The Maximum Size of C_4-Free Planar Graphs

Guofei ZHOU, Yaojun CHEN*
Department of Mathematics, Nanjing University, Nanjing 210093, P.R. CHINA

Abstract

A planar graph is called C_4-free if it has no cycles of length four. Let $f(n, C_4)$ denote the maximum size of a C_4-free planar graph with order n. In this paper, it is shown that $f(n, C_4) = \lfloor \frac{15}{7}(n-2) \rfloor - \mu$ for $n \geq 30$, where $\mu = 1$ if $n \equiv 3 \pmod 7$ or $n = 32, 33, 37$, and $\mu = 0$ otherwise.

1. Introduction

All graphs considered in this paper are finite, undirected graphs without loops and multiple edges. Let $G = (V(G), E(G))$. For $v \in V(G)$, $N_G(v)$ denotes the neighbors of v in G and $N_G[v] = N_G(v) \cup \{v\}$. The degree of v, written as $d_G(v)$, is $|N_G(v)|$. We write $N(v)$, $N[v]$ and $d(v)$ for $N_G(v)$, $N_G[v]$ and $d_G(v)$, respectively, if there is no danger of confusion. The minimum degree of G is denoted by $\delta(G)$. A cycle of length n is denoted by C_n. For $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph induced by U in G. A graph G is called *k-colorable* if there is an assignment of colors $\{1, 2, ..., k\}$ to $V(G)$ such that adjacent vertices receive distinct colors and the *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer k such that G is k-colorable. We say that G is *k-choosable* if for any prescribed list $L(v)$ of k colors associated with v, there exists an assignment of colors to its vertices such that each vertex v receives a color from $L(v)$ and adjacent vertices receive distinct colors. The *choice number* of G, denoted by $ch(G)$, is the minimum integer k such that G is k-choosable. A *planar graph* is one that can be drawn in the plane so that its edges intersect only at their ends. We say G is a *plane graph* if G has been embedded in the plane. A nonincreasing sequence of positive integers $\pi = (d_1, d_2, \ldots, d_n)$ is called *planar graphical* if there exists a planar graph with order n having π as its

*Corresponding author. Email: yaojunc@nju.edu.cn

vertex degree sequence. We use $d_{i_1}^{l_1} d_{i_2}^{l_2} \cdots d_{i_k}^{l_k}$ to denote π consisting of l_1 d_{i_1}'s, ..., l_k d_{i_k}'s, where $\sum_{j=1}^{k} l_j = n$. A graph G is called C_l*-free* if it has no cycle of length l. Let $f(n, C_l)$ denote the maximum size of a C_l-free planar graph with order n.

Let G be a plane graph with n vertices, m edges and f faces. There is an important result on plane graphs by Euler [2] who proved that $n+f-m=2$ if G is connected. The formula is known as Euler's formula which can be found in many text books on graph theory, see for instance [1], and it is still a main tool in dealing with problems on plane graphs. Another interesting topic is the vertex colorings of plane graphs. Let G be a planar graph. Thomassen [11, 12] showed that G is 5-choosable and if the girth of G is at least 5, then G is 3-choosable, where the *girth* is the length of the shortest cycle in G. Lam et al. [4] showed that G is 4-choosable if G is C_4-free and Lam et al. [5] conjectured that G is 4-choosable if G has no two triangles sharing one edge. Steinberg [10] conjectured that G is 3-colorable if G has no 4- and 5-cycles. These results and conjectures support the idea that $\chi(G)$ and $ch(G)$ are related to the cycles of short lengths such as C_3, C_4 and so on and their distribution in G, so it is of interest to consider the structural properties of a graph without some cycles of given lengths. One of such properties is the maximum size of a graph without a cycle of given length, which is also a classical problem in extremal graph theory. Turán [13] showed that the maximum size of any C_3-free graph on p vertices is at most $\lfloor \frac{p^2}{4} \rfloor$; Reiman [8] showed that the maximum size of any C_4-free graph on p vertices is at most $\frac{p}{4}(1+\sqrt{4p-3})$ and Füredi [3] showed that the maximum size of a C_4-free graph equals to $\frac{1}{2}q(q+1)^2$ if q is a prime power greater than 13 and $p = q^2+q+1$. For planar graphs, we can deduce that $m \leq 3n-6$ by Euler's formula, and the equality holds if and only if G is maximal, that is, each face is a triangle. Furthermore, if the girth of G is g, then $m \leq g(n-2)/(g-2)$. In particular, $m \leq 2n-4$ if G is C_3-free and a complete bipartite graph $K_{2,n-2}$ is an extremal graph with $2n-4$ edges for each $n \geq 3$, that is, $f(n, C_3) = 2n-4$. If G is C_4-free, how many edges can G have? In this paper, we consider the maximum size of a C_4-free plane graph with order n, $n \geq 30$.

