On the Bounds for Sizes of Least Common Multiples of Several pairs of graphs

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Abstract

G. Chartrand. et. al [3] define a graph G without isolated vertices to be a least common multiple of two graphs G_1 and G_2, if G is a graph of minimum size such that G is G_1 decomposable and G_2 decomposable. A bi-star $B_{m,n}$ is a caterpillar with spine length one. In this paper, we discuss a good lower bound for $lcm(B_{m,n}, G)$ where G is a simple graph. We also investigate $lcm(B_{m,n}, rK_2)$ and a good lower bound and an appropriate upper bound for $lcm(B_{m,n}, P_{r+1})$ for all $m \geq 1, n \geq 1$ and $r \geq 1$.

1 Introduction

In this paper we only consider simple graphs. A graph G is decomposable into the sub graphs $G_1, G_2, G_3, \cdots, G_n$ of G if no G_i for $(i = 1, 2, 3, \cdots, n)$ has isolated vertices and the edge set $E(G)$ of G can be partitioned into the sub sets $E(G_1), E(G_2), \cdots,$

ARS COMBINATORIA 131(2017), pp. 331-346

$E(G_n)$. If $G_i \cong H$ for each i, then G is H-decomposable. This concept was studied and this notation was introduced by Harary, Robinson and Wormald [5]. In [2] G. Chartrand proved that two graphs can have an arbitrarily large number of least common multiples, and all graphs G for which G and P_3 (and G, $2K_2$) have a unique least common multiple are characterized. In this paper, we discuss the Bounds for Sizes of Least Common Multiples of Several pairs of graphs.

2 Notations

The order of a graph G is $p(G)$ and the size of the graph is $q(G)$. Vertex set and edge set of the graph G is denoted by $V(G)$ and $E(G)$ respectively. A star on $n+1$ vertices is denoted by $K_{1,n}$. A path, a cycle, and a complete graph on n vertices are denoted by P_n , C_n and K_n respectively. The minimum and maximum degree of the vertices of a graph G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. In any graph G, if $\delta(G)$=$\Delta(G)$ =r(say), that is $deg(v) = r$ for every vertex v of G, then the graph G is r - regular. A spanning cycle of a graph is called a Hamilton cycle. A graph is Hamiltonian if it contains a Hamilton cycle. A Hamiltonian graph G is regular Hamilton if all the vertices of G have equal degree. If a graph G consists of $r(\geq 2)$ disjoint copies of a graph K_2, then we write $G = rK_2$. If the non pendant vertices of a Bi-star have degrees $m+1$ and $n+1$, we denote this bi-star by $B_{m,n}$. The set of all least common multiples of two graphs G_1 and G_2 is denoted by LCM(G_1,G_2) and the size of any such graph is denoted by $lcm(G_1, G_2)$. $V_r = \{v \in V(G) : deg(v) = r\}$ and $W_r = V_r \cap W$ where $W \subset V(G)$.

3 Preliminary Results

Theorem 3.1. [3, *pp.*96]. *For any two graphs G_1 and G_2, $lcm(G_1, G_2) \geq lcm(q(G_1), q(G_2))$.*

Theorem 3.2. [6, *pp.*186]. *Let G_1 and G_2 be two bipartite graphs then $lcm(G_1, G_2) \leq q(G_1).q(G_2)$.*

Corollary 3.3. [6, *pp.*186]. *Let G_1 and G_2 be two bipartite graphs. If $q(G_1)$ and $q(G_2)$ are relatively prime, then $lcm(G_1, G_2) = q(G_1).q(G_2)$.*

Theorem 3.4. [7] *A graph G of size $s(m+n+1)$ with $m \geq 1$, $n \geq 1$ and $s \geq 3$ is $B_{m,n}$ decomposable, if it satisfies the following conditions.*
(i) A set of vertices $W \subset V(G)$ with $|W| = s$ and $\langle W \rangle = C_s$.
(ii) The number of edges incident with each vertex $v \in W$ is $m+n+2$.

Corollary 3.5. [7] *A graph G of size $s(m+n+1)$ with $m \geq 1$, $n \geq 1$ and $s \geq 3$ is $B_{m,n}$ decomposable, if it satisfies the following conditions.*
(i) $W \subset V(G)$ with $|W| = s$ and $\langle W \rangle$ is $2r$- regular Hamilton decomposable, where $m+n+1 \geq r$.
(ii) The number of edges incident with each vertex $v \in W$ is $m+n+1+r$.

Theorem 3.6. [7] *A graph G of size $s(m+n+1)$ with $m \geq 1$, $n \geq 1$ and $s \geq 3$ is $B_{m,n}$ decomposable, if it satisfies the following conditions.*
(i) $W \subset V(G)$ with $|W| = 2s$ and $\langle W \rangle = C_{2s} = v_1, u_1, v_2, u_2, v_3, \cdots, v_s, u_s, v_1$.
(ii) For each i, $deg(v_i) = m+2$, and $deg(u_i) = n+1$
(iii) For each i, the vertices v_i and u_i has no common neighbor in G.

