$L(d, 1)$-labelings of the edge-multiplicity-path-replacements *

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Abstract

For a positive integers $d \geq 1$, an $L(d, 1)$-labeling of a graph G is an assignment of nonnegative integers to $V(G)$ such that the difference between labels of adjacent vertices is at least d, and the difference between labels of vertices that are distance two apart is at least 1. The span of an $L(d, 1)$-labeling of a graph G is the difference between the maximum and minimum integers used by it. The minimum span of an $L(d, 1)$-labeling of G is denoted by $\lambda_d(G)$. In [17], we obtained that $r\Delta+1 \leq \lambda(G(rP_5)) \leq r\Delta+2$, $\lambda(G(rP_k)) = r\Delta+1$ for $k \geq 6$; and $\lambda(G(rP_4)) \leq (\Delta+1)r+1$, $\lambda(G(rP_3)) \leq (\Delta+1)r+\Delta$ for any graph G with maximum degree Δ. In this paper, we will focus on $L(d, 1)$-labelings-number of the edge-multiplicity-path-replacement $G(rP_k)$ of a graph G for $r \geq 2$, $d \geq 3$ and $k \geq 3$. And we show that the class of graphs $G(rP_k)$ with $k \geq 3$ satisfies the conjecture proposed by Havet and Yu [7].
Keywords: channel assignment; $L(d, 1)$-labeling; $(d, 1)$-total labeling; the edge-multiplicity-path-replacement.

1 Introduction

In the channel assignment problem, we need to assign frequency bands to transmitters, if two transmitters are too close, interference will occur

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if they attempt to transmit on close frequencies. In order to avoid this situation, the separation of the channels assigned to them must be sufficient. Moreover, if two transmitters are close but not too close, the channels assigned must be different. In one such model, an *$L(d,1)$-labeling* of G is employed, which is an integer assignment f to the vertices of G such that for all u, v in $V(G)$, $d_G(u,v) = 1$ implies $|f(u) - f(v)| \geq d$, and $d_G(u,v) = 2$ implies $|f(u) - f(v)| \geq 1$. The minimum span of an $L(d,1)$-labeling of G is denoted by $\lambda_d(G)$. In 1992, Griggs and Yeh introduced this labeling with $d = 2$ in [3]. This notion has been studied many times and gives many challenging problems [1, 4, 5, 8–12, 14, 19, 20].

In particular, in 1995, Whittlesty, Georges and Mauro [20] studied the $L(2,1)$-labeling of the *incidence graph* of G, which is the graph obtained from G by replacing each edge by a path of length 2. The $L(2,1)$-labeling of the incidence graph of G is equivalent to an assignment of integers to each element of $V(G) \cup E(G)$ of G such that: (i) any two adjacent vertices of G receive different integers; (ii) any two adjacent edges of G receive different integers; and (iii) a vertex and its incident edge receive integers that differ by at least 2 in absolute value. This labeling is called (2,1)-total labeling of G, which was introduced by Havet and Yu in 2002 [6, 7] and generalized to the $(d,1)$-total labeling of a graph G. They [7] obtained the bound $\Delta + d - 1 \leq \lambda_d^T(G) \leq 2\Delta + d - 1$ and gave the conjecture as follows. It is sufficient to prove the conjecture for $\Delta > d$. Havet and Yu [7] completed the proof of the conjecture for $\Delta \leq 3$ by proving that $\lambda_2^T(G) \leq 6$ if $\Delta \leq 3$.

Conjecture 1.1 *[7] For any graph G, $\lambda_d^T(G) \leq \Delta + 2d - 1$.*

For $r \geq 1$ and $k \geq 3$, *the edge-multiplicity-paths-replacement $G(rP_k)$ of a graph G* is a graph obtained by replacing each edge uv with r vertex-disjoint paths P_k^i: $ux_{uv}^{i_1}x_{uv}^{i_2}\cdots x_{uv}^{i_{k-2}}v$, where $i = 1, 2, \cdots r$. Note that the vertices of G are called as the nodes of $G(rP_k)$. It is easily seen that $G(rP_{2k-1})$ is the incidence graph of $G(rP_k)$, and for $r\Delta \geq 2$, the maximum degree of $G(rP_k)$ is $r\Delta$ where Δ is the maximum degree of G. We can consider $G(rP_k)$ with $r = 1$ as the edge-path-replacement of a graph G.

In [13, 15, 16], the authors worked on $L(d, 1)$-labeling-number of the edge-path-replacement $G(P_k)$ of a graph G, which is a graph obtained by replacing each edge with a path P_k. We [17] obtained that $r\Delta + 1 \leq \lambda(G(rP_5)) \leq r\Delta + 2$, $\lambda(G(rP_k)) = r\Delta + 1$ for $k \geq 6$; and $\lambda(G(rP_4)) \leq (\Delta + 1)r + 1$, $\lambda(G(rP_3)) \leq (\Delta + 1)r + \Delta$ for any graph G with maximum degree Δ.

In this paper, we will focuses on $L(d, 1)$-labeling-number of the edge-multiplicity-path-replacement $G(rP_k)$ of a graph G for $r \geq 2$, $d \geq 3$ and $k \geq 3$. The same bounds are also used to show that the class of graphs $G(rP_k)$ with $k \geq 3$ satisfies Conjecture 1.1.

Note that $\lambda_d(G(rP_k)) \geq r\Delta + d - 1$, since G has a subgraph which is a star $K_{1,r\Delta}$. In the following section, suppose $r \geq 2$ and $d \geq 3$.