The main result of this paper is the following.

Theorem 1. Let $n \geq 30$ be an integer. Then $f(n, C_4) = \lfloor \frac{15}{7}(n-2) \rfloor - \mu$, where $\mu = 1$ if $n \equiv 3 (\mathrm{mod}\ 7)$ or $n = 32, 33, 37$, and $\mu = 0$ otherwise.

It is known that $\delta(G) \leq 5$ for any planar graph G of order n and $\delta(G) = 5$ can be achieved for all n, $n \geq 12$ and $n \neq 13$, see [6]. By Theorem 1, $\delta(G) \leq 4$ for any C_4-free planar graph G of order n. If $\delta(G) = 4$, then

$4n/2 \leq 15(n-2)/7$ by Theorem 1 and hence $n \geq 30$. One natural question is whether there is a C_4-free planar graph G such that $\delta(G) = 4$ for all n, $n \geq 30$. However, the answer to this question is negative for some integers n by the following Corollary 1. Let $\delta(n, C_4) = max\{\delta(G) \mid G$ is C_4-free planar graph of order $n\}$. Based on Theorem 1, we have the following.

Corollary 1. If $31 \leq n \leq 38$ and $n \neq 36$, then $\delta(n, C_4) = 3$.

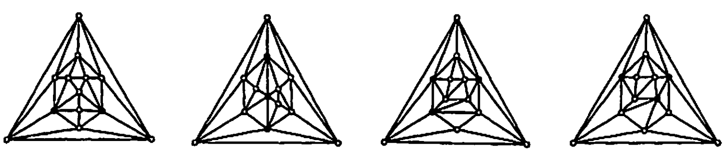
2. Upper bounds

In this section, our main task is to establish that $f(n, C_4) \leq \lfloor \frac{15}{7}(n-2) \rfloor - \mu$ for $n \geq 30$. In order to do this, we need the following two lemmas.

Lemma 1(Schmeichel and Hakimi [9]) The sequences $6^1 5^{12}$ and $6^1 5^{14}$ are not planar graphical.

The following lemma can be easily obtained by computer using software "Plantri" written by Brinkmann and McKay, see [7].

Lemma 2. Let G be a maximal planar graph of order 13. If $\pi(G) = 6^2 5^{10} 4^1$, then $G \cong T_{13}^1$ and if $\pi(G) = 6^3 5^8 4^2$, then $G \cong T_{13}^i$ for some $i \in \{2, 3, 4\}$, where T_{13}^i $(1 \leq i \leq 4)$ are shown in Figure 1.



T_{13}^1 T_{13}^2 T_{13}^3 T_{13}^4

Figure 1

Theorem 2. $f(n, C_4) \leq \frac{15}{7}(n-2)$ for $n \geq 4$.