Lemma 3.7. [4, *pp.*181]. *A graph of size at least two which has an edge adjacent to all other edges has no disconnected divisor.*

Theorem 3.8. [1, *pp.*222]. *For every graph G and every $t > 1$, $tK_2|G$ if and only if $t|e(G)$ and $\chi'(G) \leq \frac{e(G)}{t}$.*

4 Bounds for Least Common Multiples of Bi-star Versus a Simple Graph

Any two graphs has a least common multiple follows directly from the following result of Wilson [2], " For every graph F without isolated vertices and having size q, there exists a positive integer N such that if (i) $n \geq N$, (ii)$q|\frac{n(n-1)}{2}$ and (iii) $d|(n-1)$, where d =gcd($deg(v)$: $v \in V(F)$), then K_n is F- decomposable ". In general, upper bound obtained by the above result would be extremely large and no good general upper bound for $lcm(G_1, G_2)$ is known. Here we investigate a good lower bound for $lcm(B_{m,n}, G)$, where G is any simple graph.

Theorem 4.1. *For all integers $m \geq 1$, $n \geq 1$ and $(m+n+1) = k_1 d$, $lcm(B_{m,n}, G) \geq \lceil \frac{m+1}{\Delta(G)k_1} \rceil \frac{q(G)}{d}(m+n+1)$ where $d = (q(G), m+n+1)$.*

Proof. By Theorem 3.1, $lcm(B_{m,n}, G) = s\frac{q(G)(m+n+1)}{d}$ where s is a positive integer. Let H be a graph of minimum size $s\frac{q(G)(m+n+1)}{d}$ such that H is $B_{m,n}$ and G decomposable then H can be decomposed into $s\frac{(m+n+1)}{d}$ copies of G. But each of these G contributes a degree at most $\Delta(G)$ to each of the vertices of H. Thus we have $s\frac{(m+n+1)}{d}\Delta(G) \geq \Delta(H)$.
Since H is $B_{m,n}$ decomposable, $\Delta(H) \geq m+1$. So we have $s\frac{(m+n+1)}{d}\Delta(G) \geq m+1 \Rightarrow s \geq \frac{d(m+1)}{\Delta(G)(m+n+1)} = \frac{d(m+1)}{\Delta(G)k_1 d} = \frac{(m+1)}{\Delta(G)k_1} \Rightarrow s \geq \lceil \frac{m+1}{\Delta(G)k_1} \rceil$.
Hence $lcm(B_{m,n}, G) \geq \lceil \frac{m+1}{\Delta(G)k_1} \rceil \frac{q(G)}{d}(m+n+1)$. □

Theorem 4.2. *For any graph G with $\Delta(G) = 2$ and $(m+n+1) = k_1 d$, then $lcm(B_{m,n}, G) \geq \lceil \frac{d+1}{4} \rceil \frac{q(G)}{d}(m+n+1)$ where $d = (q(G), m+n+1)$.*

Proof. Without loss of generality we may choose $m \geq n$, since $B_{m,n}$ is isomorphic to $B_{n,m}$. Also $m \geq n \Rightarrow 2m+1 \geq m+n+1 \Rightarrow m \geq \frac{k_1d-1}{2} \Rightarrow m+1 \geq \frac{k_1d+1}{2} = \frac{k_1d+k_1}{2} + \frac{1-k_1}{2} \Rightarrow \frac{m+1}{2k_1} \geq \frac{d+1}{4} + \frac{1-k_1}{4k_1} \Rightarrow \lceil \frac{m+1}{2k_1} \rceil \geq \lceil \frac{d+1}{4} \rceil$, since $\left| \frac{1-k_1}{4k_1} \right| < \frac{1}{4}$. Hence by Theorem 4.1, $lcm(B_{m,n}, G) \geq \lceil \frac{m+1}{\Delta(G)k_1} \rceil \frac{q(G)}{d}(m+n+1) = \lceil \frac{m+1}{2k_1} \rceil \frac{q(G)}{d}(m+n+1) \geq \lceil \frac{d+1}{4} \rceil \frac{q(G)(m+n+1)}{d}$. □

Corollary 4.3. *Let* $(\frac{2m}{k_1} - d) + \frac{2}{k_1} \leq x \leq (\frac{2m}{k_1} - d) + 4$ *be an integer such that* $d + x \equiv 0 (mod\, 4)$, *where* $d = (q(G), m+n+1)$ *and* $(m+n+1) = k_1 d$, *then for any graph* G *with* $\Delta(G) = 2$, $lcm(B_{m,n}, G) \geq \frac{d+x}{4} \frac{q(G)}{d}(m+n+1)$.

Proof. We note that, the closed interval $[\frac{2m}{k_1} + \frac{2}{k_1}, \frac{2m}{k_1} + 4]$ contains a unique integer $\equiv 0 (\text{mod}\, 4)$. The integer we denote by $d + x$. Since $(\frac{2m}{k_1} - d) + \frac{2}{k_1} \leq x \leq (\frac{2m}{k_1} - d) + 4 \Rightarrow \frac{2m}{k_1} + \frac{2}{k_1} \leq d + x \leq \frac{2m}{k_1} + 4$ $\Rightarrow \frac{2(m+1)}{k_1} \leq d + x \leq \frac{2m+2}{k_1} + 4 - \frac{2}{k_1} \Rightarrow \frac{(m+1)}{2k_1} \leq \frac{d+x}{4} \leq \frac{m+1}{2k_1} + 1 - \frac{1}{2k_1}$ $\Rightarrow \lceil \frac{m+1}{2k_1} \rceil = \frac{d+x}{4}$, since x is an integer such that $d + x \equiv 0 (\text{mod}\, 4)$. Hence by Theorem 4.1, $lcm(B_{m,n}, G) \geq \frac{d+x}{4} \frac{q(G)}{d}(m+n+1)$. □

Theorem 4.4. *If* G *is bipartite, then* $lcm(B_{m,n}, G) \leq q(G)(m+n+1)$.

