# On Harary spectral radius with given parameters

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**Abstract.** The Harary spectral radius $\rho(G)$ of a graph $G$ is the largest eigenvalue of the Harary matrix $RD(G)$. In this paper, we determine graphs with the largest Harary spectral radius in four classes of simple connected graphs with $n$ vertices: with given matching number, vertex connectivity, edge connectivity and chromatic number, respectively.

**Keywords**: Harary matrix, matching number, vertex connectivity, chromatic number

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## 1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The distance $d_{i,j}$ is defined as the length of the shortest path between $v_i$ and $v_j$ in $G$. The diameter $d$ of a graph is the maximum distance between any two vertices of $G$. The Harary matrix $RD(G)$ [3] of $G$, is an $n \times n$ matrix $(RD_{i,j})$ such that

$$RD_{i,j} = \begin{cases} \frac{1}{d_{i,j}}, & if \;\; i \neq j, \\ 0, & if \;\; i = j. \end{cases}$$

The Harary spectral radius $\rho(G)$ of a graph $G$ is the largest eigenvalue of the Harary matrix $RD(G)$. In [4], Ivanciuc et al. have shown that $\rho(G)$ is able to produce fair QSPR models for the boiling points, molar heat capacities, vaporization enthalpies, refractive indices and densities for $C_6 - C_{10}$ alkanes. Hence it is an interesting topic to study the maximum

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eigenvalue of $RD(G)$. Note that $RD(G)$ is a real symmetric matrix, its all eigenvalues are real. Further by the Perron-Frobenius theorem, we know that there is a unique positive eigenvector corresponding to $\rho(G)$ whose entries sum to 1 if $G$ is connected.

In order to state our results, we introduce some notation and terminology. For other undefined notation we refer to Bollobás [1].

Two edges of $G$ are said to be independent if they are not adjacent in $G$. A set of pairwise independent edges in $G$ is called a matching in $G$. A matching of maximum cardinality is a maximum matching in $G$. The matching number $\beta(G)$ (or just $\beta$, for short) of the graph $G$ is the number of edges in a maximum matching. Let $\mathscr{G}_{n,\beta}$ be the set of graphs on $n$ vertices with matching number $\beta$. Let $M$ be an arbitrary matching in $G$. A path in $G$ which starts at an unmatched vertex and then contains, alternately, edges from $E(G) \setminus M$ and from $M$, is an alternating path with respect to $M$. An alternating path $P$ that ends are unmatched vertices is called an augmenting path.

The vertex connectivity of a graph $G$ is the minimum number of vertices whose deletion yields a disconnected graph, and the edge connectivity of a graph $G$ is the minimum number of edges whose deletion yields a disconnected graph. The components of a graph are its maximal connected subgraphs. Components of odd (even) order are called the odd (even) components. Denote by $\mathcal{G}_{n,\kappa}$ the set of graphs with $n$ vertices and vertex connectivity $\kappa$, and $\mathcal{G}_{n,\kappa'}$ the set of graphs with $n$ vertices and edge connectivity $\kappa'$.

Two vertices of $G$ are said to be independent if they are not adjacent in $G$. The chromatic number of a graph $G$ is the smallest number of colors needed to color the vertices of $G$ such that any two adjacent vertices have different colors. A subset of vertices assigned to the same color is called a color class, every such class forms an independent set. The Turán graph $T_{n,r}$ is a complete $r$-partite graph on $n$ vertices for which the number of vertices of vertex classes are as equal as possible. Let $\mathscr{C}_{n,r}$ be the set of graphs on $n$ vertices with chromatic number $r$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The union $G_1 \cup G_2$ is defined to be $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join $G_1 \bigvee G_2$ of $G_1$ and $G_2$ is obtained from $G_1 \cup G_2$ by joining edges from each vertex of $G_1$ to each vertex of $G_2$. We denote by $\overline{K_n}$ the graph on $n$ vertices with no edges.

Let $J_{a\times b}$ be the $a \times b$ matrix whose entries are all equal to 1 and $I_n$ be the $n \times n$ unit matrix. For simplicity, if $a = b$, we write $J_a$ instead of $J_{a\times a}$.

