PLANAR, OUTERPLANAR AND RING GRAPH COZERO-DIVISOR GRAPHS

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ABSTRACT. Let R be a commutative ring with non-zero identity. The cozero-divisor graph of R, denoted by $\Gamma'(R)$, is a graph with vertex-set $W^*(R)$, which is the set of all non-zero non-unit elements of R, and two distinct vertices a and b in $W^*(R)$ are adjacent if and only if $a \notin Rb$ and $b \notin Ra$, where for $c \in R$, Rc is the ideal generated by c. In this paper, we completely determine all finite commutative rings R such that $\Gamma'(R)$ is planar, outerplanar and ring graph.

1. INTRODUCTION

The investigation of graphs related to algebraic structures is a very large and growing area of research. One of the most important classes of graphs considered in this framework is that of Cayley graphs. These graphs have been considered, for example, in [23], [24], [25], [26] for groups and in [14], [18], [19] for semigroups. Let us refer the readers to the survey article [22] for extensive bibliography devoted to various applications of the Cayley graphs. In particular, the Cayley graphs of semigroups are related to automata theory, as explained in [17] and the monograph [16]. Several

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other classes of graphs associated with algebraic structures have been also actively investigated. For example, power graphs and divisibility graphs have been considered in [20], [21]. Graphs associated to rings have been studied with respect to several ring constructions, see [13] and [15]. The present article concentrates on zero-divisor graphs of rings that have been investigated in [4], [5], [6], [7], [8], [9].

Let R be a commutative ring with non-zero identity and let $Z^*(R)$ be the set of all non-zero zero-divisors of R. Also suppose that $W^*(R)$ is the set of all non-zero non-unit elements of R.

For an arbitrary commutative ring R, the cozero-divisor graph of R, denoted by $\Gamma'(R)$, was introduced in [1], which is a dual of the zero-divisor graph $\Gamma(R)$ "in some sense". The vertex-set of $\Gamma'(R)$ is $W^*(R)$ and, for two distinct vertices a and b in $W^*(R)$, a is adjacent to b if and only if $a \notin Rb$ and $b \notin Ra$, where Rc is an ideal generated by the element c in R. In [1] and [2], some basic results on the structure of this graph were obtained. Also, in [3], the situation where $\Gamma'(R)$ is planar, outerplanar and ring graph is investigated in the following cases:

(α) R is not local,
(β) R is a local ring such that the maximal ideal $\mathfrak{m}$ of R is a principal ideal, or $|R| \neq 2^k$ for $k \geqslant 2$.

As we mentioned at the end of [3], the remaining case is that $\mathfrak{m}$ is not principal and $|R| = 2^k$ for some $k \geqslant 3$. This is the situation that we study for planarity of $\Gamma'(R)$, and so we determine all finite commutative rings such that $\Gamma'(R)$ is planar. We also characterize all finite commutative rings such that $\Gamma'(R)$ is an outerplanar graph or a ring graph.

Throughout the paper R is a finite commutative ring with non-zero identity. Hence we have that $W^*(R) = Z^*(R)$.

2. Planar, outerplanar and ring graph Cozero-divisor graphs

Recall that a graph is said to be *planar* if it can be drawn in the plane, such that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem states that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$, where K_n is a complete graph with n vertices and $K_{m,n}$ is a complete bipartite graph, with parts of sizes m and n (cf. [10, p. 153]).

In [3], the present first and third authors characterized all finite non-local rings with planar cozero-divisor graphs as follows.

Theorem 2.1. [3, Theorem 2.5] *Let R be a non-local ring. Then $\Gamma'(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$
$$\mathbb{Z}_2 \times \mathbb{F}, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times (\mathbb{Z}_2[X]/(X^2)),$$
$$\mathbb{Z}_3 \times \mathbb{F}, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times (\mathbb{Z}_2[X]/(X^2)),$$

where $\mathbb{F}$ *is a finite field.*

If $(R, \mathfrak{m})$ is a local ring, then the following results are proved in [3].