Proof. The theorem follows from Theorem 3.2, since $B_{m,n}$ is bipartite. □

Theorem 4.5. *If* G *is bipartite and* $gcd(q(G), m+n+1) = 1$, *then* $lcm(B_{m,n}, G) = q(G)(m+n+1)$.

Proof. Result follows directly by Corollary 3.3. □

5 Bounds for Least Common Multiple of Paths Versus Bi-stars

In this section, we discuss a good lower bound and an appropriate upper bound for $lcm(B_{m,n}, P_{r+1})$ for all integers

$m \geq 1$, $n \geq 1$ and $r \geq 1$. By Corollary 4.3 we have the following results.

Theorem 5.1. *Let* $(\frac{2m}{k_1} - d) + \frac{2}{k_1} \leq x \leq (\frac{2m}{k_1} - d) + 4$ *be an integer such that* $d + x \equiv 0 (mod\ 4)$, *where* $d = gcd(m+n+1, r)$ *and* $(m+n+1) = k_1 d$, *then* $lcm(B_{m,n},\ P_{r+1}) \geq \frac{d+x}{4}\frac{r(m+n+1)}{d}$.

Proposition 5.2. *Let* r *and* $m+n+1$ *be even integers with* $r \geq 2$, $m \geq 1$ *and* $n \geq 1$. *Then there exists a graph of size* $\frac{r(m+n+1)}{2}$ *which is both* $B_{m,n}$ *and* P_{r+1} *decomposable.*

Proof. Every simple connected graph of even size is P_3 decomposable. So $B_{m,n}$ is P_3 decomposable.
Thus the statement is true if r=2.
If $r \geq 4$, then our aim is to construct a graph G of size $\frac{r(m+n+1)}{2}$ such that G is $B_{m,n}$ and P_{r+1} decomposable.
Let W=\{ $v_1, u_1, v_2, u_2, \cdots v_{\frac{r}{2}},\ u_{\frac{r}{2}}$\} be a set of r vertices. Given $m+n+1$ is even, so one of them say m is even and n is odd.
We add new vertices $a_{i,l}$ and $b_{j,l}$ for $i = 1, 2, 3, \cdots, \frac{m}{2} - 1$; $j = 1, 2, 3, \cdots, \frac{n-1}{2}$; $l = 1, 2, 3, \cdots, \frac{r}{2}, \frac{r}{2} + 1$. Then for each i, construct a path P^i of length r as $a_{i,1}, v_1, a_{i,2}, v_2, \cdots a_{i,\frac{r}{2}}, v_{\frac{r}{2}}, a_{i,\frac{r}{2}+1}$ and for each j, form another path Q^j of length r as $b_{j,1}, u_1, b_{j,2}, u_2, \cdots, b_{j,\frac{r}{2}}, u_{\frac{r}{2}}, b_{j,\frac{r}{2}+1}$. By adding new vertices $a_1, a_2, a_3, \cdots, a_{\frac{r}{2}}$, form a path P^1 of length r as $a_1, v_1, u_1, v_2, u_2, \cdots v_{\frac{r}{2}}, u_{\frac{r}{2}}$ and form another path P^2 of length r as $a_1, v_2, a_2, v_3, a_3, \cdots, a_{\frac{r}{2}}, v_1, u_{\frac{r}{2}}$.
Let $G = P^1 \cup P^2 \cup \bigcup_{i=1}^{\frac{m}{2}-1} P^i \cup \bigcup_{j=1}^{\frac{n-1}{2}} Q^j$. Size of G is $(2+\frac{m}{2} - 1 + \frac{n-1}{2})r = \frac{r(m+n+1)}{2}$.
By construction itself G is P_{r+1} decomposable. According to our construction, $\langle V(G) \setminus W \rangle$ is an empty graph. Degree of v_k $=4 + (\frac{m}{2} - 1)2 = m + 2$ and degree of $u_k$$=2 + (\frac{n-1}{2})2 = n + 1$ for $k = 1, 2, 3, \cdots \frac{r}{2}$. In G it can be seen that, each vertex $v_k \in W$ is adjacent to exactly m vertices $\notin W$ and each vertex $u_k \in W$ is adjacent to exactly $n-1$ vertices $\notin W$ and $\langle W \rangle$ is a cycle C_r: $v_1, u_1, v_2, u_2, \cdots v_{\frac{r}{2}},\ u_{\frac{r}{2}}, v_1$.
Also for each $k, 1 \leq k \leq \frac{r}{2}$, the adjacent vertices v_k, u_k have no common neighbor in G. So G satisfies all the conditions of Theorem 3.6. Hence the graph G is $B_{m,n}$ decomposable. ☐

Illustration 1.