2 $k \geq 7$

2.1 $r\Delta > d$ and $k \geq 7$

Theorem 2.1 *Suppose $r \geq 2$ and $r\Delta > d$. Then $\lambda_d(G(rP_k)) = r\Delta + d - 1$ for $k \geq 11$, $r\Delta + d - 1 \leq \lambda_d(G(rP_k)) \leq r\Delta + d$ for $7 \leq k \leq 10$.*

Proof. It suffices to give an $L(d, 1)$-labeling f of the edge-multiplicity-path-replacement $G(rP_k)$ with span $r\Delta + d - 1$ for $k \geq 11$ and $r\Delta > d$. we give an $L(d, 1)$-labeling f of the edge-multiplicity-path-replacement $G(rP_k)$ with span $r\Delta + d - 1$ as follows. Label its nodes with 0, And label their adjacent vertices $x_{uv}^{i_1}$ and $x_{uv}^{i_{k-2}}$ in $G(rP_k)$ in $[d + (i - 1)\Delta, d + i\Delta - 1]$ for each $i \in \{1, 2, \cdots, r\}$. Similar to the proof of theorem 2.2 in [16], it is easily to prove that all the paths P_k can be labeled in $[0, r\Delta + d - 1]$, since $r\Delta + d - 1 \geq 2d$ for $r\Delta > d$. Then f is obviously an $L(d, 1)$-labeling of the edge-multiplicity-path-replacement $G(rP_k)$ with span $r\Delta + d - 1$ for $k \geq 11$ and $r\Delta > d$. Hence $\lambda_d(G(rP_k)) = r\Delta + d - 1$ for $k \geq 11$ and $r\Delta > d$.

For $7 \le k \le 10$, we give an $L(d,1)$-labeling f of the edge-multiplicity-path-replacement $G(rP_k)$ with span $r\Delta + d$ as follows. Label its nodes with 0, And label their adjacent vertices $x_{uv}^{i_1}$ and $x_{uv}^{i_{k-2}}$ in $G(rP_k)$ in $[d+(i-1)\Delta+1, d+i\Delta]$ for each $i \in \{1,\ 2,\ \cdots,\ r\}$. Similar to the proof of theorem 2.3 in [16], it is easy to prove that all the paths P_k can be labeled in $[0, r\Delta + d]$, since $r\Delta + d \ge 2d + 1$ for $r\Delta > d$. Then f is obviously an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $G(rP_k)$ with span $r\Delta + d$ for $7 \le k \le 10$ and $r\Delta > d$. Hence $\lambda_d(G(rP_k)) = r\Delta + d - 1$ for $7 \le k \le 10$ and $r\Delta > d$. ∎

2.2 $r\Delta \le d$ and $k \ge 7$

Theorem 2.2 *Suppose G is a graph with order n and maximum degree $r\Delta \le d$. Then $\lambda_d(G(rP_k)) \ge d + r\Delta$.*

Proof. Suppose f is an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $G(rP_k)$ with span $d + r\Delta - 1$. Then any one vertex x with degree $r\Delta$ must be labeled with 0 or $d + r\Delta - 1$. Without loss of generality, label x by 0. Then in its adjacent vertices in $G(rP_k)$, there must exist one vertex y whose label is d. So the adjacent vertex of y can't be labeled in $[0, d + r\Delta - 1]$, since $r\Delta \le d \iff r\Delta + d - 1 < 2d$. Thus $\lambda_d(G(rP_k)) \ge d + r\Delta$. ∎

Theorem 2.3 *Suppose $r \ge 2$ and $r\Delta \le d$. Then $\lambda_d(G(rP_k)) = r\Delta + d$ for odd $k \ge 9$, and $r\Delta + d \le \lambda_d(G(rP_7)) \le r\Delta + d + 1$.*

Proof. On the one hand, by theorem 2.2, $\lambda_d(G(rP_k)) \ge r\Delta + d$.

On the other hand, we give an $L(d,1)$-labeling f of the edge-multiplicity-path-replacement $G(rP_k)$ with span $r\Delta + d$ as follows for odd $k \ge 9$. Label its nodes with 0, And label their adjacent vertices $x_{uv}^{i_1}$ and $x_{uv}^{i_{k-2}}$ in $G(rP_k)$ in $[d + (i-1)\Delta + 1, d + i\Delta]$ for each $i \in \{1,\ 2,\ \cdots,\ r\}$. Similar to the proof of theorem 3.2 in [16], it is easy to prove that all the paths P_k can be labeled in $[0, r\Delta + d]$. Then f is obviously an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $G(rP_k)$ with span $r\Delta + d$ for $r\Delta \leq d$. Hence $\lambda_d(G(rP_k)) = r\Delta + d$ for odd $k \geq 9$ and $r\Delta \leq d$.

For $k = 7$, we give an $L(d,1)$-labeling f of the edge-multiplicity-path-replacement $G(rP_7)$ with span $r\Delta + d + 1$ as follows. Label its nodes with 0, And label their adjacent vertices $x_{uv}^{i_1}$ and $x_{uv}^{i_{k-2}}$ in $G(rP_7)$ in $[d + (i-1)\Delta + 2, d + i\Delta + 1]$ for each $i \in \{1,\ 2,\ \cdots,\ r\}$. Similar to the proof of theorem 3.3 in [16], it is easy to prove that all the paths P_7 can be labeled in $[0, r\Delta + d + 1]$. Then f is obviously an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $G(rP_7)$ with span $r\Delta + d + 1$ for $r\Delta \leq d$. Hence $\lambda_d(G(rP_7)) \leq r\Delta + d + 1$ for $r\Delta \leq d$. ∎

Theorem 2.4 *Suppose $r \geq 2$ and $r\Delta \leq d$. Then $r\Delta + d \leq \lambda_d(G(rP_k)) \leq 2d$ for even $k \geq 12$, and $r\Delta + d \leq \lambda_d(G(rP_k)) \leq 2d + 1$ for $k = 8, 10$.*

Proof. On the one hand, by theorem 2.2, $\lambda_d(G(rP_k)) \geq r\Delta + d$.