Theorem 2.2. [3, Theorem 2.6] *Suppose that* $(R, \mathfrak{m})$ *is a local ring such that* $\mathfrak{m}$ *is a principal ideal. Then* $\Gamma'(R)$ *is planar.*

Theorem 2.3. [3, Theorem 2.9] *Assume that* $(R, \mathfrak{m})$ *is a local ring such that* $\mathfrak{m}$ *is not principal. If* $|R|$ *is odd, then* $\Gamma'(R)$ *is not planar.*

The only remaining case that is left as a question in [3] is the case where $\mathfrak{m}$ is not principal and $|R| = 2^k$, where $k \geqslant 3$. In this note we shall completely determine such rings with planar cozero-divisor graphs. Hence in the rest of the paper unless stated otherwise, we assume that $(R, \mathfrak{m})$ is a local ring such that $\mathfrak{m}$ is not principal and $|R| = 2^k$, where $k \geqslant 3$. Also S stands for a minimal generating set of $\mathfrak{m}$.

We first present the following lemma.

Lemma 2.4. *Suppose that* $\Gamma'(R)$ *is planar. Then* $|S| = 2$ *and, for each* $a \in S$, $|U(R)a| \leqslant 2$.

Proof. If $|S| > 2$, then, for $a, b, c \in S$, the induced subgraph of $\Gamma'(R)$ with vertex set $\{a, b, c, a+b, a+c\}$ forms the complete graph K_5, and so $\Gamma'(R)$ is not planar. So we may assume that $S = \{a, b\}$. Now, if $|U(R)a| > 2$, then we consider three distinct vertices a, ua, va in $U(R)a$. Hence the induced subgraph of $\Gamma'(R)$ with vertex sets $\{a, ua, va\}$ and $\{b, a+b, a+ub\}$ form the bipartite graph $K_{3,3}$, and so $\Gamma'(R)$ again is not planar. Thus $|U(R)a| \leqslant 2$. Similarly, we have $|U(R)b| \leqslant 2$. □

Theorem 2.5. *Assume that* $\Gamma'(R)$ *is planar. Then* $|R| \leqslant 16$. *Moreover, if* $|R/\mathfrak{m}| = 2$, *then* $|\mathfrak{m}/(0:a)| = |\mathfrak{m}/(0:b)| = 2$ *and* $(0:a) \neq (0:b)$, *where* a *and* b *belong to a minimal generating set of* $\mathfrak{m}$.

Proof. Let $S = \{a, b\}$. By Lemma 2.4, we have $|U(R)a|, |U(R)b| \leqslant 2$. Now one can easily see that $R = (U(R) + U(R)) \cup U(R)$, which implies that $|Ra| \leqslant 6$. Recall that there exists an isomorphism $R/(0:a) \cong Ra$. Now since $(0:a) \subseteq \mathfrak{m}$, we have $|R/\mathfrak{m}| \leqslant |R/(0:a)| \leqslant 4$. Similarly, we have $|R/\mathfrak{m}| \leqslant |R/(0:b)| \leqslant 4$.

We have the following two cases.

Case 1. $|R/\mathfrak{m}| = 4$. In this case, we have $|R/(0:a)| = 4 = |R/(0:b)|$. Hence $\mathfrak{m} = (0:a) = (0:b)$. Thus $\mathfrak{m} = \{ra + sb \mid r, s \in U(R)\}$, and so $|\mathfrak{m}| \leqslant 9$. Therefore we have that $|\mathfrak{m}| = 8$ or $|\mathfrak{m}| = 4$.

If $|\mathfrak{m}| = 8$, then we have $|U(R)a| = 2 = |U(R)b|$. In this situation, we have $|\{a, b, ua, ub\}| = 4$ and $|\{a + b, ua + b, a + ub, ua + ub\}| = 3$, where $u \in U(R) \setminus \{1\}$. Hence one can easily see that $K_{3,3}$ is a subgraph of $\Gamma'(R)$, and so it is not planar. Thus $|\mathfrak{m}| = 4$, and so $|R| = 16$.

