An upper bound for the crossing number of locally twisted cubes *

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Abstract

The *crossing number* of a graph G is the minimum number of pairwise intersections of edges in a drawing of G. The *n-dimensional locally twisted cubes* LTQ_n, proposed by X.F. Yang, D.J. Evans and G.M. Megson, is an important interconnection network with good topological properties and applications. In this paper, we mainly obtain an upper bound on the crossing number of LTQ_n no more than $\frac{265}{6}4^{n-4} - (n^2 + \frac{15+(-1)^{n-1}}{6})2^{n-3}$.

Keywords: *Drawing; Crossing number; Locally twisted cube; Hypercube; Interconnection network*

1 Introduction

The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. The notion of crossing number is a central one for Topological Graph Theory and has been studied extensively by mathematicians including Erdős, Guy, Turán and Tutte, et al. (see [7, 23, 24]). In the past thirty years, it turned out that crossing number has many important applications in discrete and computational geometry (see [2, 14, 19, 20, 22]).

On the other hand, the immediate applications in VLSI theory and wiring layout problems (see [1, 11, 12, 17]) also inspired the study of crossing number of some popular parallel network topologies such as hypercube and its variations. Among all the popular parallel network topologies, hypercube is the first to be studied (see [3–6, 13, 21]). An n-dimensional hypercube Q_n is a graph in which the nodes can be one-to-one labeled with 0-1 binary sequences of length n, so that the labels of any two adjacent nodes differ in exactly one bit.

Computing the crossing number was proved to be NP-complete by Garey and Johnson [8]. Thus, it is not surprising that the exact crossing numbers are known for graphs of few families and that the arguments often strongly depend on their structures (see for example [10, 15, 16, 27]). Even for hypercube, for a long time the only known result on the exact value of crossing number of Q_n has been $cr(Q_3) = 0$, $cr(Q_4) = 8$ [3]. Concerned with upper bound of crossing number of hypercube, Eggleton and Guy [4] in 1970 established a drawing of Q_n to show

$$\mathrm{cr}(Q_n) \leq \frac{5}{32}4^n - \lfloor\frac{n^2+1}{2}\rfloor 2^{n-2}. \tag{1}$$

However, a gap was found in their constructions. Erdős and Guy [7] in 1973 stated the above inequality again as a conjecture. In fact, Erdős and Guy further conjectured the *equality* of (1) holds. With regard to the latest progress of this conjecture, the interested readers are referred to [5, 6, 28].

The *n-dimensional locally twisted cube* LTQ_n proposed by X.F. Yang, D.J. Evans and G.M. Megson [26] in 2005 is an important variation of Q_n. The locally twisted cube keeps as many nice properties of hypercube as possible and is conceptually closer to traditional hypercube, while it has diameters of about half of that of a hypercube of the same size. Therefore, it would be more attractive to study the crossing number of the n-dimensional locally twisted cubes.

The *n-dimensional locally twisted cube* $LTQ_n (n \geq 2)$ is defined recursively as follows.

(a) LTQ_2 is a graph isomorphic to Q_2.

(b) For $n \geq 3$, LTQ_n is built from two disjoint copies of LTQ_{n-1} according to the following steps. Let $0LTQ_{n-1}$ denote the graph obtained by prefixing the label of each vertex of one copy of LTQ_{n-1} with 0, let $1LTQ_{n-1}$ denote the graph obtained by prefixing the label of each vertex of the other copy LTQ_{n-1} with 1, and connect each vertex $x = 0x_2x_3 \dots x_n$ of $0LTQ_{n-1}$ with the vertex $1(x_2 + x_n)x_3 \dots x_n$ of $1LTQ_{n-1}$ by an edge, where $+$ represents the modulo 2 addition.

The graphs shown in Figure 1.1 are LTQ_3 and LTQ_4, respectively.

In this paper we mainly obtain an upper bound of the crossing number of LTQ_n for $n \geq 6$,

$$cr(LTQ_n) \leq \frac{265}{6}4^{n-4} - (n^2 + \frac{15 + (-1)^{n-1}}{6})2^{n-3}.$$

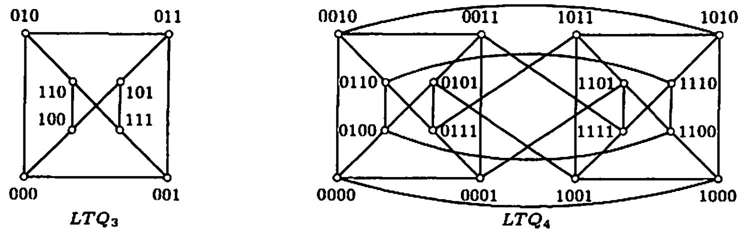


Figure 1.1: Locally twisted cubes LTQ_3 and LTQ_4

2 Upper bound for $cr(LTQ_n)$

A drawing of G is said to be a *good* drawing, provided that no edge crosses itself, no adjacent edges cross each other, no two edges cross more than once, and no three edges cross in a point. It is well known that the crossing number of a graph is attained only in *good* drawings of the graph. So, we always assume that all drawings throughout this paper are good drawings. For a good drawing D of a graph G, let $\nu_D(G)$ be the number of crossings in D. In what follows, $\nu_D(G)$ is abbreviated to ν_D when it is unambiguous.

