Rainbow connection numbers of ladders and Möbius ladders

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Abstract A path in an edge colored graph is said to be a rainbow path if no two edges on the path share the same color. An edge colored graph G is rainbow connected if there exists a $u-v$ rainbow path for any two vertices u and v in G. The rainbow connection number of a graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. For any two vertices u and v of G, a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. The strong rainbow connection number of G, denoted by $src(G)$, is the smallest number of colors that are needed in order to make G strongly rainbow connected.

In this paper, we determine the precise (strong) rainbow connection numbers of ladders and Möbius ladders. Let p be an odd prime, we show the (strong) rainbow connection numbers of Cayley graphs on the dihedral group D_{2p} of order $2p$ and the cyclic group Z_{2p} of order $2p$. In particular, an open problem posed by Li et al. in [8] is solved.

Keywords: ladder, Möbius ladder, Cayley graph, (strong) rainbow connection number

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We refer to [2] for the graph-theoretic terms not described here. Connec-

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tivity is perhaps the most fundamental graph-theoretic property. There are many ways to strengthen the connectivity property, such as requiring hamiltonicity, k-connectivity, imposing bounds on the diameter, requiring the existence of edge-disjoint spanning trees, and so on. The rainbow connection number firstly introduced by Chartrand et al.[6] is also an interesting way to quantitavely strengthen the connectivity requirement.

Let G be a nontrivial connected graph with an edge colouring $c : E(G) \to \{1, 2, \ldots, k\}, k \in \mathbb{N}$, where the adjacent edges may be coloured the same. A path is said to be a rainbow path if no two edges on the path share the same color. An edge colored graph G is *rainbow connected* if there exists a rainbow path between every pair of vertices. The *rainbow connection number* of a graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. If H is a connected spanning subgraph of a graph G, it is easy to verify that $rc(G) \leq rc(H)$. We note also the trivial fact that if C_n is a cycle with $n \geq 4$ vertices, then $rc(C_n) = src(C_n) = \lceil \frac{n}{2} \rceil$. For any two vertices u and v of G, a rainbow $u-v$ *geodesic* in G is a rainbow $u-v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v. The graph G is *strongly rainbow connected* if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. *The strong rainbow connection number* of G, denoted by $src(G)$, is the smallest number of colors that are needed in order to make G strongly rainbow connected. An easy observation is that if G is a nontrivial connected graph, we have $diam(G) \leq rc(G) \leq src(G) \leq m$, where *diam(G)* denotes the diameter of G and m is the size of G.

Recently there has been great interest in this concept and a lot of results have been published. Chakraborty et al. in [4] showed that computing $rc(G)$ is NP-Hard. Therefore, there have been various investigations towards finding good upper bounds for rainbow connection numbers in terms of other graph parameters such as connectivity, minimum degree, radius etc.([1, 3, 7, 12]) and for many special graph classes such as integral graphs, Cayley graphs, line graphs etc.([5, 8, 9]). The reader can see [10] for a survey on this topic.

In this paper, we consider the (strong) rainbow connection numbers of ladders and Möbius ladders. For an integer $h \geq 3$, the *ladder* L_h of order $2h$ is a cubic graph constructed by taking two copies of the cycle C_h on disjoint vertex sets $(u_1, u_2, \cdots, u_h)$ and $(v_1, v_2, \cdots, v_h)$, then joins the corresponding vertices u_iv_i for $1 \leq i \leq h$. The *Möbius ladder* M_h of order $2h$ is obtained from the ladder by deleting the edges u_1u_h and v_1v_h and then inserting edges u_1v_h and u_hv_1. Our main result is stated as follows.

Theorem 1.1. *For an integer* $h \geq 3$.

(i) Let $G = L_h$ *be the ladder, then* $rc(G) = src(G) = \lceil \frac{h+1}{2} \rceil$.

(ii) Let $G = M_h$ be the Möbius ladder, then $rc(G) = src(G) = \lceil \frac{h}{2} \rceil$.

