# Existence of equitable $h$-edge-colorings of type $s = 2, 3$ of $K_t$

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**Abstract**

An $h$-edge-coloring (block-coloring) of type $s$ of a graph G is a assignment of $h$ colors to the edges (blocks) of G such that for every vertex $x$ of G the edges (blocks) incident with $x$ are colored with $s$ colors. For every color $i$, $\xi_{x,i}$ ($\mathcal{B}_{x,i}$) denotes the set of all edges (blocks) incident with $x$ and colored by $i$. An $h$-edge-coloring ($h$-block-coloring) of type $s$ is equitable if for every vertex $x$ and for colors $i, j$, $\mid \mid \xi_{x,i} \mid - \mid \xi_{x,j} \mid \mid \leq 1$ ($\mid \mid \mathcal{B}_{x,i} \mid - \mid \mathcal{B}_{x,j} \mid \mid \leq 1$). In this paper we study the existence of $h$-edge-coloring of type $s = 2, 3$ of $K_t$ and then show that the solution of this problem induces the solution of the existence of a $C_4$- ${}_tK_2$-design having an equitable $h$-block-coloring of type $s = 2, 3$.

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## 1 Introduction

Let $H$ be a simple graph. An $h$-edge-coloring of type $s$ of $H$ is an assignment of $h$ colors to the edges of $H$ such that for every $x \in V(H)$ the edges incident with $x$ are colored with $s$ colors. An $h$-edge-coloring of type $s = 2$ ($s = 3$) of $H$ is called $h$-bicoloring ($h$-tricoloring) of $H$. Say $C_x$ be the set of these colors. For every $i \in C_x$, $\xi_{x,i}$ denotes the set of all edges incident with $x$ and colored with $i$. An $h$-edge-coloring of type $s$ is equitable if for every vertex $x$ and for $i, j \in C_x$, $\mid \mid \xi_{x,i} \mid - \mid \xi_{x,j} \mid \mid \leq 1$. Let $G$ be a subgraph of $H$. A $G$-$H$-design of order $v = |H|$, is a pair $\Sigma = (V, \mathcal{B})$, where V is the vertex set of $H$ and $\mathcal{B}$ is an edge-disjoint decomposition of $H$ into copies of $G$ (called block). A cycle is a connected graph in which each vertex has even degree (see [5]). A cycle is even, or odd, according as to whether the number of edges is even, or odd, respectively. Denote by $C_4$ (or 4-cycle) the cycle having 4 vertices each of degree 2. Denote by $K_t$ the complete simple graph on $t$ vertices. Let ${}_tK_x$ denote the complete multipartite graph whose vertex set may be partitioned into $X_1, \ldots, X_t$ where $|X_i| = x$ for each $i = 1, \ldots, t$. According to [1], there exist three trivial necessary conditions for existence of a decomposition of edges of ${}_tK_x$ into 4-cycles, and such a decomposition is denoted by ${}_tK_x \to C_4$:

1. $tx \geq 4$
2. Each vertex of ${}_tK_x$ has even degree, i.e., $x(t-1)$ is even.
3. The number of edges of ${}_tK_x$ ($\binom{t}{2}x^2$) is divisible by 4.