Let $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ be two vertices of LTQ_n. Denote

$$\mathscr{D}(x_1x_2\cdots x_n) = 2^{n-1}x_1 + 2^{n-2}x_2 + \cdots + 2^0x_n$$

to be the corresponding decimal number of $x_1x_2\cdots x_n$. Let

$$\theta_i(x) = x_i \quad \text{for } i \in \{1, 2, \ldots, n\}.$$

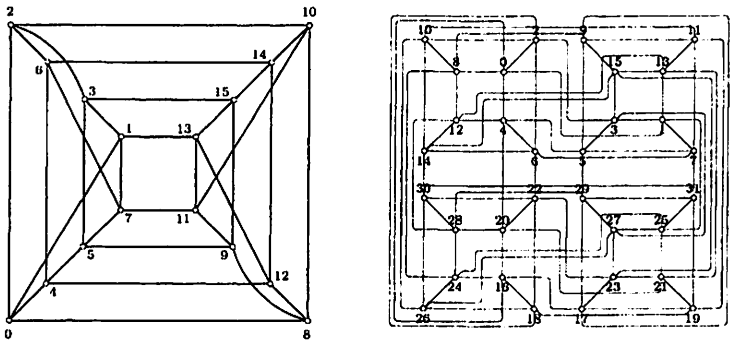
Let $\lambda(x, y)$ be the smallest positive integer $i \in \{1, 2, \ldots, n\}$ such that $\theta_i(x) \neq \theta_i(y)$. We define

$$Dim(x, y) = \begin{cases} \lambda(x, y), & \text{if } \; x \text{ and } y \text{ are adjacent;} \\ \infty, & \text{otherwise.} \end{cases}$$

In particular, for an edge $e = xy$, let $Dim(e) = Dim(x, y)$ and say the edge e lies in the $Dim(e)$-dimension. We call x an *odd vertex* if $|\{1 \leq i \leq n : x_i = 1\}| \equiv 1 \pmod 2$, and an *even vertex* if otherwise.

For the clearness of composition, in the rest of this section, any vertex $x \in V(LTQ_n)$ in figures will be represented by the corresponding decimal number $\mathscr{D}(x)$. We first give a drawing of LTQ_4 with 10 crossings and a drawing of LTQ_5 with 68 crossings as shown in Figure 2.1. Hence, we have

Proposition 1. *$cr(LTQ_4) \leq 10$ and $cr(LTQ_5) \leq 68$.*



(1)LTQ_4 (2)LTQ_5

Figure 2.1: Drawings of LTQ_4 with 10 crossings and LTQ_5 with 68 crossings

Before proving the upper bound of $cr(LTQ_n)$ for $n \geq 6$, we need to introduce some technical notations. We define two structures M^i and M_c^i, called "*meshes*" which will be used in counting the number of crossings. Consider the canonical geometry of the real plane $\mathbb{R}^2$. By $[0,1]$ we denote the closed interval joining the points $(0,0)$ and $(1,0)$ of the horizontal real axis. Let r and s be a non-horizontal pair of parallel straight lines in the real plane $\mathbb{R}^2$, such that the point $(0,0)$ belongs to r and the point $(1,0)$ belongs to s. For a positive integer n, let $\mathscr{L}_n = \{(r_i, s_i) : i \in \{1,2,\ldots,n\}\}$ be a set of non-horizontal pairs of parallel straight lines in the real plane $\mathbb{R}^2$, such that the point $(0,0)$ belongs to r_i and the point $(1,0)$ belongs to s_i.

A *mesh* with index n, denoted M^n, is the set of points of the plane consisting of the points of the n-element set $\mathscr{L}_n$ plus the points in the interval $[0,1]$. In Figure 2.2, we show as an example a drawing of each M^1, M^2, M^3 and M^5.

A *chopped mesh* with index n, denoted M_c^n, is the set of points of M^n without a pair of parallel semi-straight lines of the left-most lower semi-plane. In Figure 2.3, we show a drawing of each M_c^1, M_c^2, M_c^3 and M_c^5.

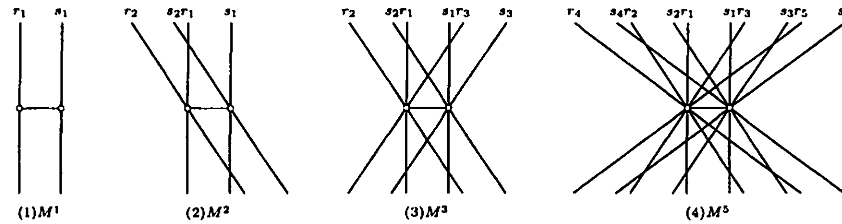


Figure 2.2: Drawings of M^1, M^2, M^3 and M^5

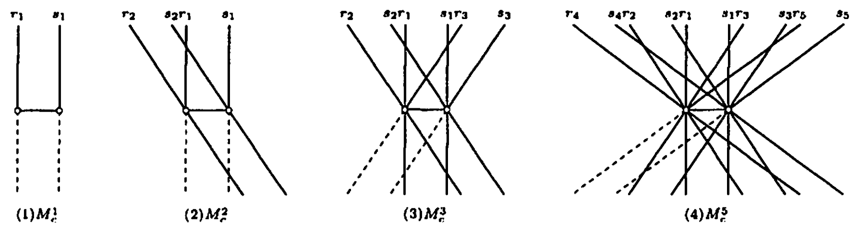


Figure 2.3: Drawings of M_c^1, M_c^2, M_c^3 and M_c^5

Lemma 2.1. *[6] For any positive integer n, there is a drawing of M^n with $n(n-1)$ crossings.*