Proof. Let G be a C_4-free plane graph of order n and size m. We use f_i to denote the number of the faces of degree i in G. Let r be the degree of the largest face in G. Assume that G has k edges not covered by triangles. Since G is C_4-free, each edge can be covered by at most one triangle and $f_4 = 0$. Thus we have $m = 3f_3 + k = 5f_5 + \cdots + rf_r - k$. Hence $f_3 = \frac{1}{3}(m-k)$ and $f_5 = \frac{1}{5}(m - 6f_6 - \cdots - rf_r + k)$. By Euler formula, we know that $n - m + f_3 + f_5 + \cdots + f_r = 2$. Replacing f_3 and f_5 with $\frac{1}{3}(m-k)$ and $\frac{1}{5}(m - 6f_6 - \cdots - rf_r + k)$, respectively, in the equality, we have $n - m + \frac{1}{3}(m-k) + \frac{1}{5}(m - 6f_6 - \cdots - rf_r + k) + f_6 + \cdots + f_r = 2$. Hence, we have

$$m = \frac{15}{7}(n-2) - \left(\frac{3}{7}f_6 + \frac{6}{7}f_7 + \cdots + \frac{3r-15}{7}f_r\right) - \frac{2k}{7}. \qquad (*)$$

By $(*)$, we have $f(n, C_4) \leq \frac{15}{7}(n-2)$. ∎

Theorem 3. Let n be a natural number and $n \equiv 3 \pmod 7$. We have $f(n, C_4) \leq \lfloor \frac{15}{7}(n-2) \rfloor - 1$.

Proof. By Theorem 2, we have $f(n, C_4) \leq \lfloor \frac{15}{7}(n-2) \rfloor$. If $n \equiv 3 \pmod 7$, then $\frac{15}{7}(n-2) = \lfloor \frac{15}{7}(n-2) \rfloor + \frac{1}{7}$. By $(*)$, we have $m = \lfloor \frac{15}{7}(n-2) \rfloor + \frac{1}{7} - (\frac{3}{7}f_6 + \frac{6}{7}f_7 + \cdots + \frac{3r-15}{7}f_r) - \frac{2k}{7}$. Because $\frac{1}{7} - (\frac{3}{7}f_6 + \frac{6}{7}f_7 + \cdots + \frac{3r-15}{7}f_r) - \frac{2k}{7}$ is an integer not exceeding $\frac{1}{7}$ and since it cannot be 0, we see that the result follows. ∎

Theorem 4. $f(32, C_4) \leq 63$.

Proof. By Theorem 2, $f(32, C_4) \leq \frac{15}{7}(32-2) = 64\frac{2}{7}$. If $f(32, C_4) = 64$, then by $(*)$, we have $k = 1$ and $f_6 = \cdots = f_r = 0$. Suppose that G is a C_4-free plane graph of order 32 and size 64. By the proof of Theorem 2, G consists of $\frac{1}{3}(m-k) = 21$ triangles and $\frac{1}{5}(m - 6f_6 - \cdots - rf_r + k) = 13$ pentagons. Let $v_0 \in V(G)$ and $d(v_0) = \delta(G)$. If $d(v_0) = 1$, then $G - v_0$ has 63 edges, which contradicts Theorem 2. If $d(v_0) = 2$, then since $k = 1$, the two edges incident to v_0 are contained in a triangle, which implies that G has a face of degree at least 6, a contradiction. Hence we have $\delta(G) \geq 3$. Let xy be the edge not covered by a triangle. If $v \in V(G) - \{x, y\}$, then since each edge incident to v is covered by exactly one triangle and the triangle covers exactly two edges incident to v, we see that $d(v)$ is even and $d(v) \geq 4$. For the same reason, $d(x)$ and $d(y)$ are odd not less than 3. Because G has 64 edges and $\sum_{v \in V(G)} d(v) = 64 \times 2 = 128$, we see that $\pi(G) = 5^1 4^{30} 3^1$ or $6^1 4^{29} 3^2$. Now, we construct a graph G^* as follows: take each pentagon of G as a vertex, two vertices are adjacent if the two pentagons share exactly one vertex or xy in G. Let A and B be the two faces of degree 5, which share xy.