A block-coloring of a $C_4$- ${}_tK_x$-design (briefly $4CD(tx)$) $\Sigma = (V, \mathcal{B})$ is a mapping $\varphi : \mathcal{B} \to \mathcal{C}$ where $\mathcal{C}$ is a set of colors. An $h$-block-coloring is a coloring in which exactly $h$ colors must be used. An $h$-block-coloring of type $s$ is a coloring of blocks such that, for each element $x \in V$, the blocks containing $x$ are colored with $s$ colors [2, 4]. Say $\mathcal{C}_x$ be the set of these colors. For a vertex $x$ and for every $i \in \mathcal{C}_x$, $\mathcal{B}_{x,i}$ is the set of all blocks incident with $x$ and colored with the $i$th color. A block-coloring of type $s$ is equitable if for every vertex $x$ and for $i, j \in \mathcal{C}_x$, $|\,|\mathcal{B}_{x,i}| - |\mathcal{B}_{x,j}|\,| \leq 1$. For $4CD(v)$ $\Sigma$ the color spectrum is defined as $\Omega_s(\Sigma) = \{h|$ there exists an h-block-coloring of type s of $\Sigma\}$, Moreover let $\Omega_s(v) = \bigcup \Omega_s(\Sigma)$, where the union is taken over the set of all $4CD(v)$s. The upper $s$-chromatic index $\overline{\chi}'_s(\Sigma)$ of $\Sigma$ is defined as $\overline{\chi}'_s(\Sigma) = \max \Omega_s(\Sigma)$, and similarly, $\overline{\chi}'_s(v) = \max \Omega_s(v)$. A graph $H$ ($4CD(tx)$) is said to be $h$-uncolorable if there is no any $h$-edge-coloring ($h$-block-coloring) of it. Note that the definition of $h$-edge-coloring of type $h$ of $H$ coincides with the definition of the edge-coloring of $H$ studied in [5].

In this paper we use the following labeling for ${}_tK_x$ (see [1]): denote by $v_1, \ldots, v_{tx}$ the vertices of ${}_tK_x$ and by $X_i = \{v_i, v_{i+t}, v_{i+2t}, \ldots, v_{i+(x-1)t}\}$, $i = 1, \ldots, t$, the $t$ disjoint independent sets of ${}_tK_x$.

**Theorem 1.1** *For all $t \geq 2$, there exists a $C_4$- ${}_tK_2$-design (or $4CD(2t)$).*

**Proof.** Let $V = \bigcup_{i=1}^{t}\{v_i, v_{i+t}\}$ and $\mathcal{B} = \{B^t_{i,j} = (v_i, v_j, v_{i+t}, v_{j+t})\} \mid i = 1, \ldots, t-1, j = i+1, \ldots, t\}$. It is easy to check that $\Sigma = (V, \mathcal{B})$ is a $4CD(2t)$. □

**Proposition 1.2** *The $4CD(2t)$ $\Sigma$ constructed in Theorem 1.1 is equivalent to the complete graph $K_t$.*

**Proof.** Take the edge $v_iv_j$ from every block $B^t_{i,j} = (v_i, v_j, v_{i+t}, v_{j+t})$ with $i = 1, \ldots, t-1$ and $j = i+1, \ldots, t$. The result is the complete graph $K_t$ on vertex set $\{v_1, \ldots, v_t\}$.
Let $K_t$ be the complete graph on vertex $V(K_t) = \{v_1, \ldots, v_t\}$. For every edge $v_iv_j \in E(K_t)$, construct the block $B^t_{i,j} = (v_i, v_j, v_{i+t}, v_{j+t})$. The result is the required $4CD(2t)$ $\Sigma$. □

**Proposition 1.3** *The* $4CD(2t)$ $\Sigma$ *constructed in Theorem* 1.1 *has an equitable* $h$*-block-coloring of type* $s$ *if and only if* $K_t$ *has an equitable* $h$*-edge-coloring of type* $s$*.*

**Proof.** Assign the same color to the edge $v_iv_j$ of $K_t$ and to the block $B^t_{i,j}$ of $\Sigma$. □

Equitable block colorings of a $C_4$- ${}_tK_1$-design for admissible values of $t$ are discussed in [2, 4]. Now we want to study the existence of equitable $h$-edge-colorings of $K_t$ and equitable $h$-block-colorings of $C_4$- ${}_tK_2$-design.

# 2 Bicoloring

In this section we will consider bicoloring.

**Lemma 2.1** [5] *A simple connected graph* $H$ *has an equitable* 2*-bicoloring if and only if it is not an odd cycle.*

From Lemma 2.1 and Proposition 1.3, we obtain the following results.