Case 2. $|R/\mathfrak{m}| = 2$. In this case, for each generator $x \in S$, either $(0 : x) = \mathfrak{m}$ or $|\mathfrak{m}/(0 : x)| = 2$. Thus we have the following three situations.

(i) If $(0 : a) = \mathfrak{m} = (0 : b)$, then $\mathfrak{m}^2 = 0$, and so $\mathfrak{m} = \{0, a, b, a + b\}$. Hence $|R| = 8$.

(ii) If $(0 : a) = \mathfrak{m}$ and $|\mathfrak{m}/(0 : b)| = 2$, then clearly $b^2 \neq 0$. Let $\mathfrak{n} = (0 : b)$. Then we have $\mathfrak{m} = \mathfrak{n} \cup (\mathfrak{n} + b)$. We claim that $b^3 = 0$. Suppose on the contrary that $b^3 \neq 0$. Then $\mathfrak{n}+b = \mathfrak{n}+b^2$. Thus $b(1-b) \in \mathfrak{n}$ and hence $b \in \mathfrak{n}$ which is a contradiction. Therefore we have $b^3 = 0$. We also have $\mathfrak{n}^2 = 0$. Now, we show that the vertices of the set $\{a, b, a + b, a + b^2, a + b + b^2\}$ induce the complete graph K_5. If $a \in R(a+b^2)$, then we have $a = r(a+b^2)$, for some non-zero element $r \in R$. Then $(1 - r)a = rb^2$. If $r \in U(R)$, then $1 - r \in \mathfrak{m}$, and so $(1 - r)a = 0$. Thus $b^2 = 0$, which is impossible. Otherwise, $1 - r \in U(R)$, and so $a \in Rb$, which is again impossible. Hence $a \notin R(a + b^2)$. Similarly $a + b^2 \notin Ra$. Therefore the vertices a and $a + b^2$ are adjacent. Now, if $a = r(a + b + b^2)$ for some non-zero element $r \in R$, then $(1 - r)a = r(b + b^2)$. If r is not unit, then $1 - r$ is unit and $a \in Rb$, which is impossible. If r is a unit element, then we have $b + b^2 = 0$ so that $b^2 + b^3 = 0$. This implies that $b^2 = 0$, which is again impossible. Thus $a \notin R(a + b + b^2)$. Similarly, $(a + b + b^2) \notin Ra$. Hence a is adjacent to $a + b + b^2$. Now, by a similar discussion, it is easy to see that the vertices of the set $\{a, b, a + b, a + b^2, a + b + b^2\}$ induce the complete graph K_5. Therefore $\Gamma'(R)$ is not planar in this situation.

If $(0 : b) = \mathfrak{m}$ and $|\mathfrak{m}/(0 : a)| = 2$, then we also get that $\Gamma'(R)$ is not planar.

(iii) Suppose that $|\mathfrak{m}/(0 : a)| = 2 = |\mathfrak{m}/(0 : b)|$. We have the following two situations:

(a) Assume that $(0 : a) = (0 : b)$. Let $\mathfrak{n} = (0 : a)$. Then clearly $a, b \notin \mathfrak{n}$. So we have $\mathfrak{m} = \mathfrak{n} \cup (\mathfrak{n} + a) = \mathfrak{n} \cup (\mathfrak{n} + b)$. Thus $\mathfrak{n} + a = \mathfrak{n} + b$. Hence $b = a + n$, for some $n \in \mathfrak{n}$. Now we have $bx = ax + nx = ax$ for all $x \in \mathfrak{m}$. In particular, we have $a^2 = ab = b^2$. We claim that $a^2 \in \mathfrak{n}$. Suppose on the contrary that $a^2 \notin \mathfrak{n}$. Then $a^2 + \mathfrak{n} = a + \mathfrak{n}$. Thus $a(1 - a) \in \mathfrak{n}$. Since $1 - a$ is unit, we have that $a \in \mathfrak{n}$, which is impossible. Hence $a^2 = ab = b^2 \in \mathfrak{n}$. Thus $Rab = \{0, ab\}$. Now one can easily see that

$$\mathfrak{m} = \{c_1a + c_2b + c_3ab \mid c_i \in \{0, 1\}, \text{ for all } 1 \leqslant i \leqslant 3\}.$$