On the other hand, label its nodes with 0. And for even $k \geq 12$, label their adjacent vertices $x_{uv}^{i_1}$ and $x_{uv}^{i_{k-2}}$ in $G(rP_k)$ in $[d + (i-1)\Delta, d + i\Delta - 1]$ for each $i \in \{1,\ 2,\ \cdots,\ r\}$. Similar to the proof of case 1-3 of theorem 2.2 in section 2 in [16], it is easy to prove that all the paths P_k can be labeled in $[0, 2d]$. Then $r\Delta + d \leq \lambda_d(G(rP_k)) \leq 2d$ for even $k \geq 12$.

For even $k = 8, 10$, label their adjacent vertices $x_{uv}^{i_1}$ and $x_{uv}^{i_{k-2}}$ in $G(rP_k)$ in $[d + (i-1)\Delta + 1, d + i\Delta]$ for each $i \in \{1,\ 2,\ \cdots,\ r\}$. Similar to the proof of case 1 and case 3 of theorem 2.3 in section 2 in [16], it is easy to prove that all the paths P_k can be labeled in $[0, 2d + 1]$. Then $r\Delta + d \leq \lambda_d(G(rP_k)) \leq 2d + 1$ for $k = 8, 10$. ∎

Theorem 2.5 *Suppose $r \geq 2$, and $r\Delta \leq d$. If G is a bipartite graph, then $\lambda_d(G(rP_k)) = r\Delta + d$ for even $k \geq 8$, otherwise, $\lambda_d(G(rP_k)) = 2d$ for even $k \geq 12$, $2d \leq \lambda_d(G(rP_k)) \leq 2d + 1$ for $k = 8, 10$.*

Proof. If G is a bipartite graph with bipartition X, Y, label all the vertices in X with 0 and their adjacent vertices in $G(P_k)$ in $[d+(i-1)\Delta+1, d+i\Delta]$. label all the vertices in Y with $d+\Delta$ and their adjacent vertices in $G(P_k)$ in $[(i-1)\Delta, i\Delta-1]$.

Similar to the proof of theorem 3.4 in section 3 in [16], it is easy to prove that all the paths P_k can be labeled in $[0, d+r\Delta]$. So for even $k \geq 8$, $\lambda_d(G(rP_k)) = d + r\Delta$ when G is a bipartite graph.

If G is not a bipartite graph, then there must exist an odd cycle in G. Note that $\lambda_d(C_l) = 2d$ for odd l. So $\lambda_d(G(rP_k)) \geq 2d$, since there must exist an odd cycle in $G(rP_k)$ for even $k \geq 8$. By theorem 2.4, $\lambda_d(G(rP_k)) \leq 2d$ for $k \geq 12$, and $\lambda_d(G(rP_k)) \leq 2d+1$ for $k = 8, 10$. Thus we obtain that $\lambda_d(G(rP_k)) = 2d$ for $k \geq 12$, and $2d \leq \lambda_d(G(rP_k)) \leq 2d+1$ for $k = 8, 10$. ∎

3 $k = 5$

In this section, we give an upper bound for $k = 5$ by the result as follows.

Theorem 3.1 *[7] $\lambda_d^T(G) \leq \chi + \chi' + d - 2$, where χ and χ' are the chromatic number and the edge chromatic number of the graph G, respectively.*

Theorem 3.2 *Suppose G is a graph with maximum degree Δ. Then $d + r\Delta - 1 \leq \lambda_d(G(rP_5)) \leq d + r\Delta$, and $\lambda_d(G(rP_5) = d + r\Delta$ for $r\Delta \leq d$.*

Proof. Since that $G(rP_5)$ is the incidence graph of $G(rP_3)$, $\lambda_d(G(rP_5)) = \lambda_d^T(G(rP_3))$. By theorem 3.1, $\lambda_d^T(G(rP_3)) \leq \chi + \chi' + d - 2$, where χ and χ' are the chromatic number and the edge chromatic number of the graph $G(rP_3)$, respectively. Note that $G(rP_3)$ is bipartite. Then we have $\chi = 2$ and $\chi' = r\Delta$ by König's Theorem. Thus $\lambda_d(G(rP_5)) \leq d+r\Delta$. By theorem 2.2, $\lambda_d(G(rP_5) = d + r\Delta$ for $r\Delta \leq d$. ∎

4 $k = 4, 6$

We first consider $L(d,1)$-labeling-number of the edge-multiplicity-path- replacement of regular graph for $k = 4, 6$.

A *factor* of a graph G is a spanning subgraph of G. A *k-factor* of G is a factor of G that is k-regular. Thus a 2-factor of G is a factor of G that is a disjoint union of cycles of G. A graph G is k-factorable if G is an edge-disjoint union of k-factors of G.

Theorem 4.1 *[18] Every regular graph with positive even degree has a 2-factor.*

Theorem 4.2 *Suppose that G is a regular graph with positive even degree $r\Delta$. Let $k = 4, 6$. Then $\lambda_d(G(rP_k)) = d + r\Delta - 1$ for $\frac{r\Delta}{2} \geq d$, and $\lambda_d(G(rP_k)) \leq 2d + \frac{r\Delta}{2} - 1$ for $\frac{r\Delta}{2} < d$.*

Proof. By theorem 4.1, $G(rP_2)$ can be decomposed into $\frac{r\Delta}{2}$ 2-factors.