In fact, ladders and Möbius ladders are connected Cayley graphs. Let Γ be a finite group, and S an inverse closed subset of Γ satisfying $1 \notin S$, the *Cayley graph* $G = \mathrm{Cay}(\Gamma, S)$ of Γ with respect to S is defined by

$$V(G) = \Gamma,$$
$$E(G) = \{\{g, h\} | g^{-1}h \in S\}.$$

Then $\mathrm{Cay}(G, S)$ is a well-defined simple regular graph of valency $|S|$. It is well-known that $\mathrm{Cay}(\Gamma, S)$ is connected if and only if S is a generating set of Γ. Throughout the remainder of this paper, we denote $\bar{S} := S \cup S^{-1}$, where $S^{-1} = \{s^{-1}, s \in S\}$.

Interconnection networks are often modeled by highly symmetric Cayley graphs. The rainbow connection number of a graph can be applied to measure the safety of a network. Thus the object of the rainbow connection number of Cayley graphs should be meaningful. In [8], the upper and lower bounds for the (strong) rainbow connection numbers of Cayley graphs on abelian groups were investigated. They also posed an open problem as follows.

Open problem : Given an abelian group Γ and a minimal generating set $S \subseteq \Gamma \backslash \{1\}$ of Γ, when some element $s \in \bar{S}$ has an *odd order*, is it true that

$$rc(\mathrm{Cay}(\Gamma, \bar{S})) = src(\mathrm{Cay}(\Gamma, \bar{S})) = \sum_{s \in S} \lceil \frac{|s|}{2} \rceil ?$$

We answer the problem in terms of the following theorem.

Theorem 1.2. *Let $G = \mathrm{Cay}(\Gamma, \bar{S})$ be a connected graph, where Γ is a finite group of order $2p$ and p is an odd prime.*

(i) If S is a minimal generating set of Γ such that $|\bar{S}| = 2$, then $rc(G) = src(G) = p$.

(ii) If S is a minimal generating set of Γ such that $|\bar{S}| = 3$, then $rc(G) = src(G) = \frac{p+1}{2}$.

In addition, if Γ is a group of order 4, we give the precise value of the (strong) rainbow connection numbers of $\mathrm{Cay}(\Gamma, \bar{S})$. Therefore, the (strong) rainbow connection numbers of Cayley graphs of order $2p$ (p is a prime) are completely discussed in this paper. As is customary, unless stated otherwise, it will be assumed that i, j are nonnegative integers and p is an odd prime in the following sections.

2 Proof of Theorem 1.1

The *Cartesian product* of graphs G and H is the graph $G\Box H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$. Now we recall a result from [11].

Lemma 2.1. *Let $G^* = G_1\Box G_2\Box \ldots \Box G_k (k \geq 2)$, where each $G_i (1 \leq i \leq k)$ is connected, then we have*

$$rc(G^*) \leq \sum_{i=1}^{k} rc(G_i).$$

Moreover, if $diam(G_i) = rc(G_i)$ for each G_i, then the equality holds.

Proof of Theorem 1.1. (i) If h is an even integer, then it is easy to obtain that $diam(G) = \frac{h}{2} + 1$. Since $G = C_h\Box K_2, rc(C_h) = diam(C_h) = \frac{h}{2}$, and $rc(K_2) = diam(K_2) = 1$ we have $rc(G) = diam(G) = \frac{h}{2} + 1$ by Lemma 2.1. Define a colouring C showed in [11, Theorem 4.4] by

$$C(e) = \begin{cases} i & \text{if } e = u_iu_{i+1} \text{ and } e = v_iv_{i+1} \quad \text{for } 1 \leq i \leq \frac{h}{2}, \\ i - \frac{h}{2} & \text{if } e = u_iu_{i+1} \text{ and } e = v_iv_{i+1} \text{ for } \frac{h+2}{2} \leq i \leq h, \\ \frac{h+2}{2} & \text{if } e = u_iv_i \quad \text{for } 1 \leq i \leq h. \end{cases}$$

An easy observation is that the coloring makes G strongly rainbow connected. Therefore, $rc(G) = src(G) = \frac{h}{2} + 1$. Now we only need to show that $rc(G) = src(G) = \frac{h+1}{2}$, where h is an odd integer(see Figure 1).