Hence $|\mathfrak{m}| \leqslant 8$. Since $\mathfrak{n} = \{0, a + b, ab, a + b + ab\}$, we have that $|\mathfrak{m}| = 8$. Clearly, $\mathfrak{m} = \{0, a, b, ab, a + b, a + ab, b + ab, a + b + ab\}$. We show that, in this case, the vertices of the sets $\{a, a + a^2, a + b\}$ and $\{b, b + b^2, a + b + b^2\}$ induce a subgraph isomorphic to $K_{3,3}$, and so $\Gamma'(R)$ is not planar. Clearly,

$a \notin R(b+b^2)$. If $b+b^2 = ra$, for some $r \in R$, then we have $b = ra - ab \in Ra$, which is impossible. So a and $b+b^2$ are adjacent. Similarly, b and $a+a^2$ are adjacent. If $a = r(a+b+b^2)$, for some $r \in R$, then we have $(1-r)a = r(b+b^2)$. If $1-r$ is unit, then $a \in Rb$, which is impossible. If r is unit, then we have $b = r^{-1}(1-r)a - ab \in Ra$, which is again impossible. Also, if $a+b+b^2 = ra$, for some $r \in R$, then we have $b \in Ra$, which is impossible. Thus a is adjacent to $a+b+b^2$. If $a+a^2 = r(a+b+b^2)$, for some $r \in R$, then we have $r(a+b+b^2) = 0$, whenever $r \in \mathfrak{m}$, which is a contradiction. Otherwise, r is unit and since $rb = a^2 + (1-r)a - rab$, $b \in Ra$, which is impossible. Similarly, we have $a+b+b^2 \notin R(a+a^2)$. So $a+a^2$ is adjacent to $a+b+b^2$. Also, one can easily see that $a+a^2$ is adjacent to $b+b^2$. Now, if $a+b+b^2 = r(a+b)$, then we have $a+b+b^2 = 0$, whenever $r \in \mathfrak{m}$, which is impossible. Otherwise, r is a unit and $(r-1)(a+b) = b^2$. Since $r-1 \in \mathfrak{m}$, we have $b^2 = 0$, which is impossible. Similarly, $(a+b) \notin R(a+b+b^2)$. Also, one can easily check that $a+b$ is adjacent to vertices b and $b+b^2$. Therefore $\Gamma'(R)$ has a subgraph isomorphic to $K_{3,3}$ so that it is not planar.