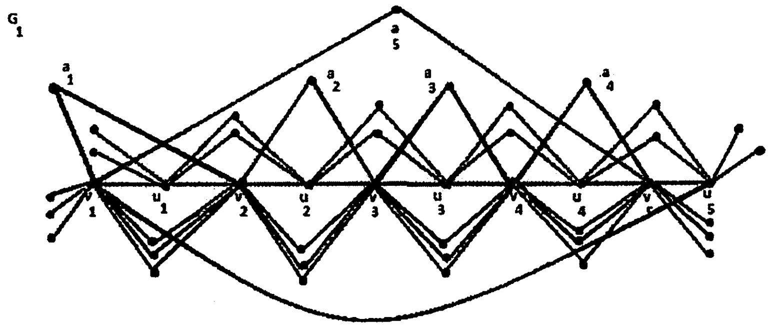

Figure 1:

The graph in Figure 1 is an example of a graph with size 70 which is P_{11} decomposable and $B_{8,5}$ decomposable.

Proposition 5.3. *Let r and t be integers with $r > 2$ and $t \geq \lceil \frac{r}{2} \rceil$, then there exists a P_{r+1} decomposable graph G of size tr such that $|W_r|$=t and the subgraph induced by $V(G) \setminus W_r$ is an empty graph.*

Proof. Let $W = v_1, v_2, v_3, \cdots, v_t$ be a set of t vertices. We consider two cases r is even and odd separately. We construct the required graph G in such a way that every edge in G, has exactly one end point in W and $\langle V(G) \setminus W \rangle$ is an empty graph.
Case 1: When $r > 2$ is even.
By adding new vertices $a_{i,j}$ for $i = 1, 2, 3, \cdots, t$ and $j = 1, 2, 3, \cdots, \frac{r}{2} + 1$ then for each i, construct a path P^i of length r as $a_{i,1}, v_i, a_{i,2}, v_{i+1}, a_{i,3}, v_{i+2}, a_{i,4} \cdots, a_{i,\frac{r}{2}}, v_{i+\frac{r}{2}-1}, a_{i,\frac{r}{2}+1}$ and the addition in subscript is taken modulo t.
Let $G = \cup_{i=1}^{t} P^i$. For constructing a path we had taken $\frac{r}{2}$ vertices from W cyclically in a given order. So for constructing t paths we include a vertex exactly $\frac{r}{2}$ times from W. Thus the number of paths passing through each vertex v_i is $\frac{r}{2}$ and in G, no vertex v_i is a pendant vertex of a path. Hence for each i,

$1 \leq i \leq t$, degree of each vertex $v_i \in W$ in G is r.
Case 2: When r is is odd.
By adding new vertices $b_{i,k}$ for $i = 1, 2, 3, \cdots, t$ and $k = 1, 2, 3, \cdots, \frac{r+1}{2}$, then for each i, construct a path Q^i of length r as $b_{i,1}$, v_i, $b_{i,2}$, v_{i+1}, $b_{i,3}$, v_{i+2}, $b_{i,4}$ $\cdots$, $b_{i,\frac{r+1}{2}}$, $v_{i+\frac{r-1}{2}}$.
Let $G = \cup_{i=1}^{t} Q^i$. For constructing a path we had taken $\frac{r+1}{2}$ vertices from W cyclically in a given order. So for constructing t paths we include a vertex exactly $\frac{r+1}{2}$ times from W.
In G it can be seen that, each vertex v_i is a pendant vertex of exactly one path. Thus the number of paths passing through each vertex v_i is $\frac{r-1}{2}$ and exactly one path ends on each vertex v_i. So for each i, $1 \leq i \leq t$, degree of each vertex $v_i \in W$ in G= $\frac{r-1}{2}$ 2 +1=r. In both case, G is P_{r+1} decomposable. Also size of G is tr and $|W|$= $|W_r|$=t. □

Illustration 2.

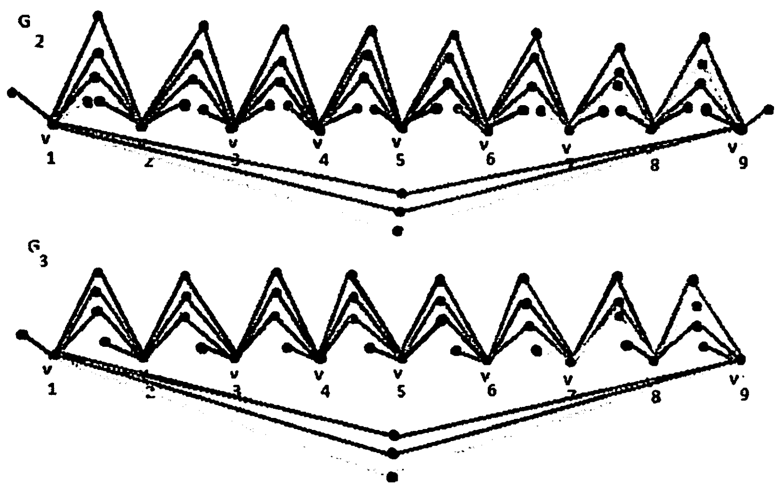

Figure 2:

The graph G_2 in Figure 2 is an example of P_9 decomposable graph of size 72. The graph G_3 in Figure 2 is an example of G_3 is a P_8 decomposable graph of size 63.

