Vertices of Degree 9 in Contraction Critical 8-Connected Graphs *

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Abstract In this paper, we have proved that if a contraction critical 8-connected graph G has no vertices of degree 8, then for every vertex x of G, either x is adjacent to a vertex of degree 9, or there are at least 4 vertices of degree 9 such that every of them is at distance 2 from x.

Key words Fragment, contractible edge, contraction critical 8-connected graph
MSC 05C40

1 Introduction

In this paper, we only consider finite undirected simple graphs. For notations and terminologies undefined here we refer the