(1) For $1 \leq i, j \leq h$, it is not hard to see that $d(u_i, u_j) \leq \frac{h-1}{2}$, $d(v_i, v_j) \leq \frac{h-1}{2}$ and $d(u_i, v_j) \leq \frac{h-1}{2} + 1 = \frac{h+1}{2}$. Hence the diameter of G is $\frac{h+1}{2}$. Then we obtain $src(G) \geq rc(G) \geq diam(G) = \frac{h+1}{2}$.

(2) Define a colouring C of the graph G by

$$C(e) = \begin{cases} i & \text{if } e = u_iu_{i+1} \text{ and } e = v_iv_{i+1} \quad \text{for } 1 \leq i \leq \frac{h+1}{2}, \\ i - \frac{h+1}{2} & \text{if } e = u_iu_{i+1},\ e = v_iv_{i+1} \text{ and } e = u_iv_i \text{ for } \frac{h+3}{2} \leq i \leq h, \\ \frac{h+1}{2} & \text{if } e = u_iv_i \quad \text{for } 1 \leq i \leq \frac{h+1}{2}. \end{cases}$$

Next we will show C is a strong rainbow $\frac{h+1}{2}$-colouring of G, that is, G contains a rainbow $x - y$ geodesic for every two vertices x and y of G.

First, observe that if $x, y \in U = \{u_1, u_2, \ldots, u_h\}$ or $x, y \in V = \{v_1, v_2, \ldots, v_h\}$, there exists a rainbow $x - y$ geodesic contained in the cycle $C_1 = (u_1, u_2, \ldots, u_h, u_1)$ or $C_2 = (v_1, v_2, \ldots, v_h, v_1)$. It remains to

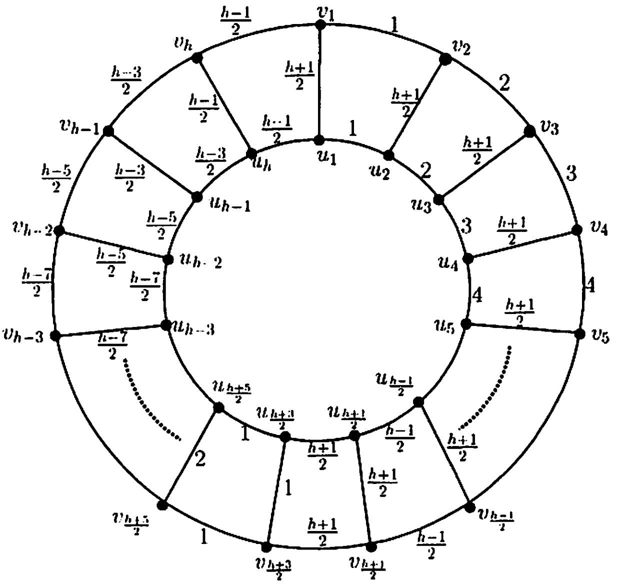

Figure 1: $G = L_h$, where $h = 1 (\text{mod } 2)$.

prove that if $x \in U$ and $y \in V$, or $x \in V$ and $y \in U$, then G contains a rainbow $x - y$ geodesic. Without loss of generality, we assume that $x \in U$ and $y \in V$. Since $d(x, y) \leq \frac{h+1}{2}$, x and y must be contained in a connected induced subgraph H of G, which satisfies:

(a) $|V(H)| = h + 1$;

(b) $V(H) = \{u_i, v_i \mid u_i \in U, v_i \in V, 1 \leq i \leq h\}$.

It is clear that H contains a $x - y$ geodesic in G and H must be one of the following four cases:

Case 1: $V(H) = \{u_1, u_2, \ldots, u_{\frac{h+1}{2}}; v_1, v_2, \ldots, v_{\frac{h+1}{2}}\}$.

Suppose that $x = u_i \in H$, $y = v_j \in H$, $1 \leq i, j \leq \frac{h+1}{2}$. If $j \leq i$, the path $v_j, v_{j+1}, \ldots, v_i, u_i$ is a rainbow $v_j - u_i$ geodesic; if $i \leq j$, the path $u_i, u_{i+1}, \ldots, u_j, v_j$ is a rainbow $u_i - v_j$ geodesic.