Proposition 5.4. *For every positive integer r, there exists a graph of size $r(m+n+1)$ which is both $B_{m,n}$ and P_{r+1} decomposable.*

Proof. $B_{m,n}$ and P_{r+1} are bi-parttite graphs. So the result follows directly by Theorem 3.1 and Theorem 3.2. □

Proposition 5.5. *Let $d = gcd(r, m+n+1)$ be an odd integer, then for all $r \geq 1$ there exists a graph of size $\frac{d+1}{2}\frac{r(m+n+1)}{d}$ which is both $B_{m,n}$ and P_{r+1} decomposable.*

Proof. If $d = 1$, then $\frac{d+1}{2}\frac{r(m+n+1)}{d} = r(m+n+1)$. So the result is true by Proposition 5.4.
If $d > 1$, then construct a graph G of size $\frac{d+1}{2}\frac{r(m+n+1)}{d}$ in the following way. Let $W=\{v_1, v_2, \cdots, v_{\frac{d+1}{2}\frac{r}{d}}\}$ be a set of $\frac{d+1}{2}\frac{r}{d}$ vertices.
By division algorithm, let $m+n+1 = ar+b$, a and b are integers with $a \geq 0$, $0 \leq b < r$. Clearly $b \equiv 0 (\text{mod } d)$.
By adding new vertices $(a_{i,j})_s$, for $i = 1, 2, 3, \cdots \frac{d+1}{2}\frac{r}{d}$, $j = 1, 2, 3, \cdots, \lceil \frac{r}{2} \rceil$ and $s = 1, 2, 3, ..., a$, then by fixing a value for s, construct $\frac{d+1}{2}\frac{r}{d}$ paths P_{r+1} of length r as in Proposition 5.3. Let this graph be H_s and degree of each vertex v_i in $H_s = r$.
Applying these process a times by taking $s = 1, 2, 3, ..., a$, we get $a\frac{d+1}{2}\frac{r}{d}$ paths P_{r+1} of length r.
Let this graph be H, then $H = \cup_{s=1}^{a} H_s$. Clearly H is P_{r+1} decomposable and size of H is $ar\frac{d+1}{2}\frac{r}{d}$.
In H, degree of each vertex in $v_i \in W$ is ar and the subgraph induced by $V(H) \setminus W$ is an empty graph.
We can form $\lceil \frac{d+1}{4}\frac{r}{d} \rceil - 1$ edge disjoint spanning cycles of the complete graph on the $\frac{d+1}{2}\frac{r}{d}$ vertices of W (which are also referred to as spanning cycles of W for simplicity).
Select one spanning cycles of W and then for each k, $1 \leq k \leq \frac{d+1}{2}$, construct a path P^k as $c_{k.1}$, $v_{(k-1)\frac{r}{d}+1}$, $v_{(k-1)\frac{r}{d}+2}$, $v_{(k-1)\frac{r}{d}+3}$, $\cdots$, $v_{k\frac{r}{d}+1}$, $c_{k.2}$, $v_{k\frac{r}{d}+2}$, $c_{k.3}$, $\cdots$, $v_{(k-1)\frac{r}{d}+\frac{d+1}{2}\frac{r}{d}}$, $c_{k.\frac{d-1}{2}\frac{r}{d}+1}$, by adding new vertices $c_{k.t}$ for $1 \leq t \leq \frac{d-1}{2}\frac{r}{d}+1$, where the addition in subscript is taken modulo $\frac{d+1}{2}\frac{r}{d}$.

Length of each path $P^k = \frac{r}{d} + 2\frac{d-1}{2}\frac{r}{d} = r$. Let this graph be F_1, then $F_1 = \cup_{k=1}^{\frac{d+1}{2}} P^k$. The number of vertices v_i occurs in a path P^k is $\frac{r}{d} + \frac{d-1}{2}\frac{r}{d} = \frac{d+1}{2}\frac{r}{d}$ and v_i is not a pendant vertex of P^k for any k. So the number of paths passing through each vertex v_i is $\frac{d+1}{2}$. Thus the degree of each vertex $v_i \in W$ in F_1 is $d+1$.
A similar construction with respect to another edge disjoint spanning cycles of W leads to a similar graph.
It is possible to select $\frac{b}{d}$ spanning cycles of W, since $\frac{d+1}{2}\frac{r}{d} > 2\frac{b}{d} \geq 2\frac{b}{d} + 1$. So by selecting $\frac{b}{d}$ spanning cycles of W, we construct a P_{r+1} decomposable graph say $F = \cup_{x=1}^{\frac{b}{d}} F_x$ of size $r\frac{d+1}{2}\frac{b}{d}$, where F_x is the graph similar to F_1.
In F, degree of each vertex $v_i \in W$ is $\frac{b}{d}(d+1)$. Let $G = H \cup F$.
Size of G is $(a\frac{d+1}{2}\frac{r}{d} + \frac{d+1}{2}\frac{b}{d})r = \frac{d+1}{2}\frac{r}{d}[ar+b] = \frac{d+1}{2}\frac{r(m+n+1)}{d}$.
By construction itself G is P_{r+1} decomposable.
In H, it can be seen that, each vertex $v_i \in W$ is adjacent to exactly ar vertices $\notin W$. In F it can be seen that, each vertex $v_i \in W$ is adjacent to exactly $(d-1)\frac{b}{d}$ vertices $\notin W$ and $C_{\frac{d+1}{2}\frac{r}{d}}$ is a sub graph of F. So in G, it can be seen that, each vertex $v_i \in W$ is adjacent to exactly $ar + (d-1)\frac{b}{d}$ vertices $\notin W$ and $deg(v_i) = ar + (d+1)\frac{b}{d}$.
Clearly, $\langle W \rangle$ is a $\frac{2b}{d}$ regular Hamilton decomposable graph.
Hence G is $B_{m,n}$ decomposable by Corollary 3.5. □

