A Fan-type result on fractional ID-k-factor-critical graphs *

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Abstract: Let G be a graph, and let $k \geq 2$ be an integer. A graph G is fractional independent-set-deletable k-factor-critical (in short, fractional ID-k-factor-critical) if $G - I$ has a fractional k-factor for every independent set I of G. In this paper, a Fan-type condition for fractional ID-k-factor-critical graphs is given.

Key words: fractional k-factor, fractional ID-k-factor-critical graph, Fan-type
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1 Introduction

Many real-world networks can be modelled by graphs or networks. An important example of such a network is a communication network with nodes and links modelling cities and communication channels, respectively. Other examples include a railroad network with nodes and links representing railroad stations and railways between two stations, respectively, or the world wide web with nodes representing web pages, and links corresponding to hyperlinks between web pages.

We study the fractional factor problem in graphs or networks, which can be considered as a relaxation of the well-known cardinality matching

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problem. The fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e., the underlying graph is bipartite).

All graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph of order n with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, the number of edges of G incident with x is called the degree of x in G and is denoted by $d_G(x)$. We write $\delta(G) = \min\{d_G(x) : x \in V(G)\}$, which is the minimum degree of G; $\Delta(G) = \max\{d_G(x) : x \in V(G)\}$, which is the maximum degree of G. Let $k \geq 1$ be an integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for all $x \in V(G)$. A fractional k-factor is a way of assigning weights to the edges of a graph G such that for each vertex the sum of the weights of the edges incident with that vertex is k. A graph G is fractional independent-set-deletable k-factor-critical (in short, fractional ID-k-factor-critical) if $G - I$ has a fractional k-factor for every independent set I of G.

For $S \subseteq V(G)$, Let $G[S]$ be the subgraph of G induced by S. We write $G - S$ for $G[V(G) \setminus S]$. For $x \in V(G)$, the neighborhood of x is the set $N_G(x) = \{y : y \in V(G), yx \in E(G)\}$. For two disjoint subsets $S, T \subseteq V(G)$, we write $e_G(S, T)$ for the number of edges in G with one end in S and the other end in T. We define the distance $d_G(x, y)$ between two vertices x and y as the minimum of the lengths of the (x, y) paths of G. We say that a graph G is Fan-type, if every pair of vertices $x, y \in V(G)$ with $d_G(x, y) = 2$ satisfies $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$.

Fourtounelli and Katerinis [1] studied the existence of k-factors in graphs. Bauer et. al. [2]proved the degree Sequences condition for the existence of k-factors. Zhou [3] investigated the fractional k-factors in graphs. Chang, Liu and Zhu [4] first introduced the concept of a fractional ID-k-factor-critical graph, and obtained a minimum degree condition for a graph to be fractional ID-k-factor-critical. Zhou [5] showed a binding number for the existence of fractional ID-k-factor-critical graphs. Zhou, Xu and Sun [6] gave an independence number and minimum degree condition on fractional ID-k-factor-critical graphs. The following results on fractional ID-k-factor-critical graphs are known.

Theorem 1 (Chang, Liu and Zhu [4]) *Let k be a positive integer and G be a graph of order n with $n \geq 6k - 8$. If $\delta(G) \geq \frac{2n}{3}$, then G is fractional ID-k-factor-critical.*

Theorem 2 (Zhou, Xu and Sun [6]) *Let G be a graph, and let k be an integer with $k \geq 1$. If*

$$\alpha(G) \leq \frac{4k(\delta(G) - k + 1)}{k^2 + 6k + 1},$$

then G is fractional ID-k-factor-critical.

In this paper, we have established a new sufficient condition on the existence of a fractional ID-k-factor-critical graph, it involves the order and the Fan-type condition of the graph. Our result is the following theorem which is an improvement of Theorems 1.

Theorem 3 *Let $k \geq 2$ be an integer and G be a graph of order n with $n \geq 6k^2 + 3k - 9 - \frac{3}{k-1}$, $\delta(G) \geq \frac{n}{3} + k$. If*

$$\max\{d_G(x), d_G(y)\} \geq \frac{2n}{3} \tag{1}$$

for any two vertices x and y of G with $d_G(x, y) = 2$, then G is fractional ID-k-factor-critical.

2 The Proof of Theorem 3

For $S, T \subseteq V(G)$, write $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$. We use heavily the following lemma to prove Theorem 3.

Lemma 1 (Liu and Zhang [7]) *Let G be a graph. Then a graph G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k - 1\}$.