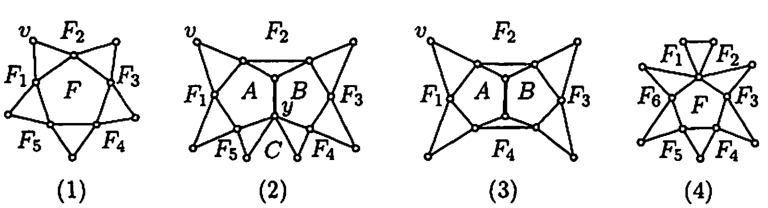


Figure 2

If $\pi(G) = 5^1 4^{30} 3^1$, we may assume that $d(x) = 3$ and $d(y) = 5$. Suppose that C is the third face of degree 5 incident to y. Let F be a face of degree 5. If $F \notin \{A, B, C\}$, then $N_{G^*}(F) = \{F_1, F_2, \ldots, F_5\}$ and $F_1 F_2 \cdots F_5$ is a C_5 in G^* since F_i and F_{i+1} share exactly one vertex (F_1 and F_2 share v), the subscripts are taken modulo 5, as shown in Figure 2(1). For the same reason, we see that A and B have degree 5 in G^* and C has degree 6 in G^*, as shown in Figure 2(2). Thus, G^* is a maximal plane graph of order 13 with $\pi(G^*) = 6^1 5^{12}$, which contradicts Lemma 1.

If $\pi(G) = 6^1 4^{29} 3^2$, then we must have $d(x) = d(y) = 3$. Let $z \in V(G)$ and $d(z) = 6$. If $z \notin V(A) \cup V(B)$, then A and B have degree 4 in G^*, as shown in Figure 2(3), the three faces of degree 5 containing z have degree 6 and form a K_3 in G^* as shown in Figure 2(4) and any other face of degree 5 has degree 5 in G^*. In this case, G^* is a maximal planar graph of order 13, $\pi(G^*) = 6^3 5^8 4^2$ and the three vertices of degree 6 form a K_3. By Lemma 2, such a graph does not exist. If $z \in V(A) \cup V(B)$, we may assume that $z \in V(A)$. Thus, A has degree 5 and B has degree 4 in G^*, the other two faces of degree 5 incident to z have degree 6 and form a K_2 in G^* and any other face of degree 5 has degree 5 in G^*. Thus, G^* is a maximal planar graph of order 13, $\pi(G^*) = 6^2 5^{10} 4^1$ and the two vertices of degree 6 form a K_2. By Lemma 2, such a graph still does not exist. Therefore, $f(32, C_4) \leq 63$. ∎

Theorem 5. $f(33, C_4) \leq 65$.

Proof. By Theorem 2, $f(33, C_4) \leq \frac{15}{7}(33-2) = 66\frac{3}{7}$. If $f(33, C_4) = 66$, then by $(*)$, we have $k = 0$ and $f_6 = 1$. Suppose that G is a C_4-free plane graph of order 33 and size 66. By the proof of Theorem 2, we see that G consists of 22 triangles, 12 pentagons and 1 face of degree 6. If $\delta(G) \leq 2$, then we have $f(32, C_4) \geq 64$, which contradicts Theorem 4. Thus, $\delta(G) \geq 3$. Since $k = 0$, we see that for any $v \in V(G)$, each edge incident to v is covered exactly by one triangle and the triangle covers exactly two edges incident to v, and hence $d(v)$ is even and $d(v) \geq 4$. Because G has 66 edges, we see that $\pi(G) = 4^{33}$. Now, Let G^* be a graph obtained from G as follows: take each face of degree at least 5 of G as a vertex, two vertices are adjacent if the two vertices share exactly one vertex in G. By a similar argument as that in the proof of Theorem 4, we see that G^* is a maximal plane graph of order 13 with $\pi(G^*) = 6^1 5^{12}$, which contradicts Lemma 1. Therefore, $f(33, C_4) \leq 65$. ∎

Theorem 6. $f(37, C_4) \leq 74$.