Lemma 2.2. *[6] For any positive integer n, there is a drawing of M_c^n with $(n-1)^2$ crossings.*

To prove the general upper bound of $cr(LTQ_n)$, we need to construct a drawing D_n of LTQ_n with the desired number of crossings. The philosophy is putting the obtained drawing D_{n-1} of LTQ_{n-1} on the given coordinate systems (see Figure 2.5), then replacing each vertex of LTQ_{n-1} by two vertices of LTQ_n and replacing each edge of LTQ_{n-1} by a bunch of two edges of LTQ_n. Hence, we need the following definitions.

Definition 2.1. *Let x be a vertex of LTQ_n, and let $e \in E(LTQ_n)$ be an edge incident with x. Assume that x is drawn precisely on some axis A. We call e an a-arc or b-arc with respect to x, provided that the edge e is drawn to be upward from A (based upon the positive direction of the axis A) or to be downward from A, respectively. In particular, let*

$$\alpha(x) = |\{e \in E(LTQ_n) : e \text{ is an a-arc with respect to } x\}|$$

and

$$\beta(x) = |\{e \in E(LTQ_n) : e \textit{ is a b-arc with respect to } x\}|.$$

For example, as shown in Figure 2.5, the three edges joining vertex 23 and vertices 17, 27, 21 are a-arcs with respect to vertex 23, and the three edges joining vertex 23 and vertices 22, 39, 15 are b-arcs with respect to vertex 23.

Definition 2.2. *Let x and y be two vertices of LTQ_n with $Dim(x, y) = n - 1$. Assume that x and y are drawn next to each other on some axis. Now we define the forward direction of x as follows: (1) if $\theta(x) = 1$ and x is an odd vertex, then the forward direction of x is coincident with the direction from y to x; (2) otherwise, the forward direction of x is coincident with the direction from x to y. (see Figure 2.4)*

(1)$\theta_n(x)$=1 and x is an odd vertex (2)otherwise

Figure 2.4: The forward direction of vertex x

Definition 2.3. *Let x and y be two adjacent vertices of LTQ_n. For $i \in \{1, 2\}$, we define $\varepsilon_i = \varepsilon_i(x, y)$ and $\zeta_i = \zeta_i(x, y)$ satisfying that $\{(\varepsilon_1, \zeta_1), (\varepsilon_2, \zeta_2)\} = \{(0, 1), (1, 0)\}$ if $Dim(x, y) = n - 1$ and $\theta_n(x) = 1$, and that $\{(\varepsilon_1, \zeta_1), (\varepsilon_2, \zeta_2)\} = \{(0, 0), (1, 1)\}$ otherwise.*

In what follows, $\varepsilon_i(x, y), \zeta_i(x, y)$ are abbreviated to ε_i, ζ_i respectively when it is unambiguous. Let $x = x_1x_2 \cdots x_n$ be a vertex of LTQ_n. We define

$$x^\delta = x_1x_2 \cdots x_{n-1}\delta x_n$$

to be a vertex of LTQ_{n+1}, where $\delta \in \{0, 1\}$.

Observation 2.1. *Let x and y be two adjacent vertices of LTQ_n. Then x^{ε_i} and y^{ζ_i} are adjacent vertices of LTQ_{n+1}, in particular;*

$$Dim(x^{\varepsilon_i}, y^{\zeta_i}) = \begin{cases} Dim(x, y), & \textit{if } \ Dim(x, y) \leq n - 1; \\ n + 1, & \textit{if } \ Dim(x, y) = n; \end{cases}$$

Observation 2.2. *Let x, y, u, v be four vertices of LTQ_n with $Dim(x, u) = Dim(y, v) = n - 1$. If x and y are adjacent, then u and v are adjacent, in particular, $Dim(u, v) = Dim(x, y)$.*

Now we are in a position to prove the general upper bound of $cr(LTQ_n)$.

Theorem 2.1. *For $n \geq 6$,*

$$cr(LTQ_n) \leq \frac{265}{6}4^{n-4} - (n^2 + \frac{15 + (-1)^{n-1}}{6})2^{n-3}.$$

Proof. To prove the theorem, we shall construct a drawing D_n of LTQ_n for any $n \geq 6$, which satisfies the following five properties.

Property 1: $\nu_{D_n} = \frac{265}{6}4^{n-4} - (n^2 + \frac{15+(-1)^{n-1}}{6})2^{n-3}$.

Property 2: *Every vertex x of LTQ_n is drawn precisely on some axis, and moreover, $|\alpha(x) - \beta(x)| \leq 1$.*

Property 3: *Let x, u be two vertices of LTQ_n with $Dim(x, u) = n-1$. Then x and u are drawn next to each other on the same axis. Moreover, $\alpha(x) = \alpha(u)$ and $\beta(x) = \beta(u)$.*

Property 4: *Let x, y, u, v be four vertices of LTQ_n with $Dim(x, u) = Dim(y, v) = n - 1$. Assume that x and y are adjacent. Then xy is an a-arc (b-arc) with respect to x if and only if uv is an a-arc (b-arc) with respect to u.*

Property 5: *Let x, y, u, v be four vertices of LTQ_n with $Dim(x, u) = Dim(y, v) = n - 1$. If $Dim(x, y) < n$ then $\nu_{D_n}(xy, uv) = 0$.*

Assume first $n = 6$. The drawing D_6 is given in Figure 2.5. It is not hard to check that Properties 2, 3, 4 and 5 hold for D_6. We verify that the number of crossings is $400 = \frac{265}{6} \cdot 4^{6-4} - (6^2 + \frac{15+(-1)^{6-1}}{6}) \cdot 2^{6-3}$, and so Property 1 holds for D_6.