Proof of Theorem 3. Let I be an independent set of G and $H = G - I$. To prove Theorem 3, we only need to verify that H has a fractional k-factor. In order to prove this by reduction to absurdity, we assume that H has no fractional k-factor. Then from Lemma 1, there exists some subset S of $V(H)$ satisfying

$$\delta_H(S, T) = k|S| + d_{H-S}(T) - k|T| \leq -1, \tag{2}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq k - 1\}$.

Claim 1. $|I| \leq \frac{n}{3}$.

Proof. The inequality obviously holds for $|I| = 1$. Hence, we may assume that $|I| \geq 2$. We consider two cases.

Case 1. $d_G(u,v) \geq 3$ for any $u, v \in I$.

In this case, for any $x \in V(G) \setminus I$, there exists at most one vertex u in I satisfying $xu \in E(G)$. (Otherwise, there exist two vertices u, v in I such that $xu, xv \in E(G)$, then $d_G(u,v) = 2$, which contradicts the condition of Case 1.) Thus, we obtain

$$\max\{d_{G-I}(x), d_{G-I}(y)\} = \max\{d_H(x), d_H(y)\} \geq \frac{2n}{3} - 1$$

for any two vertices x, y of $H = G - I$ with $d_H(x,y) = 2$. It is easy to see that

$$n \geq \max\{d_H(x), d_H(y)\} + 2 + (|I| - 1) \geq \frac{2n}{3} + |I|,$$

which implies

$$|I| \leq \frac{n}{3}.$$

Case 2. There exist $u, v \in I$ with $d_G(u,v) = 2$.

In terms of the condition of Theorem 3, we have

$$n \geq \max\{d_G(u), d_G(v)\} + |I| \geq \frac{2n}{3} + |I|,$$

which implies

$$|I| \leq \frac{n}{3}.$$

The proof of Claim 1 is complete.

Claim 2. $|T| \geq k + 1$.

Proof. Assume that $|T| \leq k$. Then by Claim 1 and the fact that $\delta(H) \geq \delta(G) - |I| \geq \frac{n}{3} + k - |I|$, we obtain

$$\begin{aligned}
\delta_H(S,T) &\geq k|S| + d_{H-S}(T) - k|T| \geq |T||S| + d_{H-S}(T) - k|T| \\
&= \sum_{x \in T}(|S| + d_{H-S}(x) - k) \geq \sum_{x \in T}(\delta(H) - k) \\
&\geq \sum_{x \in T}(\frac{n}{3} + k - |I| - k) = \sum_{x \in T}(\frac{n}{3} - |I|) \geq 0,
\end{aligned}$$

which contradicts (2). This completes the proof of Claim 2.

Claim 3. $k|T| \geq k|S| + 1$, that is, $|T| \geq |S| + 1$.

Proof. Using (2), we get

$$-1 \geq \delta_H(S,T) = k|S| + d_{H-S}(T) - k|T| \geq k|S| - k|T|,$$

that is,

$$k|T| \geq k|S| + 1,$$

and so

$$|T| \geq |S| + 1.$$

This completes the proof of Claim 3.

Claim 4. $|S| \geq 1$.

Proof. Note that $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq k-1\}$. It follows from Claim 1, $H = G - I$ and $\delta(G) \geq \frac{n}{3} + k$ that

$$|S| + k - 1 \geq |S| + d_{H-S}(x) \geq d_G(x) - |I| \geq \delta(G) - |I| \geq \frac{n}{3} + k - |I| \geq k$$

for each $x \in T$. Thus, we have $|S| \geq 1$. The proof of Claim 4 is complete.

Claim 5. $|S| + |I| < \frac{2n}{3}$.

Proof. In terms of Claim 1, Claim 3 and $|S| + |T| + |I| \leq n$, we obtain

$$kn \geq k|S| + k|T| + k|I| > 2k|S| + k|I| = 2k(|S| + |I|) - k|I| \geq 2k(|S| + |I|) - \frac{kn}{3},$$

which implies

$$|S| + |I| < \frac{2n}{3}.$$

This completes the proof of Claim 5.

Claim 6. $|S| + |I| < \frac{2n}{3} - (k-1)$.