(b) Suppose that $(0:a) \neq (0:b)$. Let $\mathfrak{n} = (0:a) \cap (0:b)$. Clearly $|\mathfrak{m}/\mathfrak{n}| = 4$. Thus $|(0:a)/\mathfrak{n}| = |(0:b)/\mathfrak{n}| = 2$. Since the elements a and $a+b$ also generate $\mathfrak{m}$, we have $|\mathfrak{m}/(0:(a+b))| = 2$. We claim that in this situation $|R| \leqslant 16$. Since $a, b, a+b \notin \mathfrak{n}$ and $|\mathfrak{m}/\mathfrak{n}| = 4$, we have $\mathfrak{m} = \mathfrak{n} \cup (\mathfrak{n}+a) \cup (\mathfrak{n}+b) \cup (\mathfrak{n}+a+b)$. Hence we have $(0:a) = \mathfrak{n} \cup (\mathfrak{n}+x)$, $(0:b) = \mathfrak{n} \cup (\mathfrak{n}+y)$ and $(0:a+b) = \mathfrak{n} \cup (\mathfrak{n}+z)$, where $\{x,y,z\} = \{a,b,a+b\}$. First suppose that there exist generators a, b of $\mathfrak{m}$ with $ab = 0$. Then if $(0:a+b) = \mathfrak{n} \cup (\mathfrak{n}+a)$ or $(0:a+b) = \mathfrak{n} \cup (\mathfrak{n}+b)$, then we get that $a, b \in \mathfrak{n}$, which is impossible. Hence we have $(0:a+b) = \mathfrak{n} \cup (\mathfrak{n}+a+b)$. Thus $(a+b)^2 = 0$ which implies that $a^3 = b^3 = a^2 + b^2 = 0$. Therefore $\mathfrak{m} = \{ra + sb + ta^2 \mid r,s,t \in R\}$. Now one can easily check that $|\mathfrak{m}| \leqslant 8$, and so $|R| \leqslant 16$.

Now, assume that, for each minimal generating set $\{c,d\}$ of $\mathfrak{m}$, we have $cd \neq 0$. Hence we have $x = a$, $y = b$ and $z = a+b$. Thus $a^2 = b^2 = 2ab = 0$. Now, in this situation we have $\mathfrak{m} = \{ra + sb + tab \mid r,s,t \in R\}$, and it is easy to see that $|\mathfrak{m}| \leqslant 8$. Therefore $|R| \leqslant 16$.

Now since $|\mathfrak{m}/\mathfrak{n}| = 4$, we have that $|\mathfrak{m}| = 8$ and $|R| = 16$. □

Now it is enough to investigate the planarity of rings R with $|R| \leqslant 16$.

Proposition 2.6. *If $(R,\mathfrak{m})$ is a local ring such that $|R| = 2^n \leqslant 8$, then $\Gamma'(R)$ is planar.*

Proof. The result is obvious as $\Gamma'(R)$ has $|\mathfrak{m}| - 1 \leqslant 3$ vertices. □

Proposition 2.7. *Let $(R,\mathfrak{m})$ be a local ring with 16 elements. Then $\Gamma'(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{F}_{16}, \mathbb{F}_4[X]/(x^2), \mathbb{Z}_2[X]/(X^4),$$

$$\mathbb{Z}_2[X,Y]/(X^2-Y^2,XY), \mathbb{Z}_2[X,Y]/(X^2,Y^2), \mathbb{Z}_4[X]/(X^2+X+1),$$
$$\mathbb{Z}_4[X]/(2X,X^3-2), \mathbb{Z}_4[X]/(X^2-2), \mathbb{Z}_4[X]/(X^2-2X-2),$$
$$\mathbb{Z}_4[X,Y]/(X^2-2,XY,Y^2-2,2X), \mathbb{Z}_4[X,Y]/(X^2,XY-2,Y^2), \mathbb{Z}_4[X]/(X^2),$$
$$\mathbb{Z}_4[X]/(X^2-2X), \mathbb{Z}_8[X]/(2X,X^2-4), \mathbb{Z}_{16}.$$

Proof. In [11], Corbas and Williams showed that the non-isomorphic local rings with identity of order 16 are precisely the following 21 rings:

$$\mathbb{F}_{16}, \mathbb{F}_4[X]/(x^2), \mathbb{Z}_2[X]/(X^4),$$
$$\mathbb{Z}_2[X,Y]/(X^3,XY,Y^2), \mathbb{Z}_2[X,Y]/(X^2-Y^2,XY), \mathbb{Z}_2[X,Y]/(X^2,Y^2),$$
$$\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2, \mathbb{Z}_4[X]/(X^2+X+1), \mathbb{Z}_4[X]/(2X,X^3-2),$$
$$\mathbb{Z}_4[X]/(X^2-2), \mathbb{Z}_4[X]/(X^2-2X-2), \mathbb{Z}_4[X,Y]/(X^2-2,XY,Y^2,2X),$$
$$\mathbb{Z}_4[X,Y]/(X^2-2,XY,Y^2-2,2X), \mathbb{Z}_4[X,Y]/(X^2,XY-2,Y^2),$$
$$\mathbb{Z}_4[X]/(2X,X^3), \mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2-2X),$$
$$\mathbb{Z}_4[X,Y]/(2,X,Y)^2, \mathbb{Z}_8[X]/(2X,X^2), \mathbb{Z}_8[X]/(2X,X^2-4), \mathbb{Z}_{16}.$$