Now assume that $n \geq 6$ and that there exists a drawing D_n of LTQ_n satisfying Properties 1, 2, 3, 4 and 5. It suffices to construct a drawing D_{n+1} of LTQ_{n+1} for which the above properties hold. The process of constructing D_{n+1} is as follows. Replace each vertex x of LTQ_n in the "small" neighborhood of x in the drawing D_n by two vertices $x^0, x^1 \in$

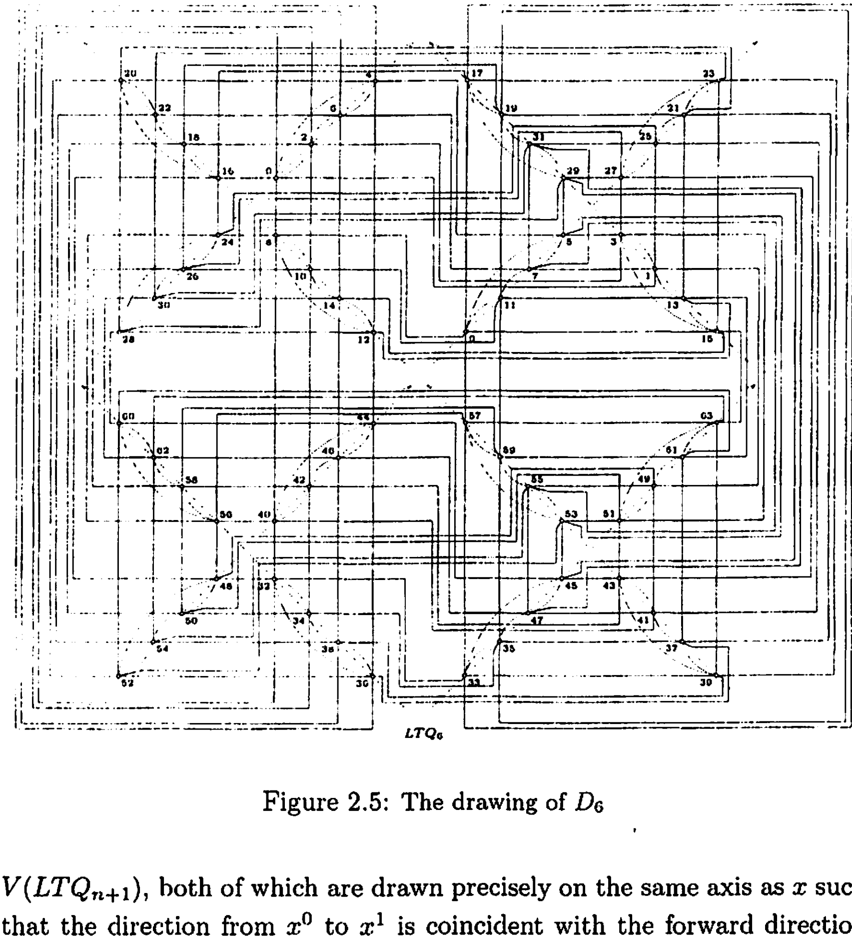


Figure 2.5: The drawing of D_6

$V(LTQ_{n+1})$, both of which are drawn precisely on the same axis as x such that the direction from x^0 to x^1 is coincident with the forward direction of x. Then join x^0 and x^1 by an *a-arc* or *b-arc* with respect to x^0 (x^1) according to $\alpha(x) \leq \beta(x)$ or not. By Observation 2.1, we need to replace each edge incident with x in LTQ_n, denoted $e = xy \in E(LTQ_n)$, by a bunch of two edges $x^{\epsilon_1}y^{\zeta_1}, x^{\epsilon_2}y^{\zeta_2} \in E(LTQ_{n+1})$ which are "parallel" or crossed each other at "infinity" (compared to the "small" neighborhoods of x and y), and drawn along the original edge e.

To illustrate the process above, we give in Figure 2.6 the extracted local drawing on vertices 9, 11, 7, 5 in D_6 and the corresponding extended drawings in D_7 and D_8. Notice that in Figure 2.6(1) the vertices 11 and 7 are odd vertices, and that $Dim(9, 11) = Dim(5, 7) = 5 = n - 1$. Hence, the forward direction of the vertex 11(7) is from 9(5) to 11(7).