In this paper, we determine graphs with the largest Harary spectral radius in four classes of simple connected graphs with $n$ vertices: with given matching number, vertex connectivity, edge connectivity and chromatic number, respectively.

We list some known results which will be used in this paper.

Let $o(G)$ be the number of odd components of $G$.

**Lemma 1.1.** *(The Tutte-Berge Formula)[2, 5, 7] Suppose $G$ is a graph on $n$ vertices with matching number $\beta$. Then there exists a subset $S_0 \subset V(G)$ on $s$ vertices in $G$ such that $n - 2\beta = \max\{o(G-S) - |S| : S \subset V(G)\} = o(G - S_0) - |S_0| = q - s$, where $q = o(G - S_0)$.*

**Lemma 1.2.** *[6] Let $G$ be a connected graph with $n \geq 2$ vertices, $m$ edges and diameter $d$. Then $\rho(G) \geq \frac{2m}{n} + \frac{1}{d}(n - 1 - \frac{2m}{n})$, with equality if and only if $G$ is a complete graph $K_n$ or $G$ is a regular graph of diameter 2.*

## 2. Main results

Let $v_k, v_l \in V(G)$ and $v_kv_l \notin E(G)$, adding edge $v_kv_l$ to $G$ does not increase distance, while it does decrease at least one distance; the distance between $v_k$ and $v_l$ is one in $G + v_kv_l$ and at least two in $G$. Let $RD(G')$ be the Harary matrix of $G'$, then $RD(G')_{i,j} \geq RD(G)_{i,j}$ for all $v_i, v_j \in V(G)$. Moreover, $1 = RD(G')_{k,l} > RD(G)_{k,l}$, by the Perron-Frobenius theorem, we conclude that

$$\rho(RD(G')) > \rho(RD(G)) \tag{2.1}$$

**Lemma 2.1.** *Let $G = K_s \bigvee (\bigcup_{i=1}^{q} K_{n_i})$ and*

$$x(G) = (\underbrace{x_1, \ldots, x_1}_{n_1}, \underbrace{x_2, \ldots, x_2}_{n_2}, \ldots, \underbrace{x_q, \ldots, x_q}_{n_q}, \underbrace{y, \ldots, y}_{s})$$

*be a unit eigenvector of $RD(G)$ corresponding to $\rho(G)$, if $n_q \geq n_i \geq 2, 1 \leq i \leq q-1$, then $n_q x_q - n_i x_i + x_i > 0$.*

*Proof.* By $RD(G)x(G) = \rho(G)x(G)$, we have

$$\begin{aligned}
\rho(G)x_i &= \frac{1}{2}n_1x_1 + \cdots + \frac{1}{2}n_{i-1}x_{i-1} + (n_i - 1)x_i + \frac{1}{2}n_{i+1}x_{i+1} + \cdots + \\
&\quad \frac{1}{2}n_{q-1}x_{q-1} + \frac{1}{2}n_qx_q + sy; \\
\rho(G)x_q &= \frac{1}{2}n_1x_1 + \frac{1}{2}n_2x_2 + \frac{1}{2}n_3x_3 + \cdots + \frac{1}{2}n_{q-1}x_{q-1} + \\
&\quad (n_q - 1)x_q + sy.
\end{aligned}$$

Then

$$x_q = \frac{\rho(G) - \frac{n_i}{2} + 1}{\rho(G) - \frac{n_q}{2} + 1} x_i.$$

Hence

$$\begin{aligned} n_q x_q - n_i x_i + x_i &= \frac{\rho(G) - \frac{n_i}{2} + 1}{\rho(G) - \frac{n_q}{2} + 1} n_q x_i - n_i x_i + x_i \\ &\geq n_q x_i - n_i x_i + x_i > 0. \end{aligned}$$

□

**Lemma 2.2.** *Let $G$ be a graph with maximal Harary spectral radius in $\mathscr{G}_{n,\beta}$, then there exist positive odd numbers $n_1, n_2, \ldots, n_q$ such that $G = K_s \bigvee (\bigcup_{i=1}^{q} K_{n_i})$ with $s = q + 2\beta - n$ and $\sum_{i=1}^{q} n_i = n - s$.*

