Roman domination edge critical graphs having precisely two cycles*

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Abstract

A Roman dominating function on a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u of G for which $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number, $\gamma_R(G)$, of G is the minimum weight of a Roman dominating function on G. A graph G is said to be *Roman domination edge critical* or just γ_R-edge critical, if $\gamma_R(G + e) < \gamma_R(G)$ for any edge $e \notin E(G)$. In this paper, we characterize all γ_R-edge critical connected graphs having precisely two cycles.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. For notation and graph theory terminology in general we follow [3].

For a graph G, let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V = V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ and $|V_i| = n_i$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \rightarrow \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$. A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a *Roman dominating function* on G if every vertex u of G for which $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The weight of a Roman dominating function f on G is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A function $f = (V_0; V_1; V_2)$ is called a $\gamma_R(G)$-function or γ_R-function if it is a Roman dominating function on G and $f(V(G)) = \gamma_R(G)$, [2, 7].

Roman domination edge critical graphs introduced by Hansberg et al. [4] and further studied in [1, 5, 6]. A graph G is said to be *Roman domination edge critical*, or just *γ_R-edge critical*, if $\gamma_R(G+e) < \gamma_R(G)$ for any $e \in E(\overline{G})$, where $\overline{G}$ denotes the complement of G.

In this paper, we continue the study of γ_R-edge critical graphs, and characterize γ_R-edge critical connected graphs having precisely two cycles. In Section 3 we state some known results which we use for the next. In Section 4 we present some preliminary results. In Section 5 we show that there is no γ_R-edge critical graph with precisely two cycles and minimum degree at least two. In Section 6 we show that there is no γ_R-edge critical graph with precisely two cycles, minimum degree one, and any support vertex of degree three. In Section 7 we prove the main result of this paper which is a full characterization of γ_R-edge critical graphs with precisely two cycles.

We recall that a *leaf* in a graph is a vertex of degree one, and a *support vertex* is one that is adjacent to a leaf. Let $L(G)$ be the set of all leaves in a graph G, and $S(G)$ be the set of all support vertices of G. Also for a graph G and a subset of vertices S we denote by $G[S]$ the subgraph of G induced by S.

2 Main result

Let H_1 be the following graph shown in Figure 1, and H_2 be a graph obtained from H_1 by removing a leaf.

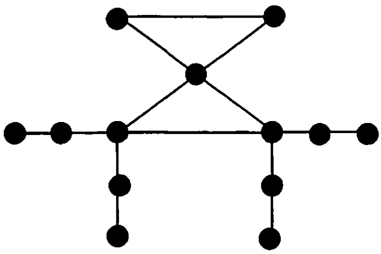

Figure 1. The graph H_1.

We will prove the following.

Theorem 1. *A graph G with precisely two cycles is γ_R-edge critical if and only if $G = H_1$ or H_2.*

3 Known results and Observations

In this section we state some known results and observations which we use for the next. The following is a fundamental theorem of

Cockayne et al. [2].

Theorem 2 ([2]). *Let $f = (V_0; V_1; V_2)$ be any γ_R-function on G. Then,*
(a) $G[V_1]$, the subgraph induced by V_1, has maximum degree 1.
(b) No edge of G joins V_1 and V_2.

As consequences of Theorem 2 we have the following.

Observation 3. *Let f be a $\gamma_R(G)$-function.*
(1) If $f(x) = 2$ for a vertex x of degree 2, then $f(z) = 0$ for any $z \in N(x)$.
(2) If $f(x) = 2$ for some leaf x, and $y \in N(x)$, then $f(z) = 0$ for any $z \in N[y] - \{x\}$.

Observation 4. *If xx_1x_2 is a path in a γ_R-edge critical graph G such that $\deg(x) \geq 2$, $\deg(x_1) = 2$ and $\deg(x_2) = 1$, then $f(x) \neq 1$ for any $\gamma_R(G)$-function f.*

Hansberg et al. [4] obtained the following results for γ_R-edge critical graphs.

Theorem 5 ([4]). *A graph G is γ_R-edge critical if and only if for any two non-adjacent vertices x, y, there is a $\gamma_R(G)$-function $f = (V_0; V_1; V_2)$ such that $\{f(x), f(y)\} = \{1, 2\}$.*

Lemma 6 ([4]). *Any support vertex in a γ_R-edge critical graph is adjacent to exactly one leaf.*

Lemma 7 ([4]). *The cycle C_n is γ_R-edge critical if and only if $n \in \{4, 5\}$.*

Hansberg et al. [5] continued the study of γ_R-edge critical graphs and obtained the following.