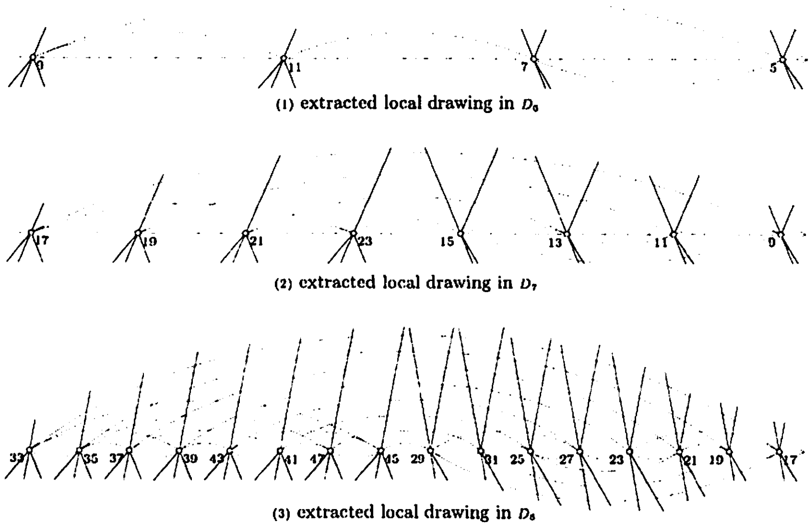


Figure 2.6: The extracted local drawings

By the process described as above, we conclude that Properties 2, 3 and 4 hold for D_{n+1}. Because that D_n has Properties 3, 4 and 5, we can verify that $\nu_{D_{n+1}}(x^{\varepsilon_1}y^{\zeta_1}, x^{\varepsilon_2}y^{\zeta_2}) = 0$ for any edge $xy \in LTQ_n$ with $Dim(xy) < n-1$ (see Figure 2.7, where $u, v \in V(LTQ_n)$ such that $Dim(u,x) = Dim(v,y) = n-1$) and that $\nu_{D_{n+1}}(x^{\varepsilon_1}y^{\zeta_1}, x^{\varepsilon_2}y^{\zeta_2}) = 0$ for any edge $xy \in LTQ_n$ with $Dim(xy) = n-1$ (see Figure 2.8). Combining with Observation 2.1, we conclude that Property 5 holds for D_{n+1}.

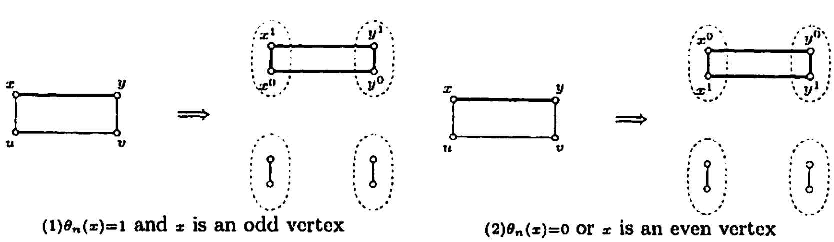
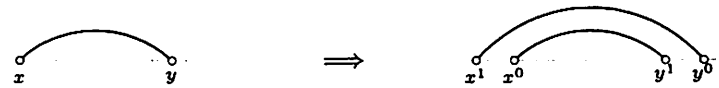


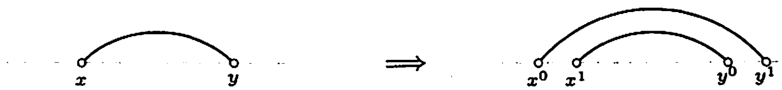
Figure 2.7: The case for $Dim(xy) < n-1$

It remains to show that Property 1 holds for D_{n+1}.

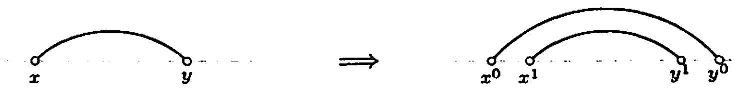
Claim A. *For any vertex x of LTQ_n, the number of crossings produced*



(1)$\theta_n(x)$=1 and x is an odd vertex



(2)$\theta_n(x)$=1 or x is an even vertex



(3)$\theta_n(x)$=0

Figure 2.8: The case for $Dim(xy) = n - 1$

in the "small" neighborhood of the new edge x^0x^1 in D_{n+1} are equal to $\frac{(n-1)^2}{4}$ for odd n and $\frac{n(n-2)}{4}$ for even n.

Proof of Claim A. Since D_{n+1} has Properties 2, 3 and 4, we conclude that the neighborhood of the new edge x^0x^1 corresponds to a drawing of $M_c^{\frac{n+1}{2}}$ for odd n, and a drawing of $M^{\frac{n}{2}}$ for even n. Then the claim follows from Lemma 2.1 and Lemma 2.2.

Claim B. $|\{xy \in E(LTQ_n) : Dim(xy) = n \text{ and } \nu_{D_{n+1}}(x^{\varepsilon_1}y^{\zeta_1}, x^{\varepsilon_2}y^{\zeta_2}) = 1\}| = 2^{n-2}$.