*Proof.* Let $M$ be a maximum matching in $G$, then $|M| = \beta$. By Lemma 1.1, there exists a subset $S_0 \subset V(G)$ on $s$ vertices in $G$ such that

$$n - 2\beta = \max\{o(G - S) - |S| : S \subset V(G)\} = q - s,$$

where $q = o(G - S_0)$. Let $G_1, G_2, \ldots, G_q$ be the odd components in $G - S_0$ with $|V(G_i)| = n_i \geq 1$ for $i = 1, 2, \ldots, q$. Clearly, $n \geq s + q = n + 2s - 2\beta$. Thus $s \leq \beta$.

**Case 1** If $s = 0$, then $G - S_0 = G$ and $n + s - 2\beta = n - 2\beta = q \leq 1$ since $G$ is connected. If $q = 0$, then $n = 2\beta$; If $q = 1$, then $n = 2\beta + 1$. In both case, by (2.1), we have $G \cong K_n$.

**Case 2** If $s \geq 1$, then $q = n + s - 2\beta \geq 1$ since $n \geq 2\beta$.

First, we claim that $G - S_0$ contains no even component.

In fact, if it doesn't hold, let $W$ be an even component of $G - S_0$. Then by adding an edge to $G$ between a vertex $v_{i_0}$ of $W$ and a vertex $v_{j_0}$ of an odd component of $G - S_0$, we obtain a graph $G'$, for which

$$n - 2\beta(G') \geq o(G - S_0) - |S_0| = o(G - S_0) - |S_0| = n - 2\beta(G),$$

note that $\beta(G) \leq \beta(G')$ since $G \subset G'$. So $\beta(G) = \beta(G')$, then $G' \in \mathscr{G}_{n,\beta}$. Let $D'$ be the Harary matrix of $G'$. Obviously, $RD(G)_{i,j} \leq RD(G')_{i,j}$ and $RD(G)_{i_0,j_0} < RD(G')_{i_0 j_0}$ for the new edge $v_{i_0} v_{j_0}$. Then by Perron-Frobenius theorem, we conclude that $\rho(G) < \rho(G')$, a contradiction.

Second, we claim that $G_1 \cong K_{n_1}$.

It is obvious for $n_1 = 1$, now we assume that $n_1 \geq 3$ in the following. If $G_1 \ncong K_{n_1}$, then there exist two vertices $v_i, v_j \in V(G_1)$ with $v_i v_j \notin E(G_1)$. Then by adding the edge $v_i v_j$ to $G$, we obtain a graph $\tilde{G}$, for which

$$n - 2\beta(\tilde{G}) \geq o(\tilde{G} - S_0) - |S_0| = o(G - S_0) - |S_0| = n - 2\beta(G),$$

then $\beta(G) \geq \beta(\tilde{G})$. Further, we have $\beta(G) = \beta(\tilde{G})$ and $\rho(G) < \rho(\tilde{G})$, also a contradiction. Similarly, we have $G_2, \ldots, G_q$, the subgraph induced by $S_0$ are all complete, and any vertex of $G_i (i = 1, 2, \ldots, q)$ is adjacent to every vertex in $S_0$. So $G = K_s \bigvee (\bigcup_{i=1}^{q} K_{n_i})$. □

**Lemma 2.3.** *If $n_q \geq n_i \geq 2, 1 \leq i \leq q-1$, $G = K_s \bigvee (\bigcup_{j=1}^{q} K_{n_j}), G' = K_s \bigvee (\bigcup_{j=1}^{i-1} K_{n_j} \cup K_{n_i - 1} \cup (\bigcup_{j=i+1}^{q-1} K_{n_j}) \cup K_{n_q+1})$, then $\rho(G') > \rho(G)$.*