If R is one of the rings $\mathbb{F}_{16}, \mathbb{F}_4[X]/(x^2), \mathbb{Z}_4[X]/(X^2+X+1), \mathbb{Z}_4[X]/(X^2-2X-2), \mathbb{Z}_4[X]/(X^2-2)$, then $|\mathfrak{m}| \leq 4$ and $\Gamma'(R)$ is planar. Also, if R is one of the rings $\mathbb{Z}_{16}$, $\mathbb{Z}_2[X]/(X^4)$ or $\mathbb{Z}_4[X]/(2X,X^3-2)$, then $\mathfrak{m}$ is principal and by Theorem 2.2, $\Gamma'(R)$ is planar. Now, a simple verification shows that all of the following rings have isomorphic planar cozero-divisor graph, which is presented in Figure 1.

$(\mathbb{Z}_2[X,Y]/(X^2-Y^2,XY), \mathfrak{m} = \{0,X,Y,X+Y,X^2,X+X^2,Y+X^2,X+Y+X^2\})$,
$(\mathbb{Z}_2[X,Y]/(X^2,Y^2), \mathfrak{m} = \{0,X,Y,X+Y,XY,X+XY,Y+XY,X+Y+XY\})$,
$(\mathbb{Z}_4[X,Y]/(X^2-2,XY,Y^2-2,2X), \mathfrak{m} = \{0,2,X,Y,X+Y,2+X,2+Y,2+X+Y\})$,
$(\mathbb{Z}_4[X,Y]/(X^2,XY-2,Y^2), \mathfrak{m} = \{0,2,X,Y,2+X,2+Y,2+X+Y,X+Y\})$,
$(\mathbb{Z}_4[X]/(X^2), \mathfrak{m} = \{0,2,X,2+X,2X,2+2X,3X,2+3X\})$,
$(\mathbb{Z}_4[X]/(X^2-2X), \mathfrak{m} = \{0,2,X,2+X,2X,2+2X,3X,2+3X\})$,
$(\mathbb{Z}_8[X]/(2X,X^2-4), \mathfrak{m} = \{0,2,4,6,X,2+X,4+X,6+X\})$,

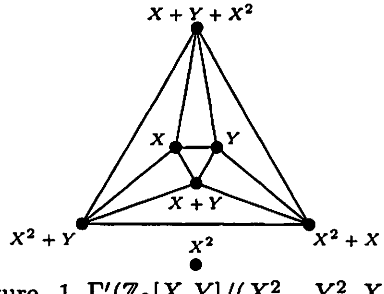

Figure. 1, $\Gamma'(\mathbb{Z}_2[X,Y]/(X^2-Y^2,XY))$

In the other remaining six rings, the cozero-divisor graphs contain a subdivision of K_5, and they are not planar. □

Now, we have the following conclusion which completely characterizes all finite commutative rings with planar cozero-divisor graphs.