**Theorem 2.2** *For every* $t \equiv 3 \pmod 4$*, the complete graph* $K_t$ *cannot be equitably* 2*-bicolored.*

**Corollary 2.3** *For every* $t \equiv 3 \pmod 4$*, any* $4CD(2t)$ *cannot be equitably* 2*-bicolored.*

**Theorem 2.4** *For every* $t \equiv 0,1,2 \pmod 4$*,* $t \geq 4$*, there exists an equitable* 2*-bicoloring of* $K_t$*.*

**Corollary 2.5** *For every* $t \equiv 0,1,2 \pmod 4$*,* $t \geq 4$*, there exists an equitable* 2*-bicolorable of* $4CD(2t)$*.*

**Theorem 2.6** *For every* $t \geq 3$*, there exists an equitable* 3*-bicoloring of* $K_t$*.*

**Proof.** The following cases may be considered:

**Case 1** $t = 2m$*. Let* $G_1 = (V_1, E)$ *and* $G_2 = (V_2, F)$ *be two* $K_m$*s with* $|V_1 \cap V_2| = 0$*. Let* $G_3$ *be the complete bipartite graph on* $V_1 \cup V_2$*. For* $i = 1,2,3$ *assign the color* $i$ *to the edges of* $G_i$*. It is easy to check that* $G_1 \cup G_2 \cup G_3$ *is a* $K_{2m}$ *having an equitable* 3*-edge-coloring of type* 2*.*

**Case 2** $t = 2m+1$*. Take the two* $K_{m+1}$*s* $G_1 = (V_1 \cup \{\infty\}, E)$ *and* $G_2 = (V_2 \cup \{\infty\}, F)$ *such that* $|V_1 \cap V_2| = 0$*. Let* $G_3$ *be the complete bipartite graph on* $V_1 \cup V_2$*. For* $i = 1,2,3$ *assign the color* $i$ *to the edges of* $G_i$*. It is easy to check that* $G_1 \cup G_2 \cup G_3$ *is a* $K_{2m+1}$ *having an equitable* 3*-edge-coloring of type* 2*.* □

**Corollary 2.7** *For all* $t$*, there exists an equitable* 3*-bicolorable* $4CD(2t)$*.*

**Proof.** The result follows from Proposition 1.3 and Theorem 2.6. □

**Theorem 2.8** *For every $t \geq 3$, the number of colors $h$ for which there exists an equitable $h$-bicoloring of $K_t$, is at most* 4.

**Proof.** Let $\phi : \mathrm{E(K_t)} \to C$ be an equitable $h$-edge-coloring of type 2 of $\mathrm{K}_t$. Let $x$ be an element of V incident with the edges of color $c \in C$. we have the following cases:

1. $t = 2k$. There are at least $k-1$ edges of color $c$ incident with $x$. Hence there are at least $k$ elements in V incident with edges of color $c$. Then $hk \leq 2t = 4k$. Thus $h \leq \lfloor \frac{4k}{k} \rfloor = 4$.

2. $t = 4k+1$. There are $2k$ edges of color $c$ incident with $x$. Hence there are at least $2k+1$ elements in V incident with edges of color $c$. Then $h(2k+1) \leq 2t = 8k+2$. Thus $h \leq \lfloor \frac{8k+2}{2k+1} \rfloor = 3$.

3. $t = 4k+3$. There are $2k+1$ edges of color $c$ incident with $x$. Hence there are at least $2k+2$ elements in V incident with edges of color $c$. Then $h(2k+2) \leq 2t = 8k+6$. Thus $h \leq \lfloor \frac{8k+6}{2k+2} \rfloor = 3$. □

**Corollary 2.9** *For the upper 2-chromatic index $\overline{\chi}'_2(2t)$ of $4CD(2t)$, $t \geq 3$ the following inequalities hold:*

- $\overline{\chi}'_2(2t) \leq 4$, *if* $t \equiv 0$ *(mod 2);*
- $\overline{\chi}'_2(2t) \leq 3$, *if* $t \equiv 1, 3$ *(mod 4).*

**Proof.** The result follows by considering Proposition 1.3 and Theorem 2.8. □

**Theorem 2.10** *The complete graph $K_t$ cannot be equitably 4-bicolored.*

**Proof.** We shall suppose that $\varphi$ be an equitable 4-bicoloring of $\mathrm{K}_t$, and we show that this leads to a contradiction. Let $\mathrm{V(K_t)} = \{v_1, v_2, \ldots, v_t\}$ and let $\mathcal{P}(\mathrm{V})$ be the family of all subsets of V. Define

$$f_\alpha : \mathrm{V} \to \mathcal{P}(\mathrm{V})$$

$$f_\alpha(v_i) = \{v_j | v_i v_j \in \mathrm{E(K_t)}, \quad \varphi(v_i v_j) = \alpha\}.$$