For $k = 4$, label all the nodes of $G(rP_4)$ with 0. For $\frac{r\Delta}{2} \geq d$, the replacement of each 2-factor from one node to the other can be labeled as follows: $0\ (d+j)(d+\frac{r\Delta}{2}+j)\ 0$ for $j \in \{0, 1, \cdots, \frac{r\Delta}{2}-1\}$. For $\frac{r\Delta}{2} < d$, the replacement of each 2-factor from one node to the other can be labeled as follows: $0\ (d+j)(2d+j)\ 0$ for $j \in \{0, 1, \cdots, \frac{r\Delta}{2}-1\}$. Then $\lambda_d(G(rP_4)) = d + r\Delta - 1$ for $\frac{r\Delta}{2} \geq d$, and $\lambda_d(G(rP_4)) \leq 2d + \frac{r\Delta}{2} - 1$ for $\frac{r\Delta}{2} < d$.

For $k = 6$, label all the nodes of $G(rP_6)$ with 0. For $\frac{r\Delta}{2} \geq d$, the replacement of each 2-factor from one node to the other can be labeled as follows: $0\ (d+j)pq(d+\frac{r\Delta}{2}+j)\ 0$ for $j \in \{0, 1, \cdots, \frac{r\Delta}{2}-1\}$, where $p = 2d$ and $q = 1$ for $j = 0$, otherwise $p = 1$ and $q = \frac{r\Delta}{2}+1$. For $\frac{r\Delta}{2} < d$, the replacement of each 2-factor from one node to the other can be labeled as follows: $0\ (d+j)(2d+j)1(d+\frac{r\Delta}{2}+1+j)\ 0$ for $j \in \{0, 1, \cdots, \frac{r\Delta}{2}-1\}$. Then $\lambda_d(G(rP_6)) = d + r\Delta - 1$ for $\frac{r\Delta}{2} \geq d$, and $\lambda_d(G(rP_6)) \leq 2d + \frac{r\Delta}{2} - 1$ for $\frac{r\Delta}{2} < d$. ∎

Theorem 4.3 *Suppose that G is a regular graph with positive odd degree $r\Delta$. Let $k = 4, 6$. Then $\lambda_d(G(rP_k)) \leq d + r\Delta$ for $\frac{r\Delta+1}{2} \geq d$, and $\lambda_d(G(rP_k)) \leq 2d + \frac{r\Delta+1}{2} - 1$ for $\frac{r\Delta+1}{2} < d$.*

Proof. we structure a graph H by connecting each pair vertices x and x' in $G(rP_2), G(rP_2)'$, where $G(rP_2)'$ is the copy graph of $G(rP_2)$, and x' corresponding to x. It is easy to see that H is a regular graph with positive even degree $r\Delta + 1$. Since $G(rP_k)$ is a subgraph of $H(rP_k)$, the proof is over by theorem 4.2. ∎

Theorem 4.4 *Suppose that G is a graph with the maximum degree Δ. Let $k = 4, 6$. Then If Δ is even, then $\lambda_d(G(rP_k)) = d + r\Delta - 1$ for $\frac{r\Delta}{2} \geq d$, and $\lambda_d(G(rP_k)) \leq 2d + \frac{r\Delta}{2} - 1$ for $\frac{r\Delta}{2} < d$. If Δ is odd, then $\lambda_d(G(rP_k)) \leq d + r(\Delta + 1) - 1$ for $\frac{r(\Delta+1)}{2} \geq d$, and $\lambda_d(G(rP_k)) \leq 2d + \frac{r(\Delta+1)}{2} - 1$ for $\frac{r(+1)\Delta}{2} < d$.*

Proof. Note that we can obtain a regular graph with the maximum degree $r\Delta$ such that $G(rP_2)$ is its subgraph. If there exist two vertices u and v whose degrees are less than $r\Delta$, then we add the edge uv. Lastly, we obtain a new graph G_1 in which there exists at least one vertex whose degree is less than $r\Delta$.

If there exists only one vertex x in G_1 whose degree is $a(< r\Delta)$, then we structure a graph H by adding all the edges between any two copies of x in $G_1, G_1^1, \cdots, G_1^{r\Delta-a+1}$, where $G_1^1, \cdots, G_1^{r\Delta-a+1}$ is the copy graph of G_1. So we obtain a regular graph with degree $r\Delta$ such that $G(rP_2)$ is its subgraph.

By theorems 4.2 and 4.3, the proof is over. ∎

5 $k = 3$

Theorem 5.1 *For $r \geq 2$, $\lambda_d(G(rP_3)) \leq r\chi' + \chi + rd - r - 1$, where χ is the chromatic number and the edge chromatic number of the graph G.*

Proof. Let c be a vertex colouring of G with the χ integers in $[0, \chi - 1]$. Let c' be a edge colouring of G with the χ' integers in $[0, \chi' - 1]$. Then label all the nodes of $G(rP_3))$ by c. For each $i \in \{1, 2, \cdots, r\}$, label the inserted vertices $x^{i_1}_{uv}$ as $\chi - 1 + (i-1)(\chi' - 1) + id + c'(uv)$. Thus we obtain an $L(d, 1)$-labeling of the edge-multiplicity-path-replacement $G(rP_3)$ with span $r\chi' + \chi + rd - r - 1$. So $\lambda_d(G(rP_3)) \leq r\chi' + \chi + rd - r - 1$. ∎

Theorem 5.2 $\lambda_d(G(rP_3)) \leq r(\lambda_d(G(P_3)) + 1) + r - 2$.

Proof. Note that $G(rP_3) - \{x^{i_1}_{uv} : 2 \leq i \leq r, uv \in E(G)\} \cong G(P_3)$. Let f be an $L(d, 1)$-labeling of $G(rP_3) - \{x^{i_1}_{uv} : 2 \leq i \leq r, uv \in E(G)\}$ with the $\lambda_d(G(P_3)) + 1$ integers of $[0, \lambda_d(G(P_3))]$. For $2 \leq i \leq r$, label $x^{i_1}_{uv}$ with $f(x^{(i-1)_1}_{uv}) + \lambda_d(G(P_3)) + 1 + i - 2$. Then the labeling is obviously a $L(d, 1)$-labeling of $G(rP_3)$ with span $r(\lambda_d(G(P_3)) + 1) + r - 2$. Thus $\lambda_d(G(rP_3)) \leq r(\lambda_d(G(P_3)) + 1) + r - 2$.