Proof of Claim B. By Observation 2.2, there exists a partition $E_1, \ldots, E_{2^{n-2}}$ of $\{e \in LTQ_n : Dim(e) = n\}$ with $|E_i| = 2$, say

$$E_i = \{x_iy_i, u_iv_i\},$$

such that

$$Dim(x_i, u_i) = Dim(y_i, v_i) = n - 1,$$

where $i \in \{1, 2, \ldots, 2^{n-2}\}$. To prove Claim B, it suffices to show that

$$\nu_{D_{n+1}}(x_i^{\varepsilon_1}y_i^{\zeta_1}, x_i^{\varepsilon_2}y_i^{\zeta_2}) + \nu_{D_{n+1}}(u_i^{\varepsilon_1}v_i^{\zeta_1}, u_i^{\varepsilon_2}v_i^{\zeta_2}) = 1 \qquad (2)$$

for all $i \in \{1, 2, \ldots, 2^{n-2}\}$. Assume without loss of generality that $\theta_n(y_i) = \theta_n(v_i) = 1$ and v_i is an odd vertex, i.e., $\theta_n(x_i) = \theta_n(u_i) = 0$ and y_i is an even vertex. Since D_n has properties 3 and 4, we can verify (2) immediately by two cases $\nu_{D_n}(x_iy_i, u_iv_i) = 0$ and $\nu_{D_n}(x_iy_i, u_iv_i) = 1$, which are shown in Figure 2.9. This proves Claim B. □

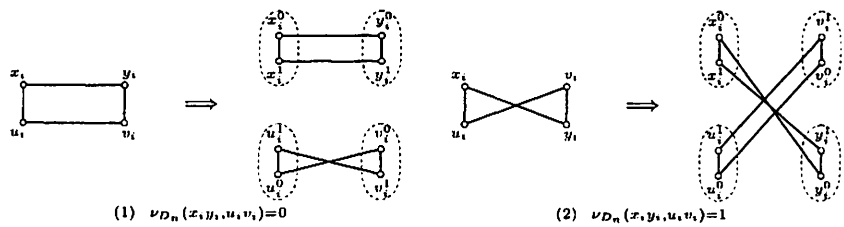


Figure 2.9: Two cases of $\nu_{D_n}(x_iy_i, u_iv_i) = 0$ and $\nu_{D_n}(x_iy_i, u_iv_i) = 1$

By the process of constructing D_{n+1}, we conclude that

$$\nu_{D_{n+1}} = 4 \cdot \nu_{D_n} + \Gamma_n + |\{xy \in E(LTQ_n) : \nu_{D_{n+1}}(x^{\varepsilon_1}y^{\zeta_1}, x^{\varepsilon_2}y^{\zeta_2}) = 1\}| \quad (3)$$

where Γ_n denotes the total number of crossings produced in the "small" neighborhoods of all new edges x^0x^1. By Claim A, we have that

$$\Gamma_n = \begin{cases} 2^n \cdot \frac{(n-1)^2}{4}, & \text{if} \quad n \equiv 1 \pmod 2; \\ 2^n \cdot \frac{n(n-2)}{4}, & \text{if} \quad n \equiv 0 \pmod 2. \end{cases} \quad (4)$$

Recall that D_{n+1} has Property 5. It follows from Observation 2.1 that $|\{xy \in E(LTQ_n) : Dim(xy) \leq n-1 \text{ and } \nu_{D_{n+1}}(x^{\varepsilon_1}y^{\zeta_1}, x^{\varepsilon_2}y^{\zeta_2}) = 1\}| = 0$. By Claim B, we have that

$$|\{xy \in E(LTQ_n) : \nu_{D_{n+1}}(x^{\varepsilon_1}y^{\zeta_1}, x^{\varepsilon_2}y^{\zeta_2}) = 1\}| = 2^{n-2}. \quad (5)$$

By (3), (4) and (5), we conclude that

$$\nu_{D_{n+1}} = \begin{cases} 4 \cdot \nu_{D_n} + 2^n \cdot \frac{(n-1)^2}{4} + 2^{n-2} = 4\nu_{D_n} + (n^2 - 2n + 2)2^{n-2}, \\ \qquad \text{if} \quad n \equiv 1 \pmod 2; \\ 4 \cdot \nu_{D_n} + 2^n \cdot \frac{n(n-2)}{4} + 2^{n-2} = 4\nu_{D_n} + (n^2 - 2n + 1)2^{n-2}, \\ \qquad \text{if} \quad n \equiv 0 \pmod 2). \end{cases}$$

Since D_n has Property 1, it is easy to verify that Property 1 holds for D_{n+1}. This completes the proof of Theorem 2.1. □

For the convenience of the reader, we offer in Figure 2.10 and 2.11 drawings for LTQ_7 and LTQ_8 obtained according to the process of constructing D_n. In the subsequent section, we shall just apply the same technique of congestions proposed by Leighton [11] to obtain a lower bound of the crossing number $cr(LTQ_n)$ of locally twisted cubes greater than $\frac{4^n}{20} - (n^2+1)2^{n-1}$.

