjoint sets. Let $V=X\cup Y$ be the vertex set. Form the following 40 kites $\{(a_{i+1},\infty,b_i)-a_i,(b_{i+4},\infty,a_{i+2})-b_i,(b_{i+3},b_{i+4},a_{i+2})-a_i,$ $(a_{i+1},\infty,b_i)-a_i,(b_{i+4},\infty,a_{i+2})-b_i,(b_{i+3},b_{i+4},a_{i+2})-a_i,$ $(b_{i+1},b_{i+3},a_i)-b_i,(a_{i+1},a_{i+2},b_i)-a_i, i\in Z_5\}$ and 5 4-cycles $\{(a_i,a_{i+1},b_{i+2},b_{i+4}), i\in Z_5\}$ on V. Take a decomposition of $2K_5$ with $(r_1,s_1)\in J(5,2)$ on X and a decomposition of $2K_5$ with $(r_2,s_2)\in J(5,2)$ on Y. Thus we obtain a decomposition of $4K_{11}$ into $40+r_1+r_2$ kites and $5+s_1+s_2$ 4-cycles.
- $\{(51,4),...,(55,0)\}\subseteq J(11,4)$:
  Let $X=\{a_i,i\in Z_5\}$ and $Y=\{b_i,i\in Z_5\}$ be two disjoint sets. Let $V=X\cup Y$ be the vertex set. Form the following 45 kites $\{a_{i+1},\infty,b_i)-a_i,(b_{i+4},\infty,a_{i+2})-b_i,(b_{i+3},b_{i+4},a_{i+2})-a_i,$ $(a_{i+1},\infty,b_i)-a_i,(b_{i+4},\infty,a_{i+2})-b_i,(b_{i+3},b_{i+4},a_{i+2})-a_i,$ $(b_{i+1},b_{i+3},a_i)-b_i,(a_{i+1},a_{i+2},b_i)-a_i,(b_i,b_{i+2},a_{i+1})-a_i, i\in Z_5\}$ on V. Take a decomposition of $2K_5$ with $(r_1,s_1)\in J(5,2)$ on X and a decomposition of $2K_5$ with $(r_2,s_2)\in J(5,2)$ on Y. Thus we obtain a decomposition of $4K_{11}$ into $45+r_1+r_2$ kites and s_1+s_2 4-cycles.

References

[1] Abueida, A. A., Daven, M.: *Multidecompositions of the complete graph.* Ars Comb. 72, 17-22 (2004).

[2] Abueida, A. A., O'Neil, T.: *Multidecomposition of* λK_m *into small cycles and claws.* Bull. Inst. Comb. Appl. 49, 32-40 (2007).

[3] Adams, P. , Bryant, D., Khodkar, A.: $3,5$-*cycle decomposition.* J. Comb. Des. 6, 91-110 (1998).

[4] Alspach, B.: *Research problems, problem* 3. Discrete Math. 36, 333-334 (1981).

[5] Alspach, B., Marshall, S.: *Even cycle decompositions of complete graphs minus a* 1-*factor.* J. Comb. Des. 2, 441-458 (1994).

[6] Alspach, B., Gavlas, H.: *Cycle decompositions of* K_n *and* $K_n - I$. J. Comb. Theory Ser. B 81, 77-99 (2001).

[7] Billington, E., Fu, H., Rodger, C. A.: *Packing* λ-*fold complete multipartite graphs with* 4-*cycles.* Graphs Combin. 21, no. 2, 169-185 (2005).

[8] Bryant, D., Maenhaut, B.: *Decomposition of complete graphs into triangles and Hamiltonian cycles.* J. Comb. Des. 12, 221-232 (2004).

[9] Bryant, D., Khodkar, A., Fu, H. L.: (m,n)*Cycle systems.* J. Stat. Plann. Inference 74 (2), 91.110 (1998).

[10] Colbourn, C.J., Dinitz, J. H.: *The CRC Handbook of Combinatorial Designs.* Chapman and Hall/CRC, Boca Raton, FL (2007). Online updates <www.emba.uvm.edu/~dinitz/newresults.html>.

[11] Danziger, P., Quattrocchi, G., Stevens, B.: *The Hamilton-Waterloo Problem for Cycle Sizes* 3 *and* 4. J. Comb. Des. 12, 221-232 (2004).

[12] Küçükçifçi, S., Lindner, C. C.: *The metamorphosis of* λ-*fold block designs with block size four into* λ-*fold kite systems.* JCMCC 40, 241-252 (2002).

[13] Heinrich, K., Horak, P., Rosa, A.: *On Alspach's conjecture.* Discrete Math. 77, 97-121 (1989).

[14] Lin, C., Shyu, T.W.: *A necessary and sufficient condition for the star decomposition of complete graphs.* J. Graph Theory 23, 361-364 (1996).

[15] Lindner, C. C., Rodger, C. A.: *Decomposition into cycles II Cycle Systems.* In: Contemporary Design Theory, A Collection of Surveys, Eds. J. H. Dinitz and D. R. Stinson, John Wiley and Son, New York, 325-369 (1992).

[16] Lo Faro, G., Tripodi, A.: *Embeddings of λ-fold kite systems, $\lambda \geq 2$* . Australas. J. Combin. 36, 143-150 (2006).

[17] Sajna, M.: *Cycle decompositions III: complete graphs and fixed length cycles.* J. Comb. Des. 10, 27-78 (2002).

[18] Sotteau, D.: *Decomposition of $K_{m,n}$ $(K^*_{m,n})$ into cycles (circuits) of lengt* $2k$. J. Comb. Theory Ser.B 30, 75-81 (1981).

[19] Shyu, T.W.: *Decomposition of complete graphs into cycles and stars.* Graphs and Combinatorics, to appear.