*Proof.* Without loss of generality, let $i = 1$, then $RD(G') =$

$$\begin{pmatrix} J_{(n_1-1)} - E_{(n_1-1)} & \cdots & \frac{1}{2}J_{(n_1-1)\times 1} & \frac{1}{2}J_{(n_1-1)\times n_q} & J_{(n_1-1)\times s} \\ \frac{1}{2}J_{n_2\times(n_1-1)} & \cdots & \frac{1}{2}J_{n_2\times 1} & \frac{1}{2}J_{n_2\times n_q} & J_{n_2\times s} \\ \cdots & & & & \\ \frac{1}{2}J_{1\times(n_1-1)} & \cdots & 0 & J_{1\times n_q} & J_{1\times s} \\ \frac{1}{2}J_{n_q\times(n_1-1)} & \cdots & J_{n_q\times 1} & J_{n_q} - E_{n_q} & J_{n_q\times s} \\ J_{s\times(n_1-1)} & \cdots & J_{s\times 1} & J_{s\times n_q} & J_{s\times s} - E_s \end{pmatrix}$$

and $RD(G) =$

$$\begin{pmatrix} J_{(n_1-1)} - E_{(n_1-1)} & \cdots & J_{(n_1-1)\times 1} & \frac{1}{2}J_{(n_1-1)\times n_q} & J_{(n_1-1)\times s} \\ \frac{1}{2}J_{n_2\times(n_1-1)} & \cdots & \frac{1}{2}J_{n_2\times 1} & \frac{1}{2}J_{n_2\times n_q} & J_{n_2\times s} \\ \cdots & & & & \\ J_{1\times(n_1-1)} & \cdots & 0 & \frac{1}{2}J_{1\times n_q} & J_{1\times s} \\ \frac{1}{2}J_{n_q\times(n_1-1)} & \cdots & \frac{1}{2}J_{n_q\times 1} & J_{n_q} - E_{n_q} & J_{n_q\times s} \\ J_{s\times(n_1-1)} & \cdots & J_{s\times 1} & J_{s\times n_q} & J_{s\times s} - E_s \end{pmatrix}$$

So $RD(G') - RD(G) =$

$$\begin{pmatrix} 0 & 0 & \cdots & -\frac{1}{2}J_{(n_1-1)\times 1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & & & & & \\ -\frac{1}{2}J_{1\times(n_1-1)} & 0 & \cdots & 0 & \frac{1}{2}J_{1\times n_q} & 0 \\ 0 & 0 & \cdots & \frac{1}{2}J_{n_q\times 1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Let

$$x(G) = (\underbrace{x_1, \ldots, x_1}_{n_1-1}, \underbrace{x_2, \ldots, x_2}_{n_2}, \ldots, \underbrace{x_{q-1}, \ldots, x_{q-1}}_{n_{q-1}}, x_1, \underbrace{x_q, \ldots, x_q}_{n_q}, \underbrace{y, \ldots, y}_{s})$$

be a unit eigenvector of $RD(G)$ corresponding to $\rho(G)$, then

$$x(G)(RD(G') - RD(G))x(G)^T = x_1(n_q x_q - n_1 x_1 + x_1),$$

by Lemma 2.1, we have $x_1(n_q x_q - n_1 x_1 + x_1) > 0$. So

$$\rho(G') \geq x(G)RD(G')x(G)^T > x(G)RD(G))x(G)^T = \rho(G).$$

□

**Theorem 2.4.** *Let $G \in \mathscr{G}_{n,\beta}$, then*

(i) *If $n = 2\beta$ or $n = 2\beta + 1$, then $\rho(G) \leq \rho(K_n)$ with equality if and only if $G \cong K_n$;*

(ii) *If $n \geq 2\beta+2$, then $\rho(G) \leq \rho(K_1 \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{n-2\beta} \cup K_{2\beta-1}))$ with equality if and only if $G \cong K_1 \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{n-2\beta} \cup K_{2\beta-1})$.*

*Proof.* (i) If $n = 2\beta$ or $n = 2\beta + 1$, by (2.1), it is easy to see $\rho(G) \leq \rho(K_n)$ and the equality holds if and only if $G \cong K_n$.