Without loss of generality, suppose that the edges incident with $v_1$ are colored with colors $A$ and $B$, such that $|f_A(v_1)| = r$ and $|f_B(v_1)| = s$, $|r - s| \leq 1$. Since $\varphi$ is an equitable 4-bicoloring, there exists a vertex $v_2$, such that $f_C(v_2) \neq \emptyset$. As edges incident with $v_1$ are colored with $A$ and $B$, edge $v_1v_2$ must be colored with $A$ or $B$. Let $v_1 \in f_A(v_2)$. Likewise, there exists a vertex $v_3$, such that $f_D(v_3) \neq \emptyset$. Similar to above, edges $v_1v_3$ and $v_2v_3$ cannot be colored $D$. Hence, $v_1$ and $v_2$ must be in $f_A(v_3)$. So $f_A(v_3) \neq \emptyset$. Suppose that a vertex $v \in f_B(v_1)$. Because $v \in f_A(v_2)$ or $v \in f_C(v_2)$, edges incident with $v$ are colored with $A$ and $B$ or $B$ and $C$. Therefore, edge $vv_3$ must be colored with $A$. Thus, $f_B(v_1) \subseteq f_A(v_3)$. Clearly, $f_B(v_1)$ does not contain $v_1$ and $v_2$. Hence, $|f_A(v_3)| \geq s+2$. Since edges incident with $v_3$ are colored with $A$ and $D$, $v_3 \in f_A(v_1)$. Therefore, $|f_D(v_3)| \leq r-2$. Hence, $||f_A(v_3)| - |f_D(v_3)|| \geq |f_A(v_3)| - |f_D(v_3)| \geq s+2-(r-2) \geq 3$. This is a contradiction. □

**Corollary 2.11** *It is not possible that 4CD $\Sigma$ is bicolorable with 4 colors.*

**Proof.** The statement follows from Proposition 1.3 and Theorem 2.10. □

**Corollary 2.12** *For every $t \geq 3$, we have $\overline{\chi}'_2(2t) = 3$.*

**Proof.** It follows by Corollaries 2.7, 2.9 and 2.11. □

# 3 Tricoloring

In this section we will consider tricolorings.

**Theorem 3.1** [5] *Let $H$ be a simple graph and let $k \geq 2$. If $k \nmid d(v)$ ($\forall v \in V(H)$) then $H$ has an equitable $k$-edge-coloring of type $k$.*

**Corollary 3.2** *For every $t \not\equiv 1 \pmod 3$, $t \geq 4$, there is an equitable 3-tricoloring of $K_t$.*

**Theorem 3.3** *For every $t \equiv 1 \pmod 3$, there is an equitable 3-tricoloring of $K_t$.*

**Proof.** This result could be easily proved as follows.

- $t \equiv 4 \pmod 6$. Let $\mathcal{F} = \{F_i \mid i = 1, \ldots, t-1\}$ be a 1-factorization of $K_t$. Partition $\mathcal{F}$ into the three parts $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ such that $|\mathcal{F}_i| = \frac{t-1}{3}$ for every $i = 1, 2, 3$. Assign the color $i$ to each edge of the 1-factors of $\mathcal{F}_i$.
- $t \equiv 1 \pmod 6$. Let $\mathcal{F} = \{F_i \mid i = 1, \ldots, t\}$ be an almost 1-factorization of $K_t$ such that $i$ is the missing vertex of $F_i$. Note that $|F_i| \equiv 0 \pmod 3$. Partition the edges of $F_1$ into the 3 classes $F_1^j$, $j = 1, 2, 3$ such that $|F_1^j| = \frac{t-1}{6}$. For $j = 1, 2, 3$, assign the color $j$ to the edges of $F_1^j$ and to the edges of $F_\alpha$ if $\alpha$ is a vertex of some edge in $F_1^j$. □