Case 2: $V(H) = \{u_{\frac{h+3}{2}}, u_{\frac{h+5}{2}}, \ldots, u_h, u_{h+1}; v_{\frac{h+3}{2}}, v_{\frac{h+5}{2}}, \ldots, v_h, v_{h+1}\}$, where $u_{h+1} = u_1$ and $v_{h+1} = v_1$.

Suppose that $x = u_i \in H$, $y = v_j \in H$, $\frac{h+3}{2} \leq i, j \leq h + 1$. Assume $j \leq i$, the path $v_j, v_{j+1}, \ldots, v_i, u_i$ is a rainbow $v_j - u_i$ geodesic; assume $i \leq j$, the path $u_i, u_{i+1}, \ldots, u_j, v_j$ is a rainbow $u_i - v_j$ geodesic.

Case 3: $V(H) = \{u_{h-s}, \ldots, u_{h-1}, u_h, u_1, \ldots, u_t; v_{h-s}, \ldots, v_{h-1}, v_h, v_1, \ldots, v_t\}$, where $s + t = \frac{h-1}{2}$, $s \geq 0$, $t \geq 2$.

Suppose that $x = u_i \in H$, $y = v_j \in H$, $i, j \in \{1, 2, \ldots t, h-s, \ldots, h\}$. If $i, j \in \{h-s, \ldots, h\}$, the rainbow $u_i - v_j$ geodesic is the same as that in *case 2*; if $i, j \in \{1, 2, \ldots, t\}$, the rainbow $u_i - v_j$ geodesic is the same as that in *case 1*; if $i \in \{1, 2, \ldots, t\}$ and $j \in \{h-s, \ldots, h\}$, the path $v_j, v_{j+1}, \ldots, v_h, v_1, \ldots, v_i, u_i$ is a rainbow $v_j - u_i$ geodesic; if $i \in \{h-s, \ldots, h\}$ and $j \in \{1, 2, \ldots, t\}$, the path $u_i, u_{i+1}, \ldots, u_h, u_1, \ldots, u_j, v_j$ is a rainbow $u_i - v_j$ geodesic;

Case 4: $V(H) = \{u_{\frac{h+1+2s}{2}}, \ldots, u_{\frac{h+1}{2}}, \ldots, u_{\frac{h+1-2t}{2}}; v_{\frac{h+1+2s}{2}}, \ldots, v_{\frac{h+1}{2}}, \ldots, v_{\frac{h+1-2t}{2}}\}$, where $s + t = \frac{h-1}{2}$, $s \geq 1$, $t \geq 0$.

Suppose that $x = u_i \in H$, $y = v_j \in H$, $i, j \in \{\frac{h+1-2t}{2}, \ldots, \frac{h+1+2s}{2}\}$. Assume $i, j \in \{\frac{h+3}{2}, \ldots, \frac{h+1+2s}{2}\}$, the rainbow $u_i - v_j$ geodesic is the same as that in *case 2*; assume $i, j \in \{\frac{h+1-2t}{2}, \ldots, \frac{h+1}{2}\}$, the rainbow $u_i - v_j$ geodesic is the same as that in *case 1*; assume $i \in \{\frac{h+3}{2}, \ldots, \frac{h+1+2s}{2}\}$ and $j \in \{\frac{h+1-2t}{2}, \ldots, \frac{h+1}{2}\}$, the path $v_j, v_{j+1}, \ldots, v_{\frac{h+1}{2}}, v_{\frac{h+3}{2}}, \ldots, v_i, u_i$ is a rainbow $v_j - u_i$ geodesic; assume $i \in \{\frac{h+1-2t}{2}, \ldots, \frac{h+1}{2}\}$ and $j \in \{\frac{h+3}{2}, \ldots, \frac{h+1+2s}{2}\}$, the path $u_i, u_{i+1}, \ldots, u_{\frac{h+1}{2}}, u_{\frac{h+3}{2}}, \ldots, u_j, v_j$ is a rainbow $u_i - v_j$ geodesic.

Therefore, $rc(G) \leq src(G) \leq \frac{h+1}{2}$. Combining (1) and (2), we come to the conclusion that $rc(G) = src(G) = \frac{h+1}{2}$.