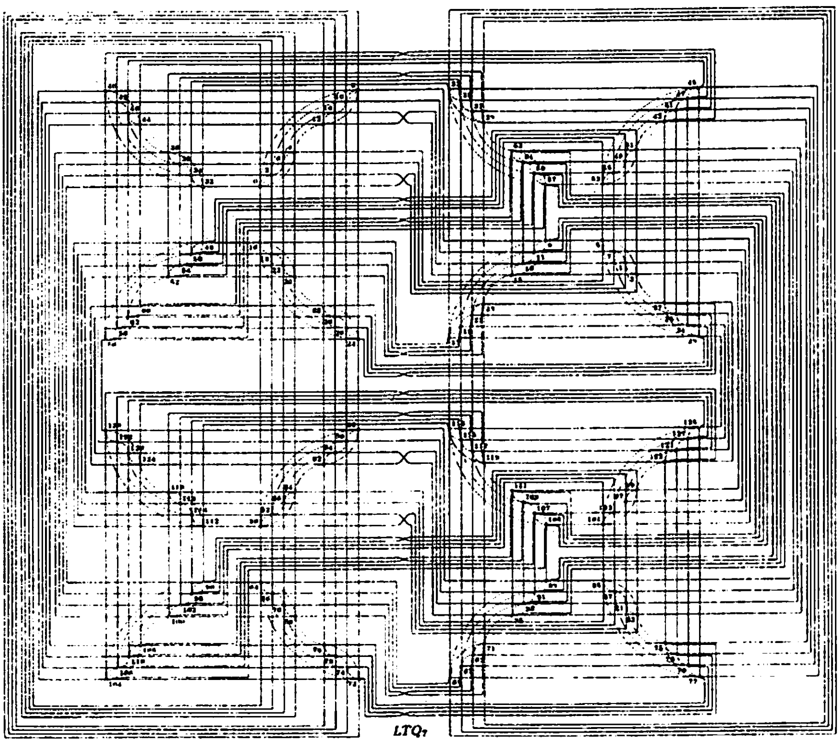


Figure 2.10: The drawing D_7

3 Lower bound for $cr(LTQ_n)$

We begin this section with the following observation.

Observation 3.1. *Let u be a vertex of LTQ_n. For any $i \in \{1, 2, \dots, n\}$, there exists exactly one vertex $u_i \in V(LTQ_n)$ such that u and u_i are adja-*

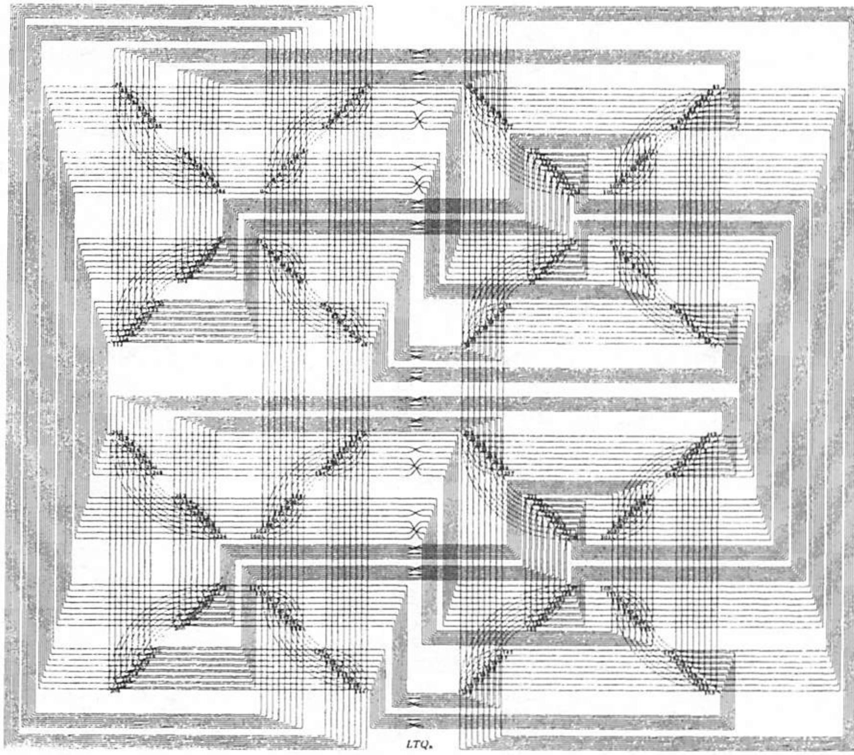


Figure 2.11: The drawing D_8

cent with $\lambda(u, u_i) = i$.

Let v be a vertex of LTQ_n. Let $\tau_v : V(LTQ_n) \setminus \{v\} \to V(LTQ_n)$ be a map defined as follows: for any vertex $u \in V(LTQ_n) \setminus \{v\}$, let $\tau_v(u)$ be the vertex of LTQ_n such that u and $\tau_v(u)$ are adjacent with $\lambda(u, \tau_v(u)) = \lambda(u, v)$.

It is easy to see that either $\tau_v(u) = v$ or $\lambda(u, v) + 1 \leq \lambda(\tau_v(u), v) \leq n$. Hence, we can define the following.

Definition 3.1. *For any two vertices* $u, v \in V(LTQ_n)$, *let* $\mathscr{P}_{u,v} = (u_0, u_1, \ldots, u_\ell)$ *be the unique path of* LTQ_n *such that* $u_0 = u$, $u_\ell = v$ *and* $\tau_v(u_i) = u_{i+1}$ *for any* $i \in \{0, 1, \ldots, \ell - 1\}$.