Proof. By Theorem 2, $f(37, C_4) \leq \frac{15}{7}(37-2) = 75$. If $f(37, C_4) = 75$, then by $(*)$, we have $k = 0$ and $f_6 = \cdots = f_r = 0$. Suppose that G is a C_4-free plane graph of order 37 and size 75. By the proof of Theorem 2, G consists of 25 triangles and 15 pentagons. If $\delta(G) = 1$, then $f(36, C_4) \geq 74$, which contradicts Theorem 2. If $\delta(G) = 2$, then since $k = 0$, the two edges incident the vertex with minimum degree v_0 are contained in a triangle, which implies that G has a face of degree at least 6, a contradiction. Thus, $\delta(G) \geq 3$. Since $k = 0$, we see that for any $v \in V(G)$, each edge incident to v is covered exactly by one triangle and the triangle covers exactly two edges incident to v, and hence $d(v)$ is even and $d(v) \geq 4$. Because G has 75 edges, we see that $\pi(G) = 4^{36}6^1$. Now, Let G^* be a graph obtained from G as follows: take each pentagon as a vertex, two vertices are adjacent if the two vertices share exactly one vertex in G. By a similar argument as that in the proof of Theorem 4, we see that G^* is a maximal plane graph of order 15 with $\pi(G^*) = 6^3 5^{12}$ and the three vertices u_1, u_2, u_3 of degree 6 form a triangle in G^*. Delete the edge u_1u_2 from G^*, then $G^* - u_1u_2$ is a planar graph with $\pi(G^* - u_1u_2) = 6^1 5^{14}$, which contradicts Lemma 1. Hence $f(37, C_4) \leq 74$. ∎

3. Lower bounds

Let $n \geq 30$ be an integer. In this section, we will establish the lower bounds for $f(n, C_4)$ by constructing some extremal graphs whose sizes are exactly the values given in Theorem 1.

We first consider the case when $n \equiv 2 \pmod 7$.

If $n = 37$, then $f(37, C_4) \geq 74$ is shown by the graph G_{37} in Figure 3. The graph G_{37} consists of 1 face of degree 6, 14 faces of degree 5, 24 faces of degree 3 and has two edges, the bold ones, not covered by triangles. By (*), the size of G_{37} is 74.

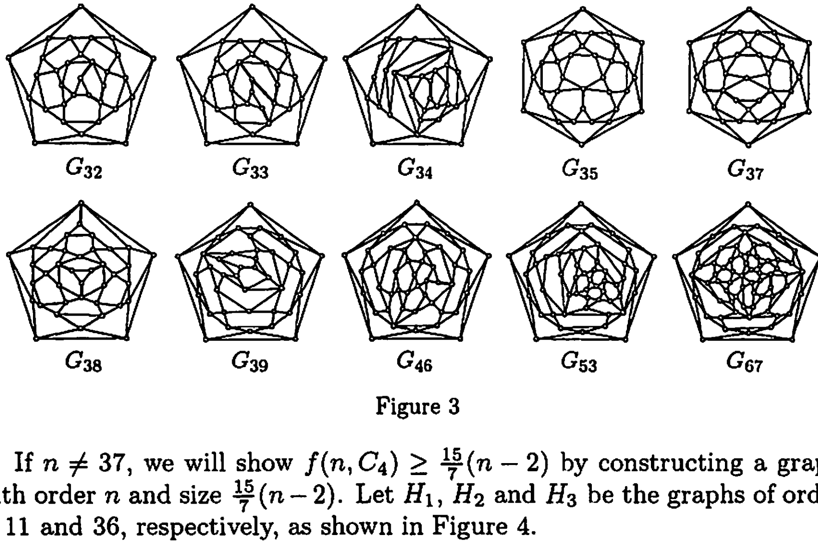


Figure 3

If $n \neq 37$, we will show $f(n, C_4) \geq \frac{15}{7}(n-2)$ by constructing a graph with order n and size $\frac{15}{7}(n-2)$. Let H_1, H_2 and H_3 be the graphs of order 5, 11 and 36, respectively, as shown in Figure 4.