(ii) ***Case 1***: h is even. Let $h = 2t$, then $2h = 4t$ (see Figure 2).

Define a colouring C of the graph G by

$$C(e) = \begin{cases} i & \text{if } e = u_iu_{i+1},\ v_iv_{i+1} \text{ and } u_iv_i \quad \text{for } 1 \leq i \leq t, \\ i-t & \text{if } e = u_iu_{i+1},\ v_iv_{i+1} \text{ and } u_iv_i \quad \text{for } t+1 \leq i \leq 2t-1, \\ t & \text{if } e = u_{2t}v_1,\ v_{2t}u_1 \text{ and } u_{2t}v_{2t}. \end{cases}$$

Next we will prove that C is a strong rainbow t-colouring of G, that is, G contains a rainbow $x-y$ geodesic for any two vertices x and y of G. Let $C_{u_i} = (u_i, u_{i+1}, \ldots, u_{2t}, v_1, v_2, \ldots, v_i, u_i)$ and $C_{v_i} = (v_i, v_{i+1}, \ldots, v_{2t}, u_1, u_2, \ldots, u_i, v_i)$ be $(2t+1)-$cycles. For any $u_i \in U = \{u_1, u_2, \ldots, u_{2t}\}$, if $w \in V(C_{u_i}) \backslash \{u_i\}$, the shorter segment between u_i and w on C_{u_i} is a rainbow $u_i - w$ geodesic in G; if $w = u_j \in V(G) \backslash V(C_{u_i})$, the rainbow $u_i - w$ geodesic in G contained in C_w; if $w = v_j \in V(G) \backslash V(C_{u_i})$, the shorter segment between u_i and w on C_w is a rainbow $u_i - w$ geodesic in G. Hence there exists a rainbow geodesic connecting u_i and any other vertex in G. With a similar argument to that of u_i, the above result also holds for any $v_i \in V = \{v_1, v_2, \ldots, v_{2t}\}$. Thus G contains a rainbow $x - y$ geodesic for any two vertices x and y of G, that is, $rc(G) \leq src(G) \leq t$.

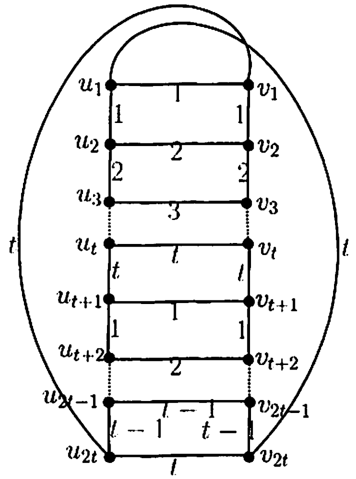

Figure 2: $G = M_h$, where $h = 2t$.

Since diam(G)= t, it follows that $rc(G) = src(G) = t$.

Case 2: h is odd. Let $h = 2t + 1$, then $2h = 4t + 2$.

Define a colouring C of the graph G by

$$C(e) = \begin{cases} i & \text{if } e = u_iu_{i+1}\ ,\ v_iv_{i+1} \text{ and } u_iv_i \quad \text{for } 1 \leq i \leq t+1, \\ i-t-1 & \text{if } e = u_iu_{i+1},\ v_iv_{i+1} \text{ and } u_iv_i \quad \text{for } t+2 \leq i \leq 2t, \\ t & \text{if } e = u_{2t+1}v_1,\ v_{2t+1}u_1 \text{ and } u_{2t+1}v_{2t+1}. \end{cases}$$

Let $C_{u_i} = (u_i, u_{i+1}, \ldots, u_{2t+1}, v_1, v_2, \ldots, v_i, u_i)$ and $C_{v_i} = (v_i, v_{i+1}, \ldots, v_{2t+1}, u_1, u_2, \ldots, u_i, v_i)$ be $(2t+2)-$cycles. Applying the same method of Case 1, for any two vertices x and y of G, there is a rainbow $x - y$ geodesic in C_{u_i} or C_{v_i}. Hence $rc(G) \leq src(G) \leq t + 1$. Since diam($G$)= t+1, it follows that $rc(G) = src(G) = t + 1$.