(ii) If $n \geq 2\beta + 2$, let $G^*$ be the graph with maximal Harary spectral radius, by Lemma 2.2, we know that $G^*$ has the form of the following: $K_s \bigvee(\cup_{i=1}^{q} K_{n_i})$ with $s = q + 2\beta - n$ and $\sum_{i=1}^{q} n_i = n - s$. Repeated by Lemma 2.3, we know that $G^* \cong K_s \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{q-1} \cup K_{n_q})$, where $q = n+s-2\beta, n_q = 2\beta-2s+1$, that is, $G^* \cong K_s \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{n+s-2\beta-1} \cup K_{2\beta-2s+1})$.

Note that

$$K_s \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{n+s-2\beta-1} \cup K_{2\beta-2s+1}) \subset K_{s-1} \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{n+s-2\beta-2} \cup K_{2\beta-2s+3}),$$

then

$$\begin{aligned} & \rho(K_s \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{n+s-2\beta-1} \cup K_{2\beta-2s+1})) \\ < \; & \rho(K_{s-1} \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{n+s-2\beta-2} \cup K_{2\beta-2s+3})). \end{aligned}$$

Further we have $G^* \cong K_1 \bigvee(\underbrace{K_1 \cup \cdots \cup K_1}_{n-2\beta} \cup K_{2\beta-1})$. □

**Theorem 2.5.** *Let $G \in \mathcal{G}_{n,\kappa}$, then $\rho(G) \leq \rho(K_\kappa \bigvee (K_1 \cup K_{n-1-\kappa}))$ with equality if and only if $G \cong K_\kappa \bigvee (K_1 \cup K_{n-1-\kappa})$.*

*Proof.* Let $G^*$ be the graph with maximal Harary spectral radius in $\mathcal{G}_{n,\kappa}$, by (2.1), we know that $G^* \cong K_\kappa \bigvee (K_{n_1} \cup K_{n_2})$ with $\kappa + n_1 + n_2 = n$ and $n_1 \leq n_2$. If $n_1 \geq 2$, let $G' = K_\kappa \bigvee (K_{n_1-1} \cup K_{n_2+1})$, by Lemma 2.3, we have $\rho(G') > \rho(K_\kappa \bigvee (K_{n_1} \cup K_{n_2}))$, a contradiction. Hence $G^* \cong K_\kappa \bigvee (K_1 \cup K_{n-1-\kappa})$. □

**Theorem 2.6.** *Let $G \in \mathcal{G}_{n,\kappa'}$, then $\rho(G) \leq \rho(K_{\kappa'} \bigvee (K_1 \cup K_{n-1-\kappa'}))$ with equality if and only if $G \cong K_{\kappa'} \bigvee (K_1 \cup K_{n-1-\kappa'})$.*

*Proof.* Let $G$ be a graph in $\mathcal{G}_{n,\kappa'}$ and $\kappa$ its vertex connectivity. It is well known that $\kappa \leq \kappa'$. If $\kappa = \kappa'$, by Theorem 2.5, we have $\rho(G) \leq \rho(K_{\kappa'} \bigvee (K_1 \cup K_{n-1-\kappa'}))$. If $\kappa < \kappa'$, by Theorem 2.5 and (2.1), we have $\rho(G) \leq \rho(K_\kappa \bigvee (K_1 \cup K_{n-1-\kappa})) < \rho(K_{\kappa'} \bigvee (K_1 \cup K_{n-1-\kappa'}))$. □

**Theorem 2.7.** *Let $G \in \mathscr{C}_{n,r}$, then $\rho(G) \leq \rho(T_{n,r})$ with equality if and only if $G \cong T_{n,r}$.*

*Proof.* Let $G^*$ be the graph with maximal Harary spectral radius in $\mathscr{C}_{n,r}$, by (2.1), we know that $G^* \cong K_{n_1,n_2,\ldots,n_r}$ with $n_1 + \cdots + n_r = n$. If there exist $i, j \in \{1, 2, \ldots, r\}$ such that $n_i - n_j \geq 2$, let

$$G' = K_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_{j-1},n_j+1,n_{j+1},\ldots,n_r}.$$