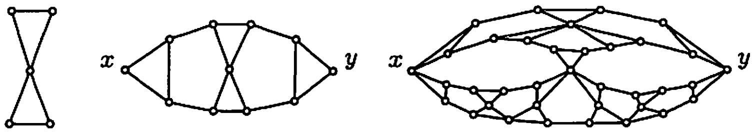
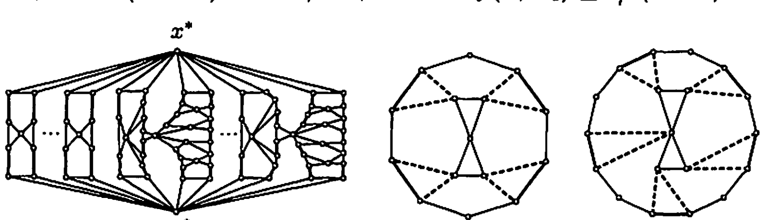


Figure 4

Let $p, q \geq 0$ be integers, $p + q \geq 1$ and $(p, q) \neq (1, 0)$. Take p copies of H_2 and q copies of H_3, identify all x into one vertex x^* and all y into one vertex y^*. Let $G(p, q)$ be the resulting graph and $\widetilde{G}(p, q)$ a planar embedding of $G(p, q)$ as shown in Figure 5. By the construction of $G(p, q)$, we see that $\widetilde{G}(p, q)$ has $p + 2q$ faces of degree 10 and q faces of degree 14. Each boundary of any face of degree 10 has 4 edges not covered by triangles and each boundary of any face of degree 14 has 5 edges not covered by triangles, that is, the bold ones in Figure 5.

Put a copy of H_1 into each face of degree 10 and 14 of $\widetilde{G}(p, q)$ and connect the vertices of H_1 to the vertices in the boundary of the face of degree 10 and 14 with dotted edges in the way as shown in Figure 5.

Let G be the final graph and $|V(G)| = n$. Since H_i for $1 \leq i \leq 3$ are C_4-free and $G(p, q)$ has no C_4 containing x^* or y^*, $G(p, q)$ is C_4-free. Noting that G has no C_4 containing the dotted edges, we see that G is C_4-free. By the construction, $n = 11p + 36q - 2(p + q - 1) + 5(p + 2q + q) = 14p + 49q + 2$. Since each edge of G is covered by a triangle and a pentagon, by $(*)$, G has $\frac{15}{7}(n - 2)$ edges. Take $p \geq 2$ and $q = 0$, then we have $n = 30 + 14(p - 2)$, and take $q = 1$ and $p \geq 0$, then $n = 51 + 14p$. This is to say that for any $n \geq 30$, $n \equiv 2 \pmod 7$ and $n \neq 37$, we have $f(n, C_4) \geq \frac{15}{7}(n - 2)$.



$\widetilde{G}(p, q)$ Face of degree 10 Face of degree 14

Figure 5

Next, we consider the case when $n \equiv 3 \pmod 7$.

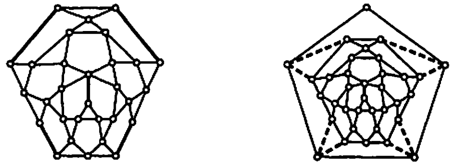
If $n = 38$, then $f(38, C_4) \geq 76$ is shown by the graph G_{38} in Figure 3. The graph G_{38} consists of 14 faces of degree 5, 24 faces of degree 3 and has four edges, the bold ones, not covered by triangles. By $(*)$, the size of G_{38} is 76.