Combining Case 1 and Case 2, we have $rc(G) = src(G) = \lceil \frac{h}{2} \rceil$. This completes the proof of the theorem. □

3 Proof of Theorem 1.2

The cyclic group Z_n is the group of order n with generator c that satisfies $c^n = 1$. Let Γ be a finite group. If $a \in \Gamma$, the subgroup $\langle a \rangle$ is called the cyclic subgroup generated by a. We denote the number of elements of $\langle a \rangle$ by $|a|$. *The dihedral group* D_{2n} is the group of order $2n$ with generators

a and b that satisfy $a^n = b^2 = 1$ and $b^{-1}ab = a^{-1}$. Now we show three useful and easy lemmas that are needed in order to establish the proof of Theorem 1.2.

Lemma 3.1. *If p is an odd prime, then every group Γ of order $2p$ is isomorphic to either the cyclic group Z_{2p} or the dihedral group D_{2p}.*

Lemma 3.2. *Let Γ be a finite group. Then Γ is isomorphic to the dihedral group if and only if Γ is generated by two elements of order 2.*

If S is a minimal generating set of D_{2p}, then either $\bar{S} = \{ba^i, ba^j\}$, where $0 \le i \ne j \le p-1$; or $\bar{S} = \{a^i, a^{p-i}, ba^j\}$, where $1 \le i \le p-1$ and $0 \le j \le p-1$. If S is a minimal generating set of $Z_{2p} = \langle c \rangle$, then either $\bar{S} = \{c^i, c^{2p-i}\}$, where $(i, 2p) = 1$ and $1 \le i \le 2p-1$; or $\bar{S} = \{c^p, c^j, c^{2p-j}\}$, where $(j, 2p) = 2$ and $2 \le j \le 2p-2$. The following lemma is immediate.

Lemma 3.3. *(i) Let $G_1 = \mathrm{Cay}(D_{2p}, \bar{S_1})$ and $G_2 = \mathrm{Cay}(D_{2p}, \bar{S_2})$. For $\bar{S_1} = \{a, a^{p-1}, b\} \ne \bar{S_2} = \{a^i, a^{p-i}, ba^j\}$, we have $G_1 \cong G_2$.*

(ii) Let $G_1 = \mathrm{Cay}(Z_{2p}, \bar{S_1})$ and $G_2 = \mathrm{Cay}(Z_{2p}, \bar{S_2})$. For $\bar{S_1} = \{c^p, c^2, c^{2p-2}\} \ne \bar{S_2} = \{c^p, c^j, c^{2p-j}\}$, we have $G_1 \cong G_2$.

With the above lemmas established, we are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Applying the above results, if S is a minimal generating set of Γ, then $|\bar{S}| = 2$ or 3.

(i) Since $\Gamma = \langle \bar{S} \rangle$, we must have G is a cycle. Therefore, $rc(G) = src(G) = p$.

(ii) If $\Gamma \cong D_{2p}$, we only need to show that $rc(G) = src(G) = \frac{p+1}{2}$ by Lemma 3.3, where $G = \mathrm{Cay}(D_{2p}, \bar{S})$ and $\bar{S} = \{a, a^{p-1}, b\}$. We shall consider the following bijection.

$$\theta : L_p \to G$$

such that $\theta(u_i) = a^{i-1}$ and $\theta(v_i) = ba^{p-i+1}$. Let $u, v \in V(L_p)$ be any two adjacent vertices. If $u = u_i$ and $v = u_{i+1}$, then $\theta(u)^{-1}\theta(v) = a \in \bar{S}$; If $u = v_i$ and $v = v_{i+1}$, then $\theta(u)^{-1}\theta(v) = a^{i-1-p}b^{-1}ba^{p-i+2} = a^{p-1} \in \bar{S}$; If $u = u_i$ and $v = v_i$, then $\theta(u)^{-1}\theta(v) = a^{1-i}ba^{p-i+1} = ba^{2p} = b \in \bar{S}$. Thus $\theta(u) \sim \theta(v)$, that is, θ is an isomorphism between L_p and G. We have $rc(G) = src(G) = \frac{p+1}{2}$ by applying Theorem 1.1.