Lemma 8 ([5]). *If x, y are two support vertices of degree 2 in a γ_R-edge critical graph and $z \in N(x) \cap N(y)$, then z is not a support vertex, and $\deg(z) \geq 3$.*

Lemma 9 ([5]). *If x, y are two support vertices of degree two in a γ_R-edge critical graph G, and $z \in N(x) \cap N(y)$ is a vertex with $\deg(z) \geq 4$, then $G[N(z) \setminus \{x, y\}]$ is a complete graph.*

Lemma 10 **([5]).** *If x, y are two adjacent support vertices of degree 3 in a γ_R-edge critical graph, and $z_1 \in N(x)\backslash\{y\}$, $z_2 \in N(y)\backslash\{x\}$ are two vertices with $\deg(z_1) \geq 2$ and $\deg(z_2) \geq 2$, then either $z_1 = z_2$ or $z_1 \in N(z_2)$.*

Lemma 11 **([5]).** *Let u and v be two support vertices in a γ_R-edge critical graph such that $\deg_G(u) = \deg_G(v) = 2$. If $u_1 \neq v$ is a non leaf adjacent to u, and $v_1 \neq u, u_1$ is a non leaf adjacent to v, then u_1 is adjacent to v_1.*

Lemma 12 **([5]).** *If u is a support vertex with $\deg_G(u) = 2$ in a γ_R-edge critical graph G, and v is a leaf such that u and v have a common neighbor z, then $G[N(z) \setminus \{u, v\}]$ is a complete graph.*

Lemma 13 **([5]).** *If $x_1x_2x_3x_4x_5$ is a path in a γ_R-edge critical graph G such that $\deg_G(x_2) = \deg_G(x_3) = \deg_G(x_4) = 2$, then x_1 is adjacent to x_5.*

Lemma 14 **([5]).** *If x is a support vertex in a γ_R-edge critical graph, and y, z are two vertices adjacent to x such that $\deg(y) = \deg(z) = 2$, then y is adjacent to z.*

4 Preliminary results

In this section we present some preliminary results.

Lemma 15. *If a graph G contains a cycle $v_1v_2v_3v_4v_1$ as an induced subgraph such that $\deg(v_i) = 2$ for $i = 1, 2, 3$ and $\deg(v_4) > 2$, then G is not γ_R-edge critical.*

Proof. Assume that a graph G contains a cycle $v_1v_2v_3v_4v_1$ as an induced subgraph such that $\deg(v_i) = 2$ for $i = 1, 2, 3$ and $\deg(v_4) > 2$. Let $x \in N(v_4) - \{v_1, v_3\}$. Suppose that G is γ_R-edge critical. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x), f(v_1)\} = \{1, 2\}$. If $f(x) = 1$, then $f(v_1) + f(v_2) + f(v_3) + f(v_4) \geq 3$ and g defined on $V(G)$ by $g(a) = f(a)$ if $a \notin \{v_1, v_2, v_3, v_4, x\}$, $g(v_4) = 2$, $g(v_2) = 1$, and $g(v_1) = g(v_3) = g(x) = 0$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus $f(x) = 2$. By Theorem 2,

$f(v_4) = 0$. Now h defined on $V(G)$ by $h(v_2) = 2$, $h(v_1) = h(v_3) = 0$, and $f(u) = f(u)$ if $u \notin N[v_2]$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. □

Lemma 16. *If a graph G contains a cycle $v_1v_2v_3v_4v_5v_1$ as an induced subgraph such that* $\deg(v_i) = 2$ *for* $i = 1, 2, 3, 4$ *and* $\deg(v_5) = 3$, *then G is not γ_R-edge critical.*

Proof. Assume that a graph G contains a cycle $v_1v_2v_3v_4v_5v_1$ as an induced subgraph such that $\deg(v_i) = 2$ for $i = 1, 2, 3, 4$ and $\deg(v_5) = 3$. Let $x \in N(v_5) - \{v_1, v_4\}$. Suppose that G is γ_R-edge critical. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x), f(v_2)\} = \{1, 2\}$. If $f(x) = 1$, then $f(v_1) + f(v_2) + f(v_3) + f(v_4) \geq 4$ and g defined on $V(G)$ by $g(a) = f(a)$ if $a \notin \{v_1, v_2, v_3, v_4, x\}$, $g(v_5) = g(v_3) = 2$, and $g(v_1) = g(v_2) = g(v_4) = g(x) = 0$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus $f(x) = 2$. Now $f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_5) \geq 4$, and h defined on $V(G)$ by $h(v_2) = 2$, $h(v_1) = h(v_3) = 0$, $h(v_4) = 1$, and $f(u) = f(u)$ if $u \notin N[v_2] \cup \{v_4\}$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. □

Lemma 17. *If a γ_R-edge critical graph G contains an induced cycle C of length three with precisely one vertex (say x) of degree at least three, then $G[N(x) - V(C)]$ is complete, and any vertex of $N(x) - V(C)$ is of degree at least three.*