Illustration 3.
In figure 3, $r = 15, m+n+1 = 20, d = 5$ and $\frac{d+1}{2}\frac{r}{d} = 9$. The graph G_4 is an example of P_{16} and $B_{m,n}$ decomposable. That is G_4 is $B_{m,n} = \{B_{10,9}, B_{11,8}, B_{12,7}, B_{13,6}, B_{14,5}, B_{15,4}, B_{16,3}, B_{17,2}, B_{18,1}\}$ decomposable.

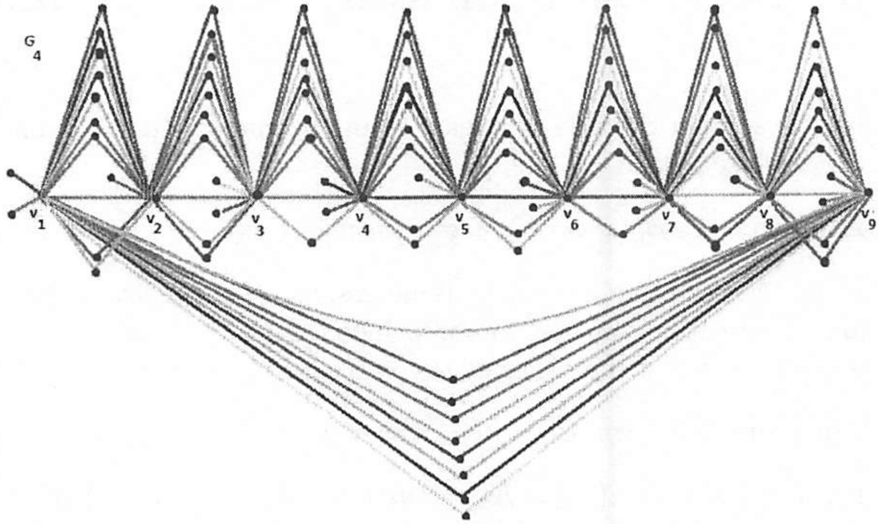

Figure 3:

Theorem 5.6. *Let* $d = (m+n+1, r)$, *then for all* $r \geq 1$,

$$lcm(B_{m,n},\ P_{r+1}) \leq \begin{cases} \frac{r(m+n+1)}{2}, \text{ if } d \text{ is even} \\ \frac{d+1}{2}\frac{r}{d}(m+n+1), \text{ if } d \text{ is odd} \end{cases}.$$

Proof. If d is even, then r should be even. So by Proposition 5.2, there exists a graph of size $\frac{r(m+n+1)}{2}$ which is both $B_{m,n}$ and P_{r+1} decomposable. If d is odd, then by Proposition 5.5, there exists a graph of size $\frac{d+1}{2}\frac{r(m+n+1)}{d}$ which is both $B_{m,n}$ and P_{r+1} decomposable. These complete the proof of the theorem. □

Corollary 5.7. *Let* $(\frac{2m}{k_1} - d) + \frac{2}{k_1} \leq x \leq (\frac{2m}{k_1} - d) + 4$ *be an integer such that* $d + x \equiv 0 (mod 4)$, *where* $d = gcd(m+n+1, r)$ *and* $(m+n+1) = k_1 d$, *then for all positive integers* m, n *and* r,

$$\frac{d+x}{4}\frac{r(m+n+1)}{d} \leq lcm(B_{m,n},\ P_{r+1}) \leq \begin{cases} \frac{r}{2}(m+n+1), & \text{if } d \text{ is even;} \\ \frac{d+1}{2}\frac{r}{d}(m+n+1), & \text{if } d \text{ is odd.} \end{cases}$$

Proof. It follows directly by Corollary 5.1 and Theorem 5.6. □

6 Least Common Multiple of $B_{m,n}$ and rK_2.

In this section the size of a least common multiple of $B_{m,n}$ and rK_2 is determined.