Proof. We may assume that $|S| + |I| \geq \frac{2n}{3} - (k-1)$. It follows from (2), $|S| + |T| + |I| \leq n$ and Claim 1 that

$$\begin{aligned} d_{H-S}(T) &\leq k|T| - k|S| - 1 \leq k(n - |S| - |I|) - k|S| - 1 \\ &= kn - 2k(|S| + |I|) + k|I| - 1 \leq kn - 2k(|S| + |I|) + \frac{kn}{3} - 1 \\ &= \frac{4kn}{3} - 2k(|S| + |I|) - 1 \leq \frac{4kn}{3} - 2k(\frac{2n}{3} - (k-1)) - 1 \\ &= 2k(k-1) - 1. \end{aligned}$$

Combining this with $n \geq 6k^2 + 3k - 9 - \frac{3}{k-1}$, Claim 1 and Claim 3, we have

$$\begin{aligned} \frac{d_{H-S}(T)}{|T|} &\leq \frac{2k(k-1) - 1}{|S| + 1} = \frac{2k(k-1) - 1}{(|S| + |I|) + 1 - |I|} \\ &\leq \frac{2k(k-1) - 1}{\frac{2n}{3} - (k-1) + 1 - \frac{n}{3}} = \frac{2k(k-1) - 1}{\frac{n}{3} - (k-1) + 1} \leq 1 - \frac{1}{k}. \end{aligned}$$

Combining this with Claim 2, we obtain

$$d_{H-S}(T) \leq (1 - \frac{1}{k})|T| = |T| - \frac{1}{k}|T| < |T| - 1. \qquad (3)$$

Let $T_0 = \{x : x \in T, d_{H-S}(x) = 0\}$. Note that $|T_0| \geq 2$ holds by (3). For any $x \in T_0$, $d_G(x) \leq d_H(x) + |I| \leq |S| + |I| < \frac{2n}{3}$ by $H = G - I$ and Claim

5. Since T_0 is an independent set of G and G satisfies the hypothesis of Theorem 3, the neighborhoods of the vertices in T_0 are disjoint. Hence, we have

$$|S| + |I| \geq |\bigcup_{x \in T_0} N_G(x)| \geq \delta(G)|T_0| \geq (\frac{n}{3} + k)|T_0|. \qquad (4)$$

According to (3) and the definition of T_0, we have that $(1 - \frac{1}{k})|T| \geq d_{H-S}(T) \geq |T| - |T_0|$, that is, $|T_0| \geq \frac{1}{k}|T|$. Combining this with (4), we obtain

$$|S| + |I| \geq |(\frac{n}{3} + k)|T_0| \geq (\frac{n}{3k} + 1)|T|. \qquad (5)$$

Using (5), Claim 1 and Claim 2, we have

$$|S| \geq (\frac{n}{3k} + 1)|T| - |I| = |T| + \frac{n}{3k}|T| - |I| > |T|,$$

which contradicts Claim 3. The proof of Claim 6 is complete.

Claim 7. $e_H(S,T) \leq k|S|$.

Proof. Note that $d_{H-S}(x) \leq k - 1$ for each $x \in T$. According to $H = G - I$ and Claim 6, we have

$$d_G(x) \leq d_H(x) + |I|d_{H-S}(x) + |S| + |I| < k - 1 + \frac{2n}{3} - (k-1) = \frac{2n}{3} \quad (6)$$

for each $x \in T$. Using (6) and the hypothesis of this theorem, $G[N_G(s) \cap T]$ is a complete induced subgraph of G for each $s \in S$. Note that $H = G - I$. Hence, $H[N_H(s) \cap T]$ is a complete induced subgraph of H for any $s \in S$. Obviously, $S \neq \emptyset$ by Claim 4. It follows from $d_{H-S}(x) \leq k - 1$ for any $x \in T$ that $e_H(s,T) \leq \Delta(H[T]) + 1 \leq k$. Hence $e_H(S,T) \leq k|S|$ holds. The proof of Claim 7 is complete.

In terms of Claim 7 and $\delta(H) \geq \delta(G) - |I| \geq (\frac{n}{3} + k) - \frac{n}{3} = k$, we have

$$\begin{aligned} \delta_H(S,T) &= k|S| + d_{H-S}(T) - k|T| = k|S| + d_H(T) - k|T| - e_H(S,T) \\ &\geq k|S| - e_H(S,T) \geq 0, \end{aligned}$$

which contradicts (2). Finally, Theorem 3 is proved.

3 Remark

The lower bound on the condition (1) is best possible in the sense that we cannot replace $\frac{2n}{3}$ by $\frac{2n}{3} - 1$, which is shown in the following example:

We construct a graph $G = ktK_1 \vee ktK_1 \vee (kt + 1)K_1$, where $k \geq 2$ is an integer and t is sufficiently large positive integer. Then it follows that $n = 3kt + 1$ and

$$\frac{2n}{3} > \max\{d_G(x), d_G(y)\} > \frac{2n}{3} - 1$$

for any two vertices vertices x and y in $(kt+1)K_1$ with $d_G(x,y)=2$. We choose $I=V(ktK_1)$. Let $H=G-I$. It is easy to see that H has no fractional k-factor, and so G is not fractional ID-k-factor-critical.

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