∎

6 Some graphs

In this section, we consider the edge-multiplicity-path-replacement of paths and cycles with $k = 3, 4, 6$.

6.1 $P_2(P_k)$ with $k = 3, 4, 6$

Note that the graph with order n and maximum degree $\Delta = 1$ is the path P_2. For $k \geq 3$, $\lambda_d(P_2(2P_k)) = r\Delta + d = r + d$, since that $P_2(2P_k) \cong C_{2k-2}$. We next consider $r \geq 3$ for $\Delta = 1$.

Theorem 6.1 *Suppose* $r \geq 3$. *Then* $\lambda_d(P_2(rP_6)) = r + d - 1$ *for* $r - 1 > d$, $\lambda_d(P_2(rP_6)) = r + d$ *for* $r - 1 \leq d$.

Proof. For $r-1 > d$, Label one node u with 0, the other one v with $r+d-1$. And label the r paths as $0(i + d)1(i + d + 1)(i + 1)(r + d - 1)$, where $i =$

$1, 2, \cdots, r-2$, and $0d(d+r-1)(r-1)0(r+d-1), 0(r+d-1)d0(r-1)(r+d-1)$. Then $\lambda_d(P_2(rP_6)) = r + d - 1$ for $r - 1 > d$.

For $r - 1 \leq d$, suppose that g is an $L(d, 1)$-labeling of the edge-multiplicity-path-replacement $P_2(rP_6)$ with span $r + d - 1$. Then the two nodes must be labeled with 0 or $r + d - 1$. Let the node labeled with 0 be u, and its adjacent vertex labeled with d and $d + r - 1$ be x and y, respectively. If $r \leq d$, then the adjacent vertex of x cannot be labeled in $[0, r + d - 1]$. If $r \leq d + 1$, then the replacing path including y cannot be labeled in $[0, r + d - 1]$. Thus $\lambda_d(P_2(rP_6)) \geq r + d$. It suffices to give an $L(d, 1)$-labeling of the edge-multiplicity-path-replacement $P_2(rP_6)$ with span $r + d$ for $r \geq 3$. Label one node u with 0, the other one v with $r + d$. And label the r paths as $0(i + d)1(d + r - 1)(i - 2)(r + d)$, where $i \in [2, r]$, otherwise, $0(d + 1)1(d + r - 1)(r - 1)(r + d)$. Hence $\lambda_d(P_2(rP_6)) = r + d$ for $r \geq 3$. ∎

Theorem 6.2 *Suppose $r \geq 3$. Then $\lambda_d(P_2(rP_4)) = r + d - 1$ for $r \geq 2d$, and $\lambda_d(P_2(rP_4)) = r + d + 1$ for $r < 2d$.*

Proof. It suffices to give an $L(d, 1)$-labeling of the edge-multiplicity-path-replacement $P_2(rP_4)$ with span $r + d - 1$ forr $\geq 2d$. Label one node with 0, the other one with $r + d - 1$. And label the r paths as $0(i + d - 1)(i + 2d - 1)0$ for $i = 1, 2, \cdots, r - d$, and $0(r + i - 1)(r + i - 1 - d)0$ for $i = 1, 2, \cdots, d$. Hence $\lambda_d(P_2(rP_4)) = r + d - 1$ for $r \geq 2d$.

For $r < 2d$, suppose that g is an $L(d, 1)$-labeling of the edge-multiplicity-path-replacement $P_2(rP_4)$ with span $r + d$. Then the two nodes must be labeled with 0, 1, $r + d - 1$ or $r + d$. By exhaustive case discussion on the labels of two nodes, the graph $P_2(rP_4)$ cannot be labeled in $[0, r + d]$. Thus $\lambda_d(P_2(rP_4)) \geq r + d$.

It suffices to give an $L(d, 1)$-labeling of the edge-multiplicity-path- replacement $P_2(rP_4)$ with span $r + d + 1$ for $r < 2d$. Label one node u with 0, the other one v with $r+d+1$. And label the r paths as $0(i+d)i(r+d+1)$, where $i = 1, 2, \cdots, r$. Hence $\lambda_d(P_2(rP_4)) = r+d+1$ for $r < 2d$. ∎

Theorem 6.3 *Suppose $r \geq 3$. Then $\lambda_d(P_2(rP_3)) = r + d$.*

Proof. Suppose that g is an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_2(rP_3)$ with span $r+d-1$. Then one node must be labeled with 0, and the other one must be labeled with $r+d-1$. The reader may prove that the graph $P_2(rP_3)$ cannot be labeled in $[0, r+d-1]$. Thus $\lambda_d(P_2(rP_3)) \geq r + d$. It suffices to give an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_2(rP_3)$ with span $r+d$ for $r \geq 3$. Label one node u with 0, the other one v with 1. And label the r paths as $0(i+d)1$, where $i = 1, 2, \cdots, r$. Hence $\lambda_d(P_2(rP_3)) = r + d$ for $r \geq 3$. ∎

6.2 $P_n(P_k)$ with $n \geq 3$ and $k = 3, 4, 6$

Note that the graph with order n and maximum degree $\Delta = 2$ is the path P_n or the cycle C_n where $n \geq 3$.

By theorem 6.8, we obtain the results as follows for $k = 4, 6$, since $P_n(rP_k) \subseteq C_n(rP_k)$.