**Corollary 3.4** *For all $t$, there exists an equitable 3-tricolorable $4CD(2t)$.*

**Proof.** The assertion is concluded by considering Proposition 1.3 and Theorems 3.2 and 3.3. □

**Theorem 3.5** *For all $t \geq 4$, there exists an equitable 4-tricoloring of $K_t$.*

**Proof.** The proof is divided into three parts, depending on $t$.

1. $t = 3m$. Let $G_i = (V_i, E_i)$, $i = 1,2,3$, be three $K_m$s with $V_1, V_2$ and $V_3$ mutually disjoint. For every $1 \le i < j \le 3$, construct the complete bipartite graph on $V_i \cup V_j$. Denote these graphs by $G_i$, $i = 4,5,6$. For every $i = 1,2,3$, assign the color $i$ to the edges of $G_i$ and assign the colors 4, 2 and 1 to the edges of $G_4$, $G_5$ and $G_6$, respectively. It is easy to check that $\bigcup_{i=1}^{6} G_i$ is a $K_{3m}$ having an equitable 4-edge-coloring of type 3.

2. $t = 3m + 1$. Let $G_i = (V_i \cup \{\infty\}, E_i)$, $i = 1,2,3$, be three $K_{m+1}$s with $V_1, V_2$ and $V_3$ mutually disjoint. For every $1 \le i < j \le 3$, construct the complete bipartite graph on $V_i \cup V_j$. Denote these graphs by $G_i$, $i = 4,5,6$. For every $i = 1,2,3$, assign the color $i$ to the edges of $G_i$ and assign the colors 4, 2 and 1 to the edges of $G_4$, $G_5$ and $G_6$, respectively. It is easy to check that $\bigcup_{i=1}^{6} G_i$ is a $K_{3m+1}$ having an equitable 4-edge-coloring of type 3.

3. $t = 3m + 2$. Let $G_i = (V_i \cup \{\infty_1, \infty_2\}, E_i)$, $i = 1,2,3$, be three $K_{m+2}$s with $V_1, V_2$ and $V_3$ mutually disjoint. Remove from $E_2$ and $E_3$ the edge $\infty_1\infty_2$ and give the same names $G_2$ and $G_3$ to the resulting graphs. For every $1 \le i < j \le 3$, construct the complete bipartite graph on $V_i \cup V_j$. Denote these graphs by $G_i$, $i = 4,5,6$. For every $i = 1,2,3$, assign the color $i$ to the edges of $G_i$ and assign the colors 4, 2 and 1 to the edges of $G_4$, $G_5$ and $G_6$, respectively. It is easy to check that $\bigcup_{i=1}^{6} G_i$ is a $K_{3m+2}$ having an equitable 4-edge-coloring of type 3. □

**Remark 3.6** *If we assign color* 5 *to the edges of $G_5$ in the proof of previous Theorem, then we obtain a* 5*-tricolorable $K_t$.*

**Theorem 3.7** *For every $t \ge 4$, there is a* 6*-tricolorable $K_t$.*

**Proof.** We consider the following three parts:

1. $t = 3m$. Let $G_i = (V_i, E_i)$, $i = 1,2,3$, be three $K_m$s with $V_1, V_2$ and $V_3$ mutually disjoint. For every $1 \le i < j \le 3$, construct the complete bipartite graph on $V_i \cup V_j$. Denote these graphs by $G_i$, $i = 4,5,6$. For every $i = 1,2,\ldots,6$, assign the color $i$ to the edges of $G_i$. It is easy to check that $\bigcup_{i=1}^{6} G_i$ is a $K_{3m}$ having an equitable 6-edge-coloring of type 3.

2. $t = 3m + 1$. Let $G_i = (V_i \cup \{\infty\}, E_i)$, $i = 1,2,3$, be three $K_{m+1}$s with $V_1, V_2$ and $V_3$ mutually disjoint. For every $1 \le i < j \le 3$, construct the complete bipartite graph on $V_i \cup V_j$. Denote these graphs by $G_i$, $i = 4,5,6$. For every $i = 1,2,\ldots,6$, assign the color $i$ to the edges of $G_i$. It is easy to check that $\bigcup_{i=1}^{6} G_i$ is a $K_{3m+1}$ having an equitable 6-edge-coloring of type 3.