Proof. Assume that a γ_R-edge critical graph G contains a cycle C : xv_1v_2x as an induced subgraph such that $\deg(v_1) = \deg(v_2) = 2$ and $\deg(v_3) \geq 3$. Assume that $G[N(x) - V(C)]$ is not complete, and let $a, b \in G[N(x) - V(C)]$ be two non-adjacent vertices. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a), f(b)\} = \{1, 2\}$. Without loss of generality let $f(a) = 1$. Then $f(v_1) + f(v_2) + f(x) \geq 2$, and g defined on $V(G)$ by $g(x) = 2$, $g(a) = g(v_1) = g(v_2) = 0$, and $g(u) = f(u)$ if $u \neq \{x, a, v_1, v_2\}$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus $G[N(x) - V(C)]$ is complete. Now let $a \in N(x) - V(C)]$. By Theorem 5, there is a $\gamma_R(G)$-function h such that $\{h(a), h(v_1)\} = \{1, 2\}$. If $h(a) = 1$, then h_1 defined on $V(G)$ by $h_1(x) = 2$, $h_1(u) = 0$ if $u \in N(x)$, and $h_1(u) = h(u)$ if $u \notin N[x]$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus $h(a) = 2$ and $h(v_1) = 1$. If $\deg(a) = 1$, then h_1, as defined above, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Suppose that $\deg(a) = 2$. Let $b \in N(a) - \{x\}$. Clearly $h(v_1)+h(v_2)+h(x) \geq 2$. Now h_2 defined on $V(G)$ by $h_2(x) = 2$, $h_2(u) = 0$ if $u \in N(x)$, $h_2(b) = \max\{1, h(b)\}$, and $h_2(u) = h(u)$ if $u \notin N[x] \cup \{b\}$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. □

Lemma 18. *If x, y are two support vertices of a γ_R-edge critical graph G, then there is no path $xa_1a_2...a_ty$ between x and y such that* $\deg(a_i) = 2$ *for $i = 1, 2, ..., t$.*

Proof. Let x, y be two support vertices of a γ_R-edge critical graph G, x_1 be a leaf adjacent to x and y_1 be a leaf adjacent to y. Assume that there is a path $P : xa_1a_2...a_ty$ between x and y such that $\deg(a_i) = 2$ for $i = 1, 2, ..., t$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x_1), f(y_1)\} = \{1, 2\}$. Assume, without loss of generality, that $f(x_1) = 2$. Then by Observation 3, $f(x) = f(a_1) = 0$. If $t = 1$, then $f(y) = 2$ contradicting Theorem 2. Thus $t \geq 2$. Then $f(a_2) = 2$, contradicting Observation 3. □

Lemma 19. *Let G be a γ_R-edge critical graph G with precisely two cycles. If $C_1 : xx_1x_2x_3x_4x$ is a cycle in G such that* $\deg(x) = 4$ *and* $\deg(x_i) = 2$ *for $i = 1, 2, 3, 4$, and C_2 is the another cycle such that $V(C_1) \cap V(C_2) = \{x\}$, then there is a vertex $y \in N(x) \cap V(C_2)$ such that* $\deg(y) \geq 3$ *and y is not a support vertex.*

Proof. Let $V(C_2) \cap N(x) = \{y, z\}$. Assume that both y and z are support vertices. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(y_1), f(z_1)\} = \{1, 2\}$, where y_1 is the leaf adjacent to y, and z_1 is the leaf adjacent to z. Without loss of generality assume that $f(y_1) = 2$. Then $f(x)+f(x_1)+...+f(x_4) \geq 4$. Now g defined on $V(G)$ by $g(y_1) = g(x_2) = g(x_3) = 1$, $g(x) = 2$, $g(x_1) = g(x_4) = g(y) = 0$, and $g(u) = f(u)$ if $u \in V(G) - \{x, x_1, ..., x_4, y, y_1\}$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus we assume that z is not a support vertex. If $\deg(z) \geq 3$ then the proof is complete. Thus assume that $\deg(z) = 2$. If $\deg(y) = 2$, then by Lemma 17, $|V(C_2)| \geq 4$. By Theorem 5, there is a $\gamma_R(G)$-function g_1 such that $\{g_1(y), g_1(z)\} = \{1,2\}$, and this easily produce a contradiction. Thus $\deg(y) \geq 3$. We show that y is not a support vertex. Assume that y is a support vertex. Let y_1 be the leaf adjacent to y. If $|V(C_2)| = 3$, then by Theorem 5, there is a $\gamma_R(G)$-function g_2 such that $\{g_2(y_1), g_2(z)\} = \{1,2\}$, and this easily produces a contradiction. Thus $|V(C_2)| \geq 4$. Let $w \in N(y) - \{x, y_1\}$. If $d(w,z) > 1$, then by Theorem 5, there is a $\gamma_R(G)$-function f_1 such that $\{f_1(w), f_1(z)\} = \{1,2\}$ and we obtain a contradiction. Thus w is adjacent to z. By Theorem 5, there is a $\gamma_R(G)$-function f_2 such that $\{f_2(y_1), f_2(z)\} = \{1,2\}$. This easily produces a contradiction. Thus y is not a support vertex and the proof is completed. □

5 Graphs with no leaf

In this section we characterize γ_R-edge critical graphs with precisely two cycles and minimum degree at least two.