If $n \neq 38$, then $n-1 \neq 37$ and $n-1 \equiv 2 \pmod 7$. Thus, by the construction given in the case when $n \equiv 2 \pmod 7$, there exists a C_4-free plane graph G of order $n-1$ and size $\frac{15}{7}((n-1)-2)$. By $(*)$, $k=0$ and $f_6 = \cdots = f_r = 0$. Since $k=0$, $d(v)$ is even. Since $f_6 = \cdots = f_r = 0$, we have $\delta(G) \geq 4$. By Theorem 2, $\delta(G) = 4$. If $v \in V(G)$ and $d(v) = 4$, then because $k = 0$ and G is C_4-free, we see that $G[N[v]] = H_1$, where H_1 is the graph shown in Figure 4. Assume that $N(v) = \{v_1, v_2, v_3, v_4\}$ and $v_1v_2, v_3v_4 \in E(G)$. Now, let G^* be a graph obtained from G by splitting v into two vertices v', v'' such that v' is adjacent to v_1, v_2, v'' is adjacent to v_3, v_4 and $v'v''$ is an edge in G^*. Since G is C_4-free, $G^* - v'v''$ is still C_4-free. Noting that $v_iv_j \notin E(G)$ for $i \in \{1,2\}$ and $j \in \{3,4\}$, G^* has no C_4 containing $v'v''$ and hence G^* is C_4-free. Because G^* has $\frac{15}{7}((n-1)-2)+1 = \lfloor \frac{15}{7}(n-2) \rfloor - 1$ edges, we have $f(n, C_4) \geq \lfloor \frac{15}{7}(n-2) \rfloor - 1$. Therefore, we have $f(n, C_4) \geq \lfloor \frac{15}{7}(n-2) \rfloor - 1$ for $n \equiv 3 \pmod 7$.

Thirdly, we consider the case when $n \equiv 4 \pmod 7$.

If $n = 32$, then $f(32, C_4) \geq 63$ is shown by the graph G_{32} in Figure 3. The graph G_{32} consists of 1 face of degree 6, 12 faces of degree 5, 20 faces of degree 3 and has three edges, the bold ones, not covered by triangles. By $(*)$, the size of G_{32} is 63.

If $n \in \{39, 46, 53, 67\}$, then $f(n, C_4) \geq \lfloor \frac{15}{7}(n-2) \rfloor$ are shown by the graphs G_n for $n \in \{39, 46, 53, 67\}$, respectively. Each of the four graphs consists of faces of degree 3 and 5, and has exactly one edge, the bold one, not covered by triangle. By $(*)$, the size of each graph is $\lfloor \frac{15}{7}(n-2) \rfloor$.



(1) (2)

Figure 6

If $n \notin \{32, 39, 46, 53, 67\}$, then $n - 30 \geq 30$, $n - 30 \neq 37$ and $n - 30 \equiv 2 \pmod 7$. By the arguments in the case when $n \equiv 2 \pmod 7$, we see that there exists a C_4-free plane graph G with order $n - 30$ and size $\frac{15}{7}((n - 30) - 2)$. By $(*)$, $k = 0$ and $f_6 = \cdots = f_r = 0$. Let $C = C_5$ be a face of degree 5 in G and H_4 a graph of order 30 which is shown in Figure 6(1). Embed H_4 into the face C, connect the vertices of H_4 to the vertices of C with dotted edges in the way as given in Figure 6(2). Denote by G^* the resulting graph. Since G is C_4-free, H_4 is C_4-free and G^* has no C_4 containing dotted edges, G^* is still C_4-free. Because each face of G^* is a triangle or a pentagon and G^* has only one edge, the bold one, as shown in Figure 8(2), which is not covered by a triangle, by (*), the size of G^* is $\lfloor \frac{15}{7}(n-2) \rfloor$. Thus we have $f(n, C_4) \geq \lfloor \frac{15}{7}(n-2) \rfloor$ for $n \equiv 4 \pmod 7$ and $n \neq 32$.

Finally, we consider the cases when $n \equiv 5, 6, 0, 1 \pmod 7$.