Theorem 2.8. *Let R be an arbitrary finite commutative ring. Then $\Gamma'(R)$ is planar if and only if R is a local ring with principal maximal ideal, R is a local ring of order eight, or R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2[X,Y]/(X^2-Y^2,XY), \mathbb{Z}_2[X,Y]/(X^2,Y^2), \mathbb{Z}_4[X]/(X^2+X+1),$$

$$\mathbb{Z}_4[X,Y]/(X^2-2,XY,Y^2-2,2X), \mathbb{Z}_4[X,Y]/(X^2,XY-2,Y^2),$$

$$\mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2-2X), \mathbb{Z}_8[X]/(2X,X^2-4),$$

$$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,$$

$$\mathbb{Z}_2\times\mathbb{F}, \mathbb{Z}_2\times\mathbb{Z}_4, \mathbb{Z}_2\times(\mathbb{Z}_2[X]/(X^2)),$$

$$\mathbb{Z}_3\times\mathbb{F}, \mathbb{Z}_3\times\mathbb{Z}_4, \mathbb{Z}_3\times(\mathbb{Z}_2[X]/(X^2)),$$

where $\mathbb{F}$ is a finite field.

Let G be a graph with n vertices and q edges. We recall that a chord is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of G. We say that C is a primitive cycle if it has no chords. Also, a graph G has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The number frank(G) is called the free rank of G and it is the number of primitive cycles of G. Also, the number rank$(G) = q - n + r$ is called the cycle rank of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. By [12, Proposition 2.2], we have rank$(G) \leqslant$ frank(G). A graph G is called a ring graph if it satisfies in one of the following equivalent conditions (see [12]).

(i) rank(G) = frank(G),
(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Also, an undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.
Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

The following two theorems were proved in [3].

Theorem 2.9. [3, Theorem 3.1] *Let R be a non-local ring. Then $\Gamma'(R)$ is a ring graph if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{F}, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times (\mathbb{Z}_2[X]/(X^2)), \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Theorem 2.10. [3, Theorem 3.2] *Let R be a non-local ring. Then $\Gamma'(R)$ is outerplanar if and only if R is one of the following rings:*

$$\mathbb{Z}_2 \times \mathbb{F}, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times (\mathbb{Z}_2[X]/(X^2)), \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Now since we determined all finite commutative rings with planar cozero-divisor graphs, we can characterize all finite commutative rings such that their cozero-divisor graphs are ring graph and outerplanar. Note that, all local rings of order 16 with planar cozero-divisor graphs, such that their maximal ideal is not principal, contain a subdivision of $K_{2,3}$, and so they are not outerplanar, while one can easily check that they are ring graphs.

Now, by Theorem 2.8 in conjunction with Theorems 2.9 and 2.10, we have the following result.

Theorem 2.11. *Let R be an arbitrary finite commutative ring.*

(i) $\Gamma'(R)$ *is ring graph if and only if R is a local ring with principal maximal ideal, R is a local ring of order eight, or R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2[X,Y]/(X^2-Y^2,XY), \mathbb{Z}_2[X,Y]/(X^2,Y^2), \mathbb{Z}_4[X]/(X^2+X+1),$$

$$\mathbb{Z}_4[X,Y]/(X^2-2,XY,Y^2-2,2X), \mathbb{Z}_4[X,Y]/(X^2,XY-2,Y^2),$$

$$\mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2-2X), \mathbb{Z}_8[X]/(2X,X^2-4),$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{F}, \mathbb{Z}_2 \times \mathbb{Z}_4,$$

$$\mathbb{Z}_2 \times (\mathbb{Z}_2[X]/(X^2)), \mathbb{Z}_3 \times \mathbb{Z}_3.$$

(ii) $\Gamma'(R)$ *is outerplanar if and only if R is a local ring with principal maximal ideal, R is a local ring of order eight, or R is isomorphic to one of the following rings:*

$$\mathbb{F}_{16}, \Gamma'(\mathbb{Z}_2[X]/(X^4)), \Gamma'(\mathbb{Z}_4[X]/(2X,X^3-2)), \Gamma'(\mathbb{Z}_{16}),$$

$$\mathbb{Z}_2 \times \mathbb{F}, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times (\mathbb{Z}_2[X]/(X^2)), \mathbb{Z}_3 \times \mathbb{Z}_3.$$

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