Theorem 6.4 *Suppose $r \geq 2$ and $k = 4, 6$. Then $\lambda_d(P_n(rP_k)) = 2r+d-1$ for $r \geq d$, and $\lambda_d(P_n(rP_k)) = r + 2d - 1$ for $r < d$.*

Theorem 6.5 *Suppose $r \geq 2$. Then $\lambda_d(P_n(rP_6)) = 2r+d-1$ for $r < d < 2r$, and $\lambda_d(P_n(rP_6)) = 2r + d$ for $2r \leq d$.*

Proof. For $r < d < 2r$, it suffices to give an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_n(rP_6)$ with span $2r+d-1$. Label its nodes with 0 and $2r+d-1$, alternately. And label the r paths as $0d(2d)0(2r-1)(2r+d-1)$ and $0(d+i)1(2r+d-2)(2r-i-1)(2r+d-1)$, where $i = 1, 2, \cdots, r-1$, or $(2r+d-1)(i-1)(d+r)p(r+d+i)0$, where $i = 1, 2, \cdots, r$, and $p = 2$ for $r = 2$, otherwise $p = 1$. Hence $\lambda_d(P_n(rP_6)) = 2r + d - 1$ for $r < d < 2r$.

For $2r \leq d$, suppose that g is an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_n(rP_6)$ with span $2r + d - 1$. Then all the nodes with degree $2r$ must be labeled with 0 or $2r + d - 1$. Suppose that u is the node labeled with 0, and its adjacent vertex labeled with d is x. Then the adjacent vertex of x cannot be labeled in $[0, 2r + d - 1]$, since $2r \leq d$. Thus $\lambda_d(P_n(rP_6)) \geq 2r + d$. It suffices to give an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_n(rP_6)$ with span $2r + d$ for $2r \leq d$. Label its nodes with 0 and $2r + d - 1$, alternately. And label the r paths as $0(d+i)1(2r+d)(r+i-1)(2r+d-1)$ or $(2r+d-1)(i-1)(d+r-1)1(r+d+i)0$, where $i = 1, 2, \cdots, r$. Hence $\lambda_d(P_n(rP_6)) = 2r + d$ for $2r \leq d$. ∎

Theorem 6.6 *Suppose $r \geq 2$. Then $\lambda_d(P_n(rP_4)) = 2r+d-1$ for $r = d-1$, and $\lambda_d(P_n(rP_4)) = 2r + d$ for $r < d - 1$.*

Proof. For $r = d - 1$, it suffices to give an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_n(rP_4)$ with span $2r+d-1$. Label its nodes with $0, 0, 2r + d - 1, 2r + d - 1$, alternately. And label the r paths from 0 to 0 as $0d(2d)0$ or $0(2d + i - 1)(d + i - 1)0$, where $i = 1, 2, \cdots, r - 1$; the r paths from 0 to $2r + d - 1$ as $0(2d + i - 3)(d + i - 3)(2r + d - 1)$, where $i = 1, 2, \cdots, r + 1$ and $i \neq 3$; the r paths from $2r + d - 1$ to $2r + d - 1$ as $(2r+d-1)d0(2r+d-1)$ and $(2r+d-1)(i-1)(d+i-1)(2r+d-1)$, where $i = 1, 2, \cdots, r - 1$; the r paths from $2r + d - 1$ to 0 as $(2r + d - 1)i(d + i)0$, where $i = 1, 2, \cdots, r$. Hence $\lambda_d(P_n(rP_4)) = 2r + d - 1$ for $r = d - 1$.

For $r < d - 1$, suppose that g is an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_n(rP_4)$ with span $2r + d - 1$. Then all the nodes with degree $2r$ must be labeled with 0 or $2r + d - 1$. Suppose that u is labeled with 0, and its adjacent vertex labeled with d and $d + r - 1$ is x and y, respectively. If $2r \leq d$, then the adjacent vertex of x cannot be labeled in $[0, 2r + d - 1]$. For $r + 1 \leq d \leq 2r$ and $n \geq 4$, let v be the neighbouring nodes of u and degree $2r$. Then the node v cannot be labeled by $2r+d-1$. Thus the label of v is also 0. It is not difficult to see that the replacing paths between u and v cannot be labeled in $[0, 2r+d-1]$, since $r < d-1$. Thus $\lambda_d(P_n(rP_4)) \geq 2r+d$ for $n \geq 4$. For $r+1 \leq d \leq 2r$ and $n=3$, the node v can be labeled in $[0, d-1]$.Similarly, the replacing paths between u and v cannot be labeled in $[0, 2r+d-1]$, since $r < d-1$. Thus $\lambda_d(P_3(rP_4)) \geq 2r+d$. So it suffices to give an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_n(rP_4)$ with span $2r+d$.

Label its nodes with $0, 2r+d-1, 1, 2r+d$, alternately. And label the r paths from 0 to $d+2r-1$ as $0(d+2r)r(d+2r-1)$, and $0(d+i)i(d+2r-1)$, where $i = 1, 2, \cdots, r-1$; the r paths from $d+2r-1$ to 1 as $(d+2r-1)0(d+2r)1$ and $(d+2r-1)(r+i-1)(d+r+i-1)1$, where $i = 1, 2, \cdots, r-1$; the r paths from 1 to $d+2r$ as $1(d+1)0(d+2r)$,$1(d+2r-1)(2r-1)(d+2r)$ and $1(d+i)i(d+2r)$, where $i = 2, 3, \cdots, r-1$; the r paths from $d+2r$ to 0 as $(d+2r)1(d+2r-1)0$ and $(d+2r)(r+i-1)(d+r+i-1)0$, where $i = 1, 2, \cdots, r-1$. Hence $\lambda_d(P_n(rP_4)) = 2r+d$ for $r < d-1$. ∎

Theorem 6.7 *Suppose $r \geq 2$. Then $\lambda_d(P_n(rP_3)) = 2r+d-1$ for $3 \leq n \leq 4$, and $\lambda_d(P_n(rP_3)) = 2r+d$ for $n \geq 5$.*

Proof. Suppose that g is an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_n(rP_3)$ with span $2r+d-1$ for $n \geq 5$. Then all the nodes with degree $2r$ must be labeled with 0 or $2r+d-1$. The reader may prove that the graph $P_n(rP_3)$ cannot be labeled in $[0, 2r+d-1]$ for $n \geq 5$. Thus $\lambda_d(P_n(rP_3)) \geq 2r+d$. It suffices to give an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_n(rP_3)$ with span $2r+d$ for $r \geq 2$. Label its nodes with 0 and 1, alternately. And label the r paths as $0(2i+d-1)1$ or $1(2i+d)0$, where $i = 1, 2, \cdots, r$. Hence $\lambda_d(P_n(rP_3)) = 2r+d$ for $r \geq 2$.