For $n = 33, 34, 35$, the lower bounds of $f(n, C_4)$ are shown by the graphs G_{33}, G_{34} and G_{35} in Figure 3, respectively. If $n = 36$, then put a vertex into a face of degree 5 whose boundary has one edge not covered by triangle in G_{35} and connect the new vertex to the ends of the edge not covered by triangle. The new graph obtained from G_{35} has 36 vertices and $\lfloor \frac{15}{7}(35-2) \rfloor + 2 = \lfloor \frac{15}{7}(36-2) \rfloor$ edges and hence $f(36, C_4) \geq \lfloor \frac{15}{7}(36-2) \rfloor$. Now, assume that $n \geq 40$ and $n \equiv i \pmod 7$, where $i = 5, 6, 0, 1$. Obviously, $n-1, n-2, n-3, n-4 \equiv 4 \pmod 7$ for $i = 5, 6, 0, 1$, respectively. By the arguments in the case when $n \equiv 4 \pmod 7$, we see that there exists a C_4-free planar graph G with order $n-l$ and size $\lfloor \frac{15}{7}((n-l)-2) \rfloor$, having only one edge not covered by a triangle, where $1 \leq l \leq 4$. Let $\widetilde{G}$ be a planar embedding of G and C a face of degree 5 whose boundary has one edge not covered by a triangle. Put l new vertices into C and connect the new vertices to the ends of the bold edge with dotted edges in the way as shown in Figure 7(1)-(4) for $l = 1, 2, 3, 4$, respectively, where the bold edge is the only edge of G not covered by triangle. Assume that the resulting graph is G^*. Since G^* has no C_4 containing dotted edges and G is C_4-free, G^* is C_4-free and has $\lfloor \frac{15}{7}((n-l)-2) \rfloor + 2l = \lfloor \frac{15}{7}(n-2) \rfloor$ edges. Therefore, we have $f(33, C_4) \geq 65$ and $f(n, C_4) \geq \lfloor \frac{15}{7}(n-2) \rfloor$ for $n \equiv 5, 6, 0, 1 \pmod 7$ and $n \neq 33$.

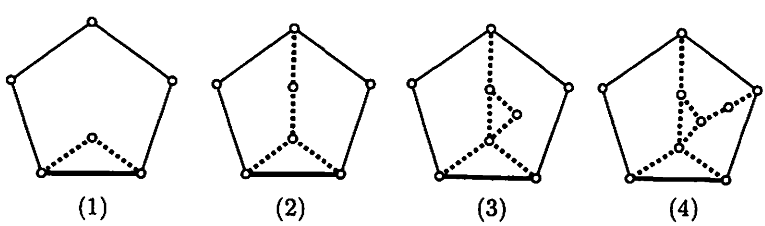


Figure 7

4. Proofs of main results

Proof of Theorem 1. For each $n \geq 30$, the upper bounds for $f(n, C_4)$ given in Section 2 equals to the lower bounds for $f(n, C_4)$ given in Section 3, so the result of Theorem 1 follows. ∎

Proof of Corollary 1. Let G be a C_4-free planar graph of order n and size m with $\delta(G) = \delta(n, C_4)$. If $n \in \{31, 32, 33\}$, then we have $\delta(n, C_4) \leq 3$ since $m \leq f(n, C_4) = 2n-1$ by Theorem 1. On the other hand, the C_4-free planar graphs of order $n \in \{31, 32, 33\}$ given in Section 3 have minimum degree 3 and hence $\delta(n, C_4) = 3$ for $n \in \{31, 32, 33\}$. If $n \in \{34, 35, 37, 38\}$, then since $f(n, C_4) = 2n$ by Theorem 1, we have $\delta(G) \leq 4$. If $\delta(G) = 4$, then G is 4-regular and $m = f(n, C_4)$. By $(*)$, $m = f(34, C_4)$ if and only if $k = 2$ and $f_6 = \cdots = f_r = 0$; $m = f(35, C_4)$ if and only if $k = 1$, $f_6 = 1$ and $f_7 = \cdots = f_r = 0$; $m = f(37, C_4)$ if and only if $k = 2$, $f_6 = 1$ and $f_7 = \cdots = f_r = 0$; $m = f(38, C_4)$ if and only if $k = 1$, $f_7 = 1$ and $f_6 = f_8 = \cdots = f_r = 0$ or $k = 1$, $f_6 = 2$ and $f_7 = \cdots = f_r = 0$. In each case, G has a vertex v which is incident to exactly one edge not covered by triangle. Since each other edge incident to v is covered exactly by one triangle, we see that $d(v)$ is odd which is a contradiction since G is 4-regular. On the other hand, the C_4-free planar graphs G_{34}, G_{35}, G_{37}, G_{38} in Figure 1 have minimum degree 3 and hence $\delta(n, C_4) = 3$ for $n \in \{34, 35, 37, 38\}$.∎

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