Lemma 6.1. *$B_{m,n}$ is rK_2 decomposable if and only if $r = 1$.*

Proof. Clearly the condition is necessary. The sufficient part follows directly from Lemma 3.7, since the central edge of $B_{m,n}$ is adjacent to all the other edges of $B_{m,n}$ □

Theorem 6.2. *$lcm(B_{m,n}, rK_2)= r(m+n+1)$, if $r \leq 2$.*

Proof. Clearly $lcm(B_{m,n}, K_2)= (m+n+1)$. If $r = 2$, then by Lemma 6.1, $lcm(B_{m,n}, 2K_2) > (m+n+1)$. Also by Theorem 4.4, $lcm(B_{m,n}, 2K_2) \leq 2(m+n+1)$. Hence $lcm(B_{m,n}, 2K_2)=2(m+n+1)$. □

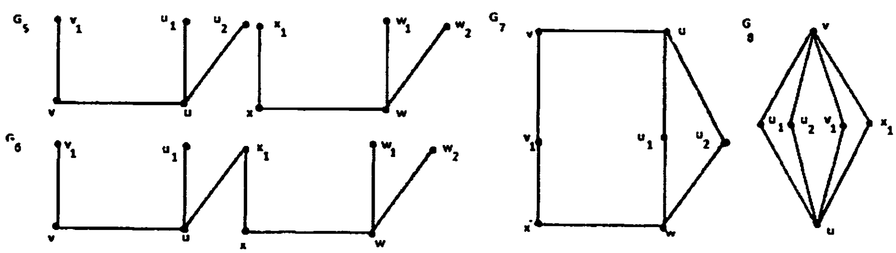

Figure 4:

Every least common multiple of two connected graphs is connected but least common multiple of a connected and a disconnected graph need not be connected. For example consider the graphs $B_{2,1}$ and $2K_2$.
By Theorem 3.2, $lcm(B_{2,1}, 2K_2)$=8. Clearly G_5, G_6, G_7 and G_8 are both $B_{2,1}$ and $2K_2$ decomposable graphs of size 8. So G_5, G_6, G_7 and G_8 are belongs to LCM($B_{2,1}$, $2K_2$) and all are

connected except G_5.
Now we turn our attention to $lcm(B_{m,n}, rK_2)$ for $r > 2$. We use the characterization of rK_2 decomposition of a graph G for finding a least common multiple of $B_{m,n}$ and rK_2. For this purpose it is necessary to find the edge chromatic number of $B_{m,n}$. The edge chromatic number $\chi'(B_{m,n}) = \Delta(B_{m,n}) = m+1$, since $m \geq n$. So $\chi'(rB_{m,n}) = m + 1$.

Theorem 6.3. *For any integer $t \geq 1$, $tB_{m,n}$ is rK_2 decomposable, if $t \equiv 0 (mod\ r)$.*

Proof. $\chi'(tB_{m,n}) = m + 1 < m + n + 1 \leq \frac{t(m+n+1)}{r} = \frac{e(tB_{m,n})}{r}$.
So the theorem follows directly from Theorem 3.8. □

Remark 6.4. *The converse of the above theorem need not be true.*

Theorem 6.5. *Let $d = gcd(m + n + 1, r)$. Then $tB_{m,n}$ is rK_2 decomposable if and only if t can be expressed in the form $t = \frac{kr}{d}$ where k is an integer with $k \geq \frac{d(m+1)}{m+n+1}$.*

Proof. Suppose that $t = \frac{kr}{d}$ where k is an integer with $k \geq \frac{d(m+1)}{m+n+1}$. Also $e(tB_{m,n}) = t(m + n + 1) = \frac{kr}{d}(m + n + 1) \Rightarrow e(tB_{m,n}) \equiv 0(\text{mod } r)$.
$\chi'(tB_{m,n}) = m + 1 \leq \frac{k(m+n+1)}{d} = \frac{kr(m+n+1)}{dr} = \frac{t(m+n+1)}{r} = \frac{e(tB_{m,n})}{r}$.
Hence $tB_{m,n}$ is rK_2 decomposable by Theorem 3.8. Conversely, if $tB_{m,n}$ is rK_2 decomposable, then we have, $r|t(m + n + 1)$.
$r|t(m+n+1) \Rightarrow \frac{r}{d}|t\frac{m+n+1}{d} \Rightarrow t \equiv 0(\text{mod } \frac{r}{d})$, since $gcd(\frac{m+n+1}{d}, \frac{r}{d}) = 1$. That is $t = k\frac{r}{d}$, for some positive integer k.
$rK_2|tB_{m,n} \Rightarrow \chi'(tB_{m,n}) \leq \frac{e(tB_{m,n})}{r} \Rightarrow (m + 1) \leq t\frac{m+n+1}{r} \Rightarrow m + 1 \leq k\frac{r}{d}\frac{m+n+1}{r} \Rightarrow k \geq \frac{d(m+1)}{m+n+1}$. □

Remark 6.6. $2(m+1) = m+m+2 \geq m+n+1+1 \Rightarrow \frac{m+1}{m+n+1} \geq \frac{1}{2} + \frac{1}{2(m+n+1)} \Rightarrow \frac{d(m+1)}{m+n+1} \geq \frac{d}{2} + \frac{d}{2(m+n+1)}$.
So if k satisfies the condition $k \geq \frac{d(m+1)}{m+n+1}$, then $k > \frac{d}{2} \Rightarrow k \geq \lceil \frac{d+1}{2} \rceil$.