For $3 \leq n \leq 4$, it suffices to give an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $P_4(rP_3)$ with span $2r+d-1$. Orderly label its nodes wit 1, 0, $2r+d-1$ and $2r+d-2$. And if $r \leq d$, then we label the inserted vertices between 1 and 0 in $[d+r, d+2r-1]$, the inserted vertices between 0 and $2r+d-1$ in $[d, d+r-1]$, and the inserted vertices between $2r+d-1$ and $2r+d-2$ in $[0, r-1]$. If $r > d$, then we label the inserted vertices between 1 and 0 in $[d, r-1] \cup [2r, d+2r-1]$, the inserted vertices between 0 and $2r+d-1$ in $[r, 2r-1]$, and the inserted vertices between $2r+d-1$ and $2r+d-2$ in $[0, r-1]$. Hence $\lambda_d(P_n(rP_3)) = 2r+d-1$ for $3 \le n \le 4$. ∎

6.3 $C_n(rP_k)$ with $n \ge 3$ and $k = 3, 4, 6$

By theorem 4.2, we obtain the results as follows for $k = 4, 6$.

Theorem 6.8 *Suppose $r \ge 2$ and $k = 4, 6$. Then $\lambda_d(C_n(rP_k)) = 2r+d-1$ for $r \ge d$, and $\lambda_d(C_n(rP_k)) \le r+2d-1$ for $r < d$.*

Thus we only need to consider the case $2 \le r < d$ for $C_n(rP_k)$ with $k = 4, 6$.

Theorem 6.9 *Suppose $2 \le r < d$. If n is even, then $\lambda_d(C_n(rP_6)) = 2r+d$ for $2r \le d$, and $\lambda_d(C_n(rP_6)) = 2r+d-1$ for $r < d < 2r$. If n is odd, then $\lambda_d(C_n(rP_6)) = 2r+d$ for $2r \le d$, and $\lambda_d(C_n(rP_6)) \le 2r+d$ for $r < d < 2r$.*

Proof. Since $P_n(rP_6) \subseteq C_n(rP_6)$, and for even n, the labeling for $r < d < 2r$ in theorem 6.5 also works on $C_n(rP_6)$. So for even n, $\lambda_d(C_n(rP_6)) = 2r+d$ for $2r \le d$, and $\lambda_d(C_n(rP_6)) = 2r+d-1$ for $r < d < 2r$.

If n is odd, then we modify the labeling for $2r \le d$ in theorem 6.5 as follows: change just one node labeled 0 by 1, and for all the vertices labeled with 1 which are distance two to the modified node, modify all the labels with 0. Then the modified labeling works on $C_n(rP_6)$. Thus for odd n, $\lambda_d(C_n(rP_6)) = 2r+d$ for $2r \le d$, and $\lambda_d(C_n(rP_6)) \le 2r+d$ for $r < d < 2r$.

∎

Theorem 6.10 *Suppose $2 \leq r < d$. If $n \equiv 0(\mod 4)$, then $\lambda_d(C_n(rP_4)) = 2r+d-1$ for $r = d-1$, and $\lambda_d(C_n(rP_4)) = 2r+d$ for $r < d-1$. If $n \not\equiv 0(\mod 4)$, then $\lambda_d(C_n(rP_4)) \leq \min\{2r+d+1, r+2d-1\}$ for $r < d-1$, and $\lambda_d(C_n(rP_4)) \leq r+2d-1 = 2r+d$ for $r = d-1$.*

Proof. Since $P_n(rP_4) \subseteq C_n(rP_4)$, and for $n \equiv 0(\mod 4)$, the labeling in theorem 6.6 also works for $C_n(rP_4)$. Then for $n \equiv 0(\mod 4)$, $\lambda_d(C_n(rP_4)) = 2r+d-1$ for $r = d-1$, and $\lambda_d(C_n(rP_4)) = 2r+d$ for $r < d-1$.

If $n \not\equiv 0(\mod 4)$, $\lambda_d(C_n(rP_4)) \leq r+2d-1 = 2r+d$ for $r = d-1$ by theorem 6.8.

For $n \not\equiv 0(\mod 4)$ and $r < d-1$, label its nodes with 0 and $2r+d+1$, alternately. Label the r paths as $0(d+i)i(2r+d+1)$ or $(2r+d+1)(r+i)(d+r+i)0$ for $i = 1, 2, \cdots, r$. And change just one node u labeled 0 by 1, and modify the labels of all the vertices which are labeled by 1 and distance two to u with 0. Then the modified labeling works on $C_n(rP_4)$. Thus $\lambda_d(C_n(rP_4)) \leq \min\{2r+d+1, r+2d-1\}$ for $r < d-1$. ∎

We next study the $L(d,1)$-labeling of $C_n(rP_3)$.