Note that

$$\lambda(u_0, v) < \lambda(u_1, v) < \cdots < \lambda(u_{\ell-1}, v). \tag{6}$$

For any two vertices $v, w \in V(LTQ_n)$ and integers $1 \le t_1 \le t_2 \le n$, let

$$D_v(t_1, t_2) = \{u \in V(LTQ_n) \setminus \{v\} : t_1 \le \lambda(u, v) \le t_2\},$$

and let

$$\mathcal{F}(v, w; t_1, t_2) = D_v(t_1, t_2) \cap \{u \in V(LTQ_n) \setminus \{v\} : w \text{ is in } \mathscr{P}_{u,v}\}.$$

Lemma 3.1. *Let v, w be two vertices of LTQ_n, where $d = \lambda(w, v)$. Let k be an integer such that $1 \le k \le d$. Then*

$$|\mathcal{F}(v, w; k, d)| = 2^{d-k}.$$

Proof. By induction on $d-k$. If $k = d$, it follows from (6) that $\mathcal{F}(v, w; d, d) = \{w\}$, done. Hence, we assume

$$k < d.$$

By (6), we have $\mathcal{F}(v, w; k, k) = \{u \in D_v(k, k) : \tau_v(u) \in \mathcal{F}(v, w; k + 1, d)\}$. Combining with Observation 3.1, we conclude that $|\mathcal{F}(v, w; k, k)| = |\mathcal{F}(v, w; k{+}1, d)|$. It follows from the induction hypothesis that $|\mathcal{F}(v, w; k, d)| = |\mathcal{F}(v, w; k, k)| + |\mathcal{F}(v, w; k + 1, d)| = 2 \times 2^{d-(k+1)} = 2^{d-k}$. The lemma follows. ❐

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An embedding of G_1 in G_2 is a couple of mapping (φ, κ) satisfying

$$\varphi : V_1 \to V_2$$

is an injection

$$\kappa : E_1 \to \{\text{set of all paths in } G_2\},$$

such that if $uv \in E_1$ then $\kappa(uv)$ is a path between $\varphi(u)$ and $\varphi(v)$. For any $e \in E_2$ define

$$cg_e(\varphi, \kappa) = |\{f \in E_1 : e \in \kappa(f)\}|$$

and

$$cg(\varphi, \kappa) = \max_{e \in E_2}\{cg_e(\varphi, \kappa)\}.$$

The value $cg(\varphi, \kappa)$ is called congestion.

Let $2K_m$ be the complete multigraph of m vertices, in which every two vertices are joined by two parallel edges.

Lemma 3.2. *[11] Let (φ, κ) be an embedding of G_1 in G_2 with congestion $cg(\varphi, \kappa)$. Let $\Delta(G_2)$ denote the maximal degree of G_2. Then*

$$cr(G_2) \geq \frac{cr(G_1)}{cg^2(\varphi, \kappa)} - \frac{|V_2|}{2}\Delta^2(G_2).$$

According to Erdős [7] and Kainen [9], the following lemmas are held.

Lemma 3.3. *[7]* $cr(K_{2^n}) \geq \frac{2^n(2^n-1)(2^n-2)(2^n-3)}{80}$.

Lemma 3.4. *[9]* $cr(2K_{2^n}) = 4cr(K_{2^n})$.

Now we are in a position to show the lower bound of $cr(LTQ_n)$.

Theorem 3.1. $cr(LTQ_n) > \frac{4^n}{20} - (n^2+1)2^{n-1}$.

Proof. By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we need only to construct an embedding (φ, κ) of $2K_{2^n}$ into LTQ_n with congestion $cg(\varphi, \kappa)$ at most 2^n. Let φ be an arbitrary bijection of $V(2K_{2^n})$ onto $V(LTQ_n)$. We define the mapping κ as follows. For any two vertices u and v of LTQ_n, take $\mathscr{P}_{u,v}$ and $\mathscr{P}_{v,u}$ to be the images (paths) of the two parallel edges between $\varphi^{-1}(u)$ and $\varphi^{-1}(v)$ under κ.

Let $e = xy$ be an arbitrary edge of LTQ_n, where $d = Dim(e)$. It suffices to show

$$cg_e(\varphi, \kappa) \leq 2^n.$$

Consider first the number of paths $\mathscr{P}_{u,v}$ traversing x previous y, denoted $p(x,y)$. Let $V_{x,y} = \{v \in V(LTQ_n) \setminus \{x\} : \tau_v(x) = y\}$. Note that

$$p(x,y) = \sum_{v \in V_{x,y}} |\{u \in V(LTQ_n) \setminus \{v\} : x \text{ is in } \mathscr{P}_{u,v}\}|. \tag{7}$$

We see that $v \in V_{x,y}$ if and only if,

$$\lambda(v, x) = d, \tag{8}$$

or equivalently,

$$\theta_i(v) = \theta_i(x) \text{ for all } i \in \{1, 2, \ldots, d-1\} \text{ and } \theta_d(v) = \overline{\theta_d(x)}.$$

This implies that

$$|V_{x,y}| = 2^{n-d}. \tag{9}$$

Combined with (6), (8) and Lemma 3.1, we have that for any $v \in V_{x,y}$,

$$|\{u \in V(LTQ_n) \setminus \{v\} : x \text{ is in } \mathscr{P}_{u,v}\}| = |\mathcal{F}(v, x; 1, d)| = 2^{d-1}. \tag{10}$$

By (7), (9) and (10), we have

$$p(x, y) = 2^{n-1}.$$

Similarly, the number $p(y, x)$ of paths $\mathscr{P}_{u,v}$ traversing y previous x is 2^{n-1}. Therefore,

$$cg_e(\varphi, \kappa) = p(x, y) + p(y, x) = 2^n.$$

This completes the proof of Theorem 3.1. □

4 Concluding remarks

We recall that computing the exact values of the crossing number of the hypercubes and its invariants is a NP problem as stated in the introductory section. In fact, still there exists a gap between the upper bound and the lower bound given in this paper. One may refine them by applying some other methods. The interested readers is referred to [18, 25, 28].

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