Theorem 20. *If G is a graph with precisely two cycles and $\delta(G) > 1$, then G is not γ_R-edge critical.*

Proof. Assume that G is a γ_R-edge critical graph with precisely two cycles C_1, C_2, and $\delta(G) > 1$. By Lemma 13, $|V(C_1)| \leq 5$ and $|V(C_2)| \leq 5$. By Lemma 15, $|V(C_i)| \neq 4$ for $i = 1,2$. Then $\{|V(C_1)|, |V(C_2)|\} \subseteq \{3,5\}$.

If $|V(C_1)| = 5$, then by Lemma 16, $d(C_1, C_2) = 0$. By Lemma 17, $|V(C_2)| = 5$. Let $x \in V(C_1) \cap V(C_2)$ and $N(x) \cap V(C_1) = \{y, z\}$. Then $\gamma_R(G) = \gamma_R(G+yz) = 6$, a contradiction. Thus we assume that $|V(C_1)| \neq 5$ and similarly $|V(C_2)| \neq 5$. So $|V(C_1)| = |V(C_2)| = 3$. By Lemma 17, $d(C_1, C_2) \geq 1$. Let $x \in V(C_1)$ and $y \in V(C_2)$ be the vertices with $d(x,y) = d(C_1, C_2)$. If $d(x,y) \geq 2$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x), f(y)\} = \{1,2\}$, and we easily obtain a contradiction. Thus $d(x,y) = 1$. Now $\gamma_R(G) = \gamma_R(G + ab)$, where $a \in N(x) \cap V(C_1)$, and $b \in N(y) \cap V(C_2)$, a contradiction. □

6 Graphs with any support vertex of degree at least three

In this section we characterize γ_R-edge critical graphs with precisely two cycles, minimum degree one, and any support vertex of degree at least three.

Lemma 21. *Let G be a γ_R-edge critical graph with precisely two cycles such that $\delta(G) = 1$ and any support vertex of G has degree at least three. If x and y are two support vertices of degree 3, then x is not adjacent to y.*

Proof. Let G be a γ_R-edge critical graph with precisely two cycles C_1 and C_2. Assume that there are two adjacent support vertices x, y with $\deg(x) = \deg(y) = 3$. Let x_1 be a leaf adjacent to x and y_1 be a leaf adjacent to y. By Lemma 6 we can assume that $z_1 \in N(x)$ and $z_2 \in N(y)$ are two vertices with $\deg(z_i) > 1$ for $i = 1, 2$. By Lemma 10, either $z_1 = z_2$ or $z_1 \in N(z_2)$. If $z_1 = z_2$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x_1), f(y_1)\} = \{1, 2\}$. By Observation 3, $f(x) = f(y) = f(z_1) = 0$, a contradiction. Thus $z_1 \neq z_2$, and so $z_1 \in N(z_2)$. Assume that C_1 is the cycle with vertex set $\{x, y, z_1, z_2\}$. Without loss of generality assume that $d(z_2, C_2) = d(C_1, C_2)$. We show that $\deg(z_1) \geq 3$. Suppose that $\deg(z_1) = 2$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(y_1), f(z_1)\} = \{1, 2\}$. If $f(z_1) = 2$, then by Observation 3, $f(x) = f(z_2) = 0$, and so $f(x_1) = f(y) = 1$. Now g defined on $V(G)$ by $g(u) = f(u)$ if $u \notin \{x, y, z_1, x_1, z_2\}$, $g(x) = 2$, $g(u) = 0$ if $u \in N(x)$, and $g(z_2) = 1$ is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus $f(z_1) = 1$ and $f(y_1) = 2$. These easily produce a contradiction. Thus $\deg(z_1) \geq 3$. By Lemma 6, $\deg(z_1) = 3$ and z_1 is a support vertex. Let w be the leaf adjacent to z_1. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(y_1), f(w)\} = \{1, 2\}$. Without loss of generality assume that $f(w) = 2$ and $f(y_1) = 1$. By Observation 3, $f(z_1) = f(x) = f(z_2) = 0$, and so $f(y) = 1$. But then $f(x_1) = 2$. This contradicts Observation 3. □

Lemma 22. *Let G be a γ_R-edge critical graph with precisely two cycles such that $\delta(G) = 1$ and any support vertex of G has degree at* *least three. Then G has at most one support vertex on its cycles.*

Proof. Assume that there are at least two support vertices on a cycle C_1. By Lemma 18, G has precisely two adjacent support vertices x, y on C_1. By Lemma 21, we may assume that $\deg(x) = 3$ and $\deg(y) > 3$. Thus there is a nontrivial path between x and y in which any internal vertex of the path is of degree two. This contradicts Lemma 18. □

Lemma 23. *Let G be a γ_R-edge critical graph with precisely two cycles such that $\delta(G) = 1$ and any support vertex of G has degree at least three. Then G has no support vertex on its cycles.*

Proof. Assume that x is a support vertex on a cycle C_1. Let x_1 be the leaf adjacent to x.