3. $t = 3m + 2$. Let $G_i = (V_i \cup \{\infty_1, \infty_2\}, E_i)$, $i = 1, 2, 3$, be three $K_{m+2}$s with $V_1, V_2$ and $V_3$ mutually disjoint. Remove from $E_2$ and $E_3$ the edge $\infty_1\infty_2$ and give the same names $G_2$ and $G_3$ to the resulting graphs. For every $1 \leq i < j \leq 3$, construct the complete bipartite graph on $V_i \cup V_j$. Denote these graphs by $G_i$, $i = 4, 5, 6$. For every $i = 1, 2, \ldots, 6$, assign the color $i$ to the edges of $G_i$. It is easy to check that $\bigcup_{i=1}^{6} G_i$ is a $K_{3m+2}$ having an equitable 6-edge-coloring of type 3. □

**Corollary 3.8** *For all $t \geq 4$ and for $h = 4, 5, 6$, there exist an eqitable h-tricolorable $4CD(2t)$s.*

**Proof.** The result follows from Proposition 1.3, Theorems 3.5 and 3.7 and Remark 3.6. □

**Theorem 3.9** *For every $t \geq 3$, the number of colors h for which there exists an equitable h-tricoloring of $K_t$, is at most 9.*

**Proof.** Let $\phi : E(K_t) \to \mathcal{C}$ be an equitable $h$-edge-coloring of type 3 of $K_t$. Let $x$ be an element of V incident with the edges of color $c \in \mathcal{C}$. we have the following cases:

1. $t = 3k$. There are at least $k - 1$ edges of color $c$ incident with $x$. Hence there are at least $k$ elements in V incident with edges of color $c$. Then $hk \leq 3t = 9k$. Thus $h \leq \lfloor \frac{9k}{k} \rfloor = 9$.

2. $t = 3k + 1$. There are $k$ edges of color $c$ incident with $x$. Hence there are at least $k + 1$ elements in V incident with edges of color $c$. Then $h(k + 1) \leq 3t = 9k + 3$. Thus $h \leq \lfloor \frac{9k+3}{k+1} \rfloor = 8$.

3. $t = 3k + 2$. There are at least $k$ edges of color $c$ incident with $x$. Hence there are at least $k + 1$ elements in V incident with edges of color $c$. Then $h(k + 1) \leq 3t = 9k + 6$. Thus $h \leq \lfloor \frac{9k+6}{k+1} \rfloor = 8$. □

**Theorem 3.10** *For every $t \equiv 0, 1, 2 \pmod 6$, $t \geq 6$, there exists an equitable 7-tricoloring of $K_t$.*

**Proof.** Let $t = 6k + h$, $h = 0, 1, 2$. Let $V_i$, $i = 1, 2, \ldots, 6$, be mutually disjoint sets each of size $k$. Let $W_0 = \emptyset$, $W_1 = \{\infty\}$ and $W_2 = \{\infty_1, \infty_2\}$. Let $G_i = (W_h \cup V_i \cup V_{i+3}, E_i)$, $i = 1, 2, 3$ be three $K_{2k+h}$. For $h = 2$, remove from $E_2$ and $E_3$ the edge $\infty_1\infty_2$ and give the same names $G_2$ and $G_3$ to the resulting graphs. For $i = 4, 5, 6, 7$, construct the tripartite graphs $G_i$ on vertex sets $V_1 \cup V_2 \cup V_3$, $V_1 \cup V_5 \cup V_6$, $V_2 \cup V_4 \cup V_6$ and $V_3 \cup V_4 \cup V_5$ respectively. For $i = 1, 2, \ldots, 7$ assign the color $i$ to the edges of $G_i$. It is easy to check that $\bigcup_{i=1}^{7} G_i$ is a $K_{6k+h}$ having an equitable 7-edge-coloring of type 3. □

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