Without loss of generality, set $i = 1, j = 2$, that is, $G' = K_{n_1-1,n_2+1,n_3,\ldots,n_r}$. Note that $RD(G') =$

$$\begin{pmatrix} \frac{1}{2}(J_{n_1-1} - E_{n_1-1}) & J_{(n_1-1)\times 1} & J_{(n_1-1)\times n_2} & \cdots & J_{(n_1-1)\times n_r} \\ J_{1\times(n_1-1)} & 0 & \frac{1}{2}J_{1\times n_2} & \cdots & J_{1\times n_r} \\ J_{n_2\times(n_1-1)} & \frac{1}{2}J_{n_2\times 1} & \frac{1}{2}(J_{n_2} - E_{n_2}) & \cdots & J_{n_2\times n_r} \\ \cdots & & & & \\ J_{n_r\times(n_1-1)} & J_{n_r\times 1} & J_{n_r\times n_2} & \cdots & \frac{1}{2}(J_{n_r} - E_{n_r}) \end{pmatrix},$$

and $RD(G) =$

$$\begin{pmatrix} \frac{1}{2}(J_{n_1-1} - E_{n_1-1}) & \frac{1}{2}J_{(n_1-1)\times 1} & J_{(n_1-1)\times n_2} & \cdots & J_{(n_1-1)\times n_r} \\ \frac{1}{2}J_{1\times(n_1-1)} & 0 & J_{1\times n_2} & \cdots & J_{1\times n_r} \\ J_{n_2\times(n_1-1)} & J_{n_2\times 1} & \frac{1}{2}(J_{n_2} - E_{n_2}) & \cdots & J_{n_2\times n_r} \\ \cdots & & & & \\ J_{n_r\times(n_1-1)} & J_{n_r\times 1} & J_{n_r\times n_2} & \cdots & \frac{1}{2}(J_{n_r} - E_{n_r}) \end{pmatrix},$$

then

$$RD(G') - RD(G) = \begin{pmatrix} 0 & \frac{1}{2}J_{(n_1-1)\times 1} & 0 & \cdots & 0 \\ \frac{1}{2}J_{1\times(n_1-1)} & 0 & -\frac{1}{2}J_{1\times n_2} & \cdots & 0 \\ 0 & -\frac{1}{2}J_{n_2\times 1} & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

By symmetry, we can set $x(G) = (\underbrace{x_1, \dots, x_1}_{n_1}, \underbrace{x_2, \dots, x_2}_{n_2}, \dots, x_1, \underbrace{x_r, \dots, x_r}_{n_r})^T$ be a unit eigenvector of $RD(G)$ corresponding to $\rho(G)$, then $RD(G)x(G) = \rho(G)x(G)$. Hence

$$\begin{aligned} \frac{1}{2}(n_1 - 1)x_1 + n_2x_2 + n_3x_3 + \cdots + n_rx_r &= \rho(G)x_1 \\ n_1x_1 + \frac{1}{2}(n_2 - 1)x_2 + n_3x_3 + \cdots + n_rx_r &= \rho(G)x_2, \end{aligned}$$

then $x_2 = \frac{\rho + \frac{n_1+1}{2}}{\rho + \frac{n_2+1}{2}} x_1$. Note that

$$\begin{aligned} \rho(G') - \rho(G) &\geq x(G)^T(RD(G') - RD(G))x(G) \\ &= \frac{1}{\rho + \frac{n_2+1}{2}}[(n_1 - n_2 - 1)\rho(G) + \frac{n_1}{2} - n_2 - \frac{1}{2}]x_1^2 \\ &\geq \frac{x_1^2}{\rho + \frac{n_2+1}{2}}(\rho(G) + \frac{n_1}{2} - n_2 - \frac{1}{2}). \end{aligned}$$

By Lemma 1.2, we have

$$\begin{aligned} \rho(G) &\geq \frac{2m}{n} + \frac{1}{d}(n - 1 - \frac{2m}{n}) \\ &\geq 1 + \frac{n-1}{2} > 1 + \frac{n_1 - 1}{2}. \end{aligned}$$

Hence $\rho(G') - \rho(G) > \frac{x_1^2}{\rho + \frac{n_2+1}{2}}(n_1 - n_2) > 0$, a contradiction. Hence $G^* \cong T_{n,r}$. □

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