Case 1. $\deg(x) = 3$. If any vertex in $N(x)$ is of degree two, then by Lemma 14, $|V(C_1)| = 3$, and this contradicts Lemma 17. Thus there is a vertex $y \in N(x)$ with $\deg(y) \geq 3$. By Lemma 22 any vertex of $V(C_1) - \{x, y\}$ is of degree two. By Lemma 13, $|V(C_1)| \leq 5$. Let $z \in N(y) \cap V(C_1) - \{x\}$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x_1), f(z)\} = \{1, 2\}$. By Theorem 2 and Observation 3, $f(x_1) = 1$ and $f(z) = 2$, and $f(u) = 0$ for $u \in N(z)$. If $V(C_1)| = 3$, then g defined on $V(G)$ by $g(u) = f(u)$ if $u \in V(G) - \{z, x, x_1\}$, $g(z) = g(x_1) = 0$ and $g(x) = 2$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. If $|V(C_1)| = 4$, then $f(x) = 1$, and g defined on $V(G)$ by $g(u) = f(u)$ if $u \in V(G) - \{z, x, x_1\}$, $g(z) = 1$, $g(x_1) = 0$ and $g(x) = 2$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. It remains to assume that $|V(C_1)| = 5$. Let $w \in N(x) - \{x_1, y\}$. Then $f(w) + f(x) \geq 2$. Now g defined on $V(G)$ by $g(u) = f(u)$ if $u \in V(G) - \{w, x, x_1\}$, $g(w) = g(x_1) = 0$ and $g(x) = 2$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction.

Case 2. $\deg(x) \geq 4$. By Lemma 22, any vertex of $V(C_1) - \{x\}$ is of degree two. By Lemmas 13, 15 and 17, we obtain that $|V(C_1)| = 5$. Let $N(x) \cap V(C_1) = \{y, z\}$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(y), f(z)\} = \{1, 2\}$. Without loss of generality assume that $f(z) = 2$. Let $w \in N(z) - \{x\}$. Then by Observation 3, $f(x) = f(w) = 0$. Let $b \in N(y) - \{x\}$. So $f(x_1) = f(b) = 1$. Now g defined on $V(G)$ by $g(u) = f(u)$ if $u \in V(G) - \{y, z, w, x, x_1, b\}$, $g(z) = g(y) = g(x_1) = g(b) = 0$, $g(w) = g(x) = 2$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. □

The following is proved in a similar manner as in the proof of Lemma 23, and so we omit the proof.

Lemma 24. *Let G be a γ_R-edge critical graph with precisely two cycles and $\delta(G) = 1$. If there is a vertex x on a cycle C_1 such that any vertex of $V(C_1) - \{x\}$ is a support vertex or a vertex of degree two, then G has no support vertex on C_1.*

Now we are ready to give the main result of this section.

Theorem 25. *Let G be a graph with precisely two cycles such that $\delta(G) = 1$ and any support vertex of G has degree at least three. Then G is not γ_R-edge critical.*

Proof. Assume that G is a γ_R-edge critical graph with precisely two cycles C_1 and C_2. Let $x \in V(C_1)$ and $y \in V(C_2)$ be two vertices with $d(x, y) = d(C_1, C_2)$. By Lemma 23, no vertex of C_1 or C_2 is a support vertex. Since any support vertex is of degree at least three, by Lemma 6, any vertex of $V(C_1) \cup V(C_2) - \{x, y\}$ is of degree two. By Lemmas 13 and 15, $|V(C_i)| \in \{3, 5\}$ for $i = 1, 2$. If $|V(C_1)| = 5$, then by Lemma 16, $d(x, y) = 0$, a contradiction, since $\delta(G) = 1$. So $|V(C_1)| = |V(C_2)| = 3$. If $d(x, y) > 1$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x), f(y)\} = \{1, 2\}$. Assume, without loss of generality, that $f(x) = 1$. Then $f(u) = f(w) = 1$, where $V(C_1) = \{x, u, w\}$, a contradiction. Thus $d(x, y) \le 1$. This is a contradiction, since $\delta(G) = 1$. □

7 Proof of Theorem 1

In this section we prove our main result namely Theorem 1. First it is straightforward to see that H_1 and H_2 are γ_R-edge critical. Let G

be a γ_R-edge critical graph with precisely two cycles C_1 and C_2. By Theorem 20, $\delta(G) = 1$. Let $xx_1...x_t$ be the longest path in G such that $\deg(x_t) = 1$, $\deg(x_i) = 2$ for $i = 1, 2, ..., t-1$, and $\deg(x) > 2$. By Lemma 13, $t \leq 3$, and by Theorem 25, $t \in \{2, 3\}$. Let $C(2)$ be the set of all vertices of G of degree at least two which are adjacent to a support vertex of degree two. By Lemma 11, $G[C(2)]$ is complete. We show that $t = 2$.

Fact 1. $t = 2$.