Theorem 6.11 *For $r \geq 2$, $\lambda_d(C_n(rP_3)) = 2r+d$ for even n.*

Proof. On the one hand, the labeling f in Theorem 6.7 also works for the cycle C_n with even n. Then $\lambda_d(C_n(rP_3)) \leq 2r+d$. On the other hand, $\lambda_d(C_n(rP_3)) \geq 2r+d$. Suppose that g is an $L(d,1)$-labeling of the edge-multiplicity-path-replacement $C_n(rP_3)$ with span $2r+d-1$. Then all the nodes must be labeled with 0 or $2r+d-1$. The reader may prove that the graph $C_n(rP_3)$ cannot be labeled in $[0, 2r+d-1]$. ∎

Theorem 6.12 *For $r \geq 2$ and odd $n \geq 3$, $\lambda_d(C_n(rP_3)) \leq 2r+2d-1$ for $r > d$. $\lambda_d(C_n(rP_3)) \leq 3r+d-1$ for $r \leq d$.*

Proof. We give an $L(d,1)$-labeling f of the edge-multiplicity-path- replacement $C_n(rP_3)$ with span $2r+2d-1$ as follows for $r>d$. Label orderly its nodes with $0(2r+d-1)(2r+2d-1)$. And label the inserted vertices between 0 and $2r+d-1$ in $[d, r+d-1]$, the inserted vertices between $2r+d-1$ and $2r+2d-1$ in $\{0,1,\cdots,r-1\}$, the inserted vertices between $2r+2d-1$ and 0 in $[r+d, 2r+d-1]$. Hence $\lambda_d(C_n(rP_3)) = 2r+2d-1$ for $r>d$.

For $r \leq d$, label orderly its nodes with $0(2r+d)(3r+d-1)$. And label the inserted vertices between 0 and $2r+d$ in $[d, r+d-1]$, the inserted vertices between $2r+d$ and $3r+d-1$ in $\{0,1,\cdots,r-1\}$, the inserted vertices between $3r+d-1$ and 0 in $[r+d, 2r+d-1]$. Hence $\lambda_d(C_n(rP_3)) \leq 3r+d-1$ for $r \leq d$. ∎

7 Note

In [17], we shown that the class of graphs $G(rP_k)$ satisfies Conjecture 1.1 for $d=2$ and $k \geq 3$. We close by noticing that Conjecture 1.1 is true for the class of graphs $G(rP_k)$ for $d \geq 3$ and $k \geq 3$ from Theorem 3.2, and the theorems in section 2, Since the incidence graph of $G(rP_k)$ is $G(rP_{2k-1})$.

References

[1] Erwin D.J.,Georges J.P. andMauro D.W., On labeling the vertices of products of complete graphs with distance constraints, Naval Research Logistics, 52, 138-141(2005).

[2] Frobenius G.ü ber Matrizen aus nichtnegativen Elementen[M], Akad Wiss Belin:Sitzungsberichte Preuss, 456-477 (1992).

[3] Griggs J.R. and Yeh R.K., Labeling graphs with a condition at distance two, SIAM J.Discrete Math., 5, 586-595(1992).

[4] Georges J.P. and Mauro D.W., Some results on λ_k^j-numbers of the products of complete graphs, Congr. numer. 109,141-159(1995).

[5] Georges J.P., Mauro D.W. andStein M.I., Labeling products of complete graphs with a condition at distance two, SIAM J.Discr. Math., 14,28-35(2000).

[6] Havet F., $(d, 1)$-total labeling of graphs, Workshop Graphs and Algorithms, Dijon(FRANCE),2003.

[7] Havet F. and Yu M.L., $(d, 1)$-total labeling of graphs, Technical Report 4650, INRIA,2002.

[8] Jha P.K., Optimal $L(2, 1)$-labeling of Cartesian products of cycles, with an application to independent domination, IEEE Trans. Circ. Syst.-I:Fund. Theory and Appl. 47, 1531-1534(2000).

[9] Jha P.K.,Klavžar S. andVesel A., Optimal $L(d, 1)$-labelings of certain directed products of cycles and Cartesian products of cycles, Discrete Applied Mathematics, 152, 257-265(2005).

[10] Jha P.K.,Narayanan A.,Sood P.,Sundaran K. and Sunder V.,On $L(2, 1)$-labeling of the Cartesian product of a cycle and a path, Ars Combin. 55,81-89(2000).

[11] Jha P.K., Optimal $L(2, 1)$-labelings of strong Cartesian products of cycles, IEEE Trans. Circ. Syst.-I:Fund. Theory and Appl. 48,498-500(2001).

[12] Jha P.K.,Klavžar S. and Vesel A., $L(2, 1)$-labeling of direct products of paths and cycles, To appear in Discrete Applied Mathematics, 145, 317-325(2005).

[13] Karst et al., $L(d, 1)$-labelings of the edge-path-replacement by factorization of graphs, J. Comb. Optim., 30(1), 34-41(2015).

[14] Kuo D. and Yan J.H., On $L(2,1)$-labeling of Cartesian products of paths and cycles, Discrete Math., 283,137-144(2004).

[15] Lü D.M., $L(2,1)$-labelings of the edge-path-replacement of a graph, J Comb. Optim., 26: 385-39(2013).

[16] Lü D.M., $L(d,1)$-labelings of the edge-path-replacement of a graph, J Comb. Optim., 26: 819-831(2013).

[17] Lü D.M. and Sun J.P., $L(2,1)$-labelings of the edge-multiplicity-path-replacement of a graph. J Comb. Optim.,DOI: 10.1007/s10878-014-9761-x (2014).

[18] Petersen J., Die Theorie der regularen Graphen. Acta Math. 15, 193-220(1891).

[19] Schwarz C. and Sakai D.Troxell, $L(2,1)$-labelings of Cartesian products of two cycles, DIMACS Tech.Rep.,33,2003.

[20] Whittlesey M.A., Georges J.P. andMauro D.W., On the λ-number of Q_n and related graphs, SIAM J.Discrete Math., 8, 449-506(1995).