Remark 6.7. *If $k < \frac{d(m+1)}{m+n+1}$, then $\frac{k}{d} < \frac{(m+1)}{m+n+1} < 1 \Rightarrow k < d$. There fore if $t \geq r$ and t satisfies the condition $t = \frac{kr}{d}$, then k satisfies the condition $k \geq \frac{d(m+1)}{m+n+1}$.*

Theorem 6.8. *Let $d = gcd(m+n+1, r)$, then $lcm(B_{m,n}, rK_2) \geq \lceil \frac{d(m+1)}{m+n+1} \rceil \frac{r}{d}(m+n+1)$.*

Proof. By Theorem 4.1, we have $lcm(B_{m,n}, rK_2) \geq \lceil \frac{m+1}{k_1 \Delta rK_2} \rceil \frac{r(m+n+1)}{d}$, where $m+n+1 = k_1 d$. By putting the values of $k_1 = \frac{m+n+1}{d}$ and $\Delta rK_2 = 1$, we get $lcm(B_{m,n}, rK_2) \geq \lceil \frac{d(m+1)}{(m+n+1)} \rceil \frac{r}{d}(m+n+1)$. □

Theorem 6.9. *Let $d = gcd(m+n+1, r)$, then $lcm(B_{m,n}, rK_2) = k\frac{r}{d}(m+n+1)$, where $k = \lceil \frac{d(m+1)}{m+n+1} \rceil$*

Proof. By Theorem 6.5, $tB_{m,n}$ where $t = \frac{kr}{d}$ with $k = \lceil \frac{d(m+1)}{m+n+1} \rceil$ is rK_2 decomposable. So $lcm(B_{m,n}, rK_2) \leq k\frac{r}{d}(m+n+1)$. By Theorem 6.8, $lcm(B_{m,n}, rK_2) \geq k\frac{r}{d}(m+n+1)$. Hence $lcm(B_{m,n}, rK_2) = k\frac{r}{d}(m+n+1)$, where $k = \lceil \frac{d(m+1)}{m+n+1} \rceil$. □

Remark 6.10. *Let $d = gcd(m+n+1, r)$, then the disconnected graph $\lceil \frac{d(m+1)}{m+n+1} \rceil \frac{r}{d}(m+n+1)B_{m,n} \in LCM\ (B_{m,n}, rK_2)$.*

7 Least Common Multiple of any Graph G and rK_2.

In this section the size of a least common multiple of of any graph G and rK_2 is determined.

Theorem 7.1. *For any graph G without isolated vertices and an integer $r \geq 1$, with $d = gcd(q(G), r)$, then $lcm(G, rK_2) \geq \lceil \frac{d\Delta(G)}{q(G)} \rceil \frac{r}{d} q(G)$.*

Proof. Same as Theorem 4.1. □

Theorem 7.2. *For any graph G without isolated vertices and an integer $r \geq 1$, with $d = gcd(q(G), r)$, then tG is rK_2 decomposable if and only if t can be expressed in the form $t = \frac{kr}{d}$ where k is an integer with $k \geq \frac{d\chi'(G)}{q(G)}$.*

Proof. Suppose that $t = \frac{kr}{d}$ where k is an integer with $k \geq \frac{d\chi'(G)}{q(G)}$.
$e(tG) = tq(G) = \frac{kr}{d}q(G) \Rightarrow e(tG) \equiv 0(\text{mod } r)$.
$\chi'(tG) = \chi'(G) \leq \frac{kq(G)}{d} = \frac{krq(G)}{dr} = \frac{tq(G)}{r} = \frac{e(tG)}{r}$.
Hence tG is rK_2 decomposable by Theorem 3.8.
Conversely, if tG is rK_2 decomposable, then $r|tq(G) \Rightarrow \frac{r}{d}|\frac{tq(G)}{d} \Rightarrow$

$t \equiv 0(\text{mod } \frac{r}{d})$, since $gcd(\frac{q(G)}{d}, \frac{r}{d}) = 1$. That is $t = k\frac{r}{d}$, for some positive integer k. By Theorem 3.8, $rK_2|tG \Rightarrow \chi'(tG) \leq \frac{e(tG)}{r} \Rightarrow \chi'(G) \leq t\frac{q(G)}{r} \Rightarrow \chi'(G) \leq k\frac{r}{d}\frac{q(G)}{r} \Rightarrow k \geq \frac{d\chi'(G)}{q(G)}$. □

Theorem 7.3. *For any graph G without isolated vertices and an integer $r \geq 1$, with $d = gcd(q(G), r)$, then $lcm(G, rK_2) = k\frac{r}{d}q(G)$, where $k = \lceil\frac{d\chi'(G)}{q(G)}\rceil$*

Proof. By Theorem 7.2, tG is rK_2 decomposable, where $t = \frac{kr}{d}$ with $k = \lceil\frac{d\chi'(G)}{q(G)}\rceil$. So $lcm(G, rK_2) \leq \lceil\frac{d\chi'(G)}{q(G)}\rceil\frac{r}{d}q(G)$.
Now consider any $H \in$ LCM (G, rK_2). Since we are considering only rK_2, we may assume the copies of G in H are vertex disjoint. Then it shows that $q(H) \geq \lceil\frac{d\chi'(G)}{q(G)}\rceil\frac{r}{d}q(G)$, by Theorem7.2. Hence $lcm(G, rK_2) = k\frac{r}{d}q(G)$, where $k = \lceil\frac{d\chi'(G)}{q(G)}\rceil$. □

Remark 7.4. *If G is biparttite, then $lcm(G, rK_2) = \lceil\frac{d\Delta(G)}{q(G)}\rceil\frac{r}{d}q(G)$.*

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