Proof of Fact 1. Assume that $t = 3$. Since $G[C(2)]$ is complete, x is the unique vertex with these properties. By Lemma 18, x is not a support vertex. Assume that $x \in C(2)$. Let a be a support vertex of degree two which is adjacent to x, and b be the leaf adjacent to a. Let $y \in N(x) - \{a, x_1\}$ be a vertex of degree more than one. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x_1), f(y)\} = \{1, 2\}$, and clearly $f(x_1) = 2$ and $f(y) = 1$. Then $f(a) + f(b) = 2$, and g defined on $V(G)$ by $g(x) = 2$, $g(b) = g(x_2) = g(x_3) = 1$, $g(v) = 0$ if $v \in N(x)$, and $g(v) = f(v)$ if $v \notin \{x, y, x_1, x_2, x_3, a, b\}$, is an RDF for G of weight less that $\gamma_R(G)$, a contradiction. Thus $x \notin C(2)$.

Let $a \in V(C_1)$ and $b \in V(C_2)$ be two vertices with $d(a, b) = d(C_1, C_2)$, and let P the shortest path between a and b. We consider the following cases.

- Case 1. P contains x.

  By Lemma 24, no vertex of C_i is support vertex, for $i = 1, 2$, and by Lemmas 13 and 15, $|V(C_i)| \in \{3, 5\}$, and C_i has $|V(C_i)| - 1$ vertices of degree two for $i = 1, 2$. We show that $|V(C_1)| = 5$. Suppose that $|V(C_1)| = 3$. If $d(x_1, a) \geq 2$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x_1), f(a)\} = \{1, 2\}$, and clearly $f(x_1) = 2$ and $f(a) = 1$, and then we obtain a contradiction. Thus $d(x_1, a) = 1$ and so $x_1 = a$. Let $a_1 \in V(C_1) \cap N(x)$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x_1), f(a_1)\} = \{1, 2\}$, and clearly $f(x_1) = 2$ and $f(a_1) = 1$, and we obtain a contradiction. We deduce that $|V(C_1)| = 5$, and similarly $|V(C_2)| = 5$.

From Lemma 16, we obtain that $a = b = x$. Then $\gamma_R(G) = \gamma_R(G + ab) = 8$, where $a, b \in N(x) \cap V(C_1)$, a contradiction.

- Case 2. P does not contain x. Without loss of generality assume that $x \in V(C_1)$, since $\deg(x) \geq 3$, $x \notin C(2)$ and x is not a support vertex. So any vertex of C_2 is either a support vertex, or a vertex of degree two. By Lemma 24, no vertex of C_2 is support. Now by Lemmas 13 and 15, $|V(C_2)| \in \{3, 5\}$. If $|V(C_2)| = 3$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x_1), f(b)\} = \{1, 2\}$, and clearly $f(b) = 1$. Then $f(u) = f(w) = 1$, where $V(C_2) = \{a, u, w\}$, a contradiction. Thus $|V(C_2)| = 5$. By Lemma 16, $d(C_1, C_2) = 0$. Let $y, z \in N(a) - V(C_2)$. By Lemma 19, $\deg(y) \geq 3$, and y is not a support vertex. This implies that $x = y$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x_1), f(y)\} = \{1, 2\}$, and clearly $f(x_1) = 2$. Then $f(x) = 0$, and $f(a) + f(a_1) + ... + f(a_4) \geq 4$, where $V(C_2) = \{a, a_1, ..., a_4\}$. Now g defined on $V(G)$ by $g(a) = g(a_2) = 2$, $g(y) = g(a_1) = g(a_3) = g(a_4) = 0$, and $g(u) = f(u)$ for $u \in V(G) - \{a, a_1, ..., a_4, y\}$, is an RDF for G, a contradiction. $\diamond$

Thus $t = 2$, and therefore $x \in C(2)$. Since $C(2)$ is complete, we may assume without loss of generality that for any vertex $u \in V(C_1) - \{a\}$, either $\deg(u) = 2$, or u is a support vertex. By Lemmas 13, 15 and 24, $|V(C_1)| \in \{3, 5\}$. Let $V(C_1) = \{a, a_1, .., a_l\}$, where $l = 2$ or 4. It is obvious that for any $\gamma_R(G)$-function f, $f(x) = 2$. Using this, it is a routine matter to obtain the following, and we omit the proofs.

Fact 2. If $|V(C_1)| = 3$ and C_1 has two vertices of degree two, then $d(x, C_1) = 1$.

Fact 3. If $|V(C_1)| = 5$ and C_1 has four vertices of degree two, then $d(x, C_1) \neq 0$.

Fact 4. $V(C_2) \cap C(2) \not\subseteq \{b\}$.

Proof of Fact 4. Assume that $V(C_2) \cap C(2) \subseteq \{b\}$. By Lemmas 13, 15 and 24, $|V(C_2)| \in \{3,5\}$. Furthermore, C_i has $|V(C_i)| - 1$ vertices of degree two for $i = 1,2$.