If $\Gamma \cong Z_{2p}$, we also obtain $G \cong L_p$ by the similar method. Hence $rc(G) = src(G) = \frac{p+1}{2}$. This completes the proof of the theorem. □

Remark. Let $G = \mathrm{Cay}(Z_{2p}, \bar{S})$ be a connected graph. If $S = \{c^p, c^j\}$ is a minimal generating set of Z_{2p}, where $|c^j| = p$ and $2 \le j \le 2p-2$, then $rc(G) = src(G) = \frac{p+1}{2}$ by Theorem 1.2. If the above open problem is true, we must have $rc(G) = src(G) = \frac{p+1}{2} + 1$, a contradiction. Hence the problem is not true for some special cases.

We conclude this section with the following lemma.

Lemma 3.4. *Let $G_1 = \mathrm{Cay}(Z_4, \bar{S}_1)$ and $G_2 = \mathrm{Cay}(Z_2 \times Z_2, \bar{S}_2)$.*

(i) If S_1 is a minimal generating set of Z_4, then $rc(G_1) = src(G_1) = 2$. Otherwise, $rc(G_1) = src(G_1) = 1$.

(ii) If S_2 is a minimal generating set of $Z_2 \times Z_2$, then $rc(G_2) = src(G_2) = 2$. Otherwise, $rc(G_2) = src(G_2) = 1$.

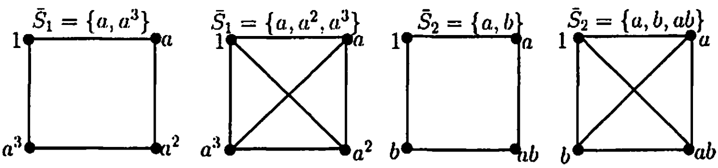

Figure 3: $G_1 = \mathrm{Cay}(Z_4, \bar{S}_1)$ and $G_2 = \mathrm{Cay}(Z_2 \times Z_2, \bar{S}_2)$.

Proof. Let $Z_4 = \langle a \rangle$ and $Z_2 \times Z_2 = \{1, a, b, ab\}$. In [8], It was shown that if S is a minimal generating set of an abelian group Γ and every element $s \in \bar{S}$ has an *even order*, then $rc(\mathrm{Cay}(\Gamma, \bar{S})) = src(\mathrm{Cay}(\Gamma, \bar{S})) = \sum_{s \in S} \frac{|s|}{2}$. If $S_1(S_2)$ is a minimal generating set of $Z_4(Z_2 \times Z_2)$, then $rc(G_1) = src(G_1) = rc(G_2) = src(G_2) = 2$. Otherwise, we have $\bar{S}_1 = \{a, a^2, a^3\}$ and $\bar{S}_2 = \{a, b, ab\}$. Hence $G_1 \cong G_2 \cong K_4$ are cliques. It follows that $rc(G_1) = src(G_1) = rc(G_2) = src(G_2) = 1$(see Figure 3). □

Let $G = \mathrm{Cay}(\Gamma, \bar{S})$ be a connected graph of order $2p$. If $\bar{S}$ contains a minimal generating set S^* of Γ such that $|\bar{S}^*| = 3$, then $rc(G) \leq rc(\mathrm{Cay}(\Gamma, \bar{S}^*)) = \frac{p+1}{2}$ by Theorem 1.2. Otherwise, $\bar{S}$ must contain a minimal generating set S^{**} of Γ such that $|\bar{S}^{**}| = 2$, then $rc(G) \leq rc(\mathrm{Cay}(\Gamma, \bar{S}^*))$ $= p$ by Theorem 1.2. Applying Lemma 3.1, every group Γ of order $2p$ is isomorphic to either the cyclic group Z_{2p} or the dihedral group D_{2p}. Since every group Γ of order p^2 (p is a prime) is abelian, the group of order 4 is isomorphic to either Z_4 or $Z_2 \times Z_2$. Combining Theorem 1.2 and Lemma 3.4, the (strong) rainbow connection numbers of Cayley graphs of order $2p$ (p is a prime) are completely investigated.

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