If $|V(C_1)| = |V(C_2)| = 5$, then by Lemma 16, $d(C_1, C_2) = 1$ and $\{a,b\} \subseteq C(2)$, or $d(C_1, C_2) = 0$, since C_1 and C_2 have no vertex of degree three. Suppose that $d(C_1, C_2) = 1$ and $\{a,b\} \subseteq C(2)$. By Lemmas 9 and 12, $\deg(a) = \deg(b) = 4$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(b_1)\} = \{1,2\}$, where $b_1 \in N(b) \cap V(C_2)$. Then $w(f) \geq 11$, while $\gamma_R(G) \leq 10$, a contradiction. Thus we assume that $d(C_1, C_2) = 0$. Let $V(C_1) \cap V(C_2) = \{a\}$, and let $b_1 \in N(a) \cap V(C_2)$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(b_1)\} = \{1,2\}$. Let $V(C_2) = \{a, b_1, ..., b_4\}$. Assume that $f(a_1) = 2$. Then $f(a_3)+f(a_4)+f(b_2)+f(b_3)+f(b_4) \geq 4$. Now g defined on $V(G)$ by $g(a) = g(a_2) = g(b_2) = 2$, $g(a_1) = g(a_3) = g(a_4) = g(b_1) = g(b_3) = g(b_4) = 0$, and $g(u) = f(u)$ if $u \in V(G) - \{a, b, a_1, ..., a_4, b_1, ..., b_4\}$, is an RDF for G of weight less than $\gamma_R(G)$, a contradiction.

If $|V(C_1)| = |V(C_2)| = 3$, then by Lemma 17, $d(C_1, C_2) = 1$, which contradicts Fact 2. Thus we may assume that $|V(C_1)| = 3$ and $|V(C_2)| = 5$. By Fact 2, $d(x, C_1) = 1$ and by Lemma 16, C_2 has a vertex of degree at least four. Consequently, $d(x, C_2) = 0$, contradicting Fact 3. ◇

Thus $V(C_2) \cap C(2) \not\subseteq \{b\}$. From Lemmas 13, 15 and 24, we obtain that $|V(C_1)| \in \{3,5\}$, and by Lemma 16 and Facts 2 and 3, $|V(C_1) \cap V(C_2)| = 1$. Let $V(C_1) \cap V(C_2) = \{a\}$. We show that $|V(C_1)| = 3$.

Fact 5. $|V(C_1)| = 3$.

Proof of Fact 5. Suppose that $|V(C_1)| = 5$. Let $\{y, z\} \subseteq N(a) \cap V(C_2)$. By Lemma 19, we may assume, without loss of generality, that $\deg(z) \geq 3$, and z is not a support vertex. Now we consider y.

- (a) If y is a support vertex of degree three, then we let $w \in$

$N(y) - \{a, y_1\}$, where y_1 is the leaf adjacent to y. If w is a support vertex of degree 3, then by Lemma 10, $|V(C_2)| = 4$ and w is adjacent to z. Let w_1 be the leaf adjacent to w. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(w_1), f(a_1)\} = \{1,2\}$, and we can easily obtain a contradiction. Now suppose that $\deg(w) = 2$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(a_4)\} = \{1,2\}$, and we obtain a contradiction. If $w \in C(2)$, then by Lemmas 9 and 12, $\deg(w) = 3$, and by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(w), f(a)\} = \{1,2\}$, and clearly $f(w) = 2$. This produces a contradiction. It remains to assume that y is adjacent to z. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(y_1)\} = \{1,2\}$, where y_1 is the leaf adjacent to y, and this easily produces a contradiction.

- (b) If $\deg(y) = 2$, then we let $h \in C(2)$ (may be $h = x$). If $d(h,z) > 1$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(h), f(z)\} = \{1,2\}$, and clearly $f(h) = 2$. This produces a contradiction. Thus $d(h,z) \leq 1$. If $z \notin C(2)$, then $h \in N(z) - V(C_2)$, and by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(h), f(y)\} = \{1,2\}$, and clearly $f(h) = 2$. This produces a contradiction. Thus $z \in C(2)$. If $|V(C_2)| \geq 6$, then we let $w \in N(y) - \{a\}$ and $u \in (N(w) \cap V(C_2)) - \{y\}$. By Lemmas 10, and 14, $\deg(w) = 2$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(z), f(a_1)\} = \{1,2\}$, and we can obtain a contradiction. Thus $|V(C_2)| \leq 5$.

  - If $|V(C_2)| = 3$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(y)\} = \{1,2\}$, and so $f(y) + f(a) + f(a_1) + ... + f(a_4) \geq 5$. Then g defined on $V(G)$ by $g(a) = g(a_2) = 2$, $g(y) = g(a_1) = g(a_3) = g(a_4) = 0$, and $g(u) = f(u)$ fore $u \in V(G) - \{y, a, a_1, ..., a_4\}$ is an RDF for G, a contradiction.
  - If $|V(C_2)| = 4$, then $V(C_2) = \{y, w, z, a\}$. If $\deg(w) = 2$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(a_4)\} = \{1,2\}$. Then $f(a) = 0$, and either $f(y) = 1$ or $f(y) + f(w) \geq 2$. If $f(y) = 1$, then g defined on $V(G)$ by $g(a) = g(a_2) = 2$, $g(a_1) = g(a_3) = g(a_4) = g(y) = 0$,

and $g(u) = f(u)$ if $u \in V(G) - \{y, a, a_1, ..., a_4\}$ is an RDF for G, a contradiction. Thus $f(y) + f(w) \geq 2$. Then g defined on $V(G)$ by $g(a) = g(a_2) = 2$, $g(w) = 1$, $g(y) = g(a_1) = g(a_3) = g(a_4) = 0$, and $g(u) = f(u)$ if $u \in V(G) - \{y, w, a, a_1, ..., a_4\}$ is an RDF for G, a contradiction. We deduce that $\deg(w) > 2$. If w is a support vertex, then similarly we obtain a contradiction. It remains to assume that $w \in C(2)$. Since $G[C(2)]$ is complete, $z \in C(2)$, and $C(2) = \{z, w\}$. By Lemmas 9 and 12, $\deg(w) = \deg(z) = 3$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(y), f(z)\} = \{1, 2\}$. Then $w(f) \geq 9$, while $\gamma_R(G) = 8$, a contradiction.

- If $|V(C_2)| = 5$, then $V(C_2) = \{y, w, z, a, u\}$, where $w \in N(y)$. If there is a support vertex in $\{u, w\}$, then by Lemmas 14, and 10, $\deg(w) = 2$ and u is a support vertex. Let u_1 be the leaf adjacent to u, and z_1 is a support vertex adjacent to z. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(z_1), f(u_1)\} = \{1, 2\}$. This easily produces a contradiction. This implies that either $\deg(u) = 2$ or $u \in C(2)$. If $\deg(u) = 2$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(u), f(a_1)\} = \{1, 2\}$. This produces a contradiction. So we assume that $u \in C(2)$. By Lemmas 9 and 12, $\deg(u) = \deg(z) = 3$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(z), f(w)\} = \{1, 2\}$. Then $w(f) \geq 10$, while $\gamma_R(G) \leq 9$, a contradiction.

- (c) If $y \in C(2)$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(y), f(a_2)\} = \{1, 2\}$, and clearly $f(y) = 2$. Then $f(a) + f(a_1) + f(a_2) + f(a_3) + f(a_4) \geq 4$. Now g defined on $V(G)$ by $g(a) = g(a_1) = g(a_3) = 0$, $g(a_2) = 2$, $g(a_4) = 1$ $g(u) = f(u)$ if $u \in V(G) - \{a, a_1, ..., a_4\}$ is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. This completes the proof of Fact 5. ◇

Thus $|V(C_1)| = 3$. Let $d(a, x) = d(C_1, x)$. By Fact 2, $d(x, a) = 1$. By Fact 4, we may assume that $x \in V(C_2)$ and $V(C_1) \cap V(C_2) = \{a\}$. If $|V(C_2)| \geq 4$, then we let $b \in (V(C_2) \cap N(a)) - \{x\}$. By Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(x), f(b)\} = \{1, 2\}$, and clearly $f(x) = 2$. Then $f(a) = 0$, and so $f(a_1)+f(a_2) \geq 2$. Now g defined on $V(G)$ by $g(a) = 2$, $g(a_1) = g(a_2) = g(b) = 0$, $g(u) = f(u)$ if $u \in V(G) - \{a, a_1, a_2, b\}$ is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus $|V(C_2)| = 3$. If $\deg(b) = 2$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(b)\} = \{1, 2\}$. Then g defined on $V(G)$ by $g(a) = 2$, $g(a_1) = g(a_2) = g(b) = 0$, and $g(u) = f(u)$ if $u \in V(G) - \{a, a_1, a_2, b\}$ is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus $\deg(b) \geq 3$. If b be a support vertex of degree three, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(b)\} = \{1, 2\}$. Then g defined on $V(G)$ by $g(a) = 2$, $g(a_1) = g(a_2) = g(b) = 0$, $g(w) = 1$, where w is the leaf adjacent to b, and $g(u) = f(u)$ if $u \in V(G) - \{a, a_1, a_2, b\}$ is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. Thus $b \in C(2)$. Since $G[C(2)]$ is complete, $C(2) = \{x, b\}$. If $\deg(b) = 3$, then by Theorem 5, there is a $\gamma_R(G)$-function f such that $\{f(a_1), f(b)\} = \{1, 2\}$, and clearly $f(b) = 2$. Then g defined on $V(G)$ by $g(a) = 2$, $g(a_1) = g(a_2) = g(b) = g(w_1) = 0$, $g(w) = 2$, w is the support vertex adjacent to b and w_1 is the leaf adjacent to w, and $g(u) = f(u)$ if $u \in V(G) - \{a, a_1, a_2, b, w, w_1\}$ is an RDF for G of weight less than $\gamma_R(G)$, a contradiction. We conclude that $\deg(b) \geq 4$. By Lemmas 9 and 12, $\deg(b) = 4$. Similarly we obtain that $\deg(x) = 4$. Now it is straightforward to see that $G \in \{H_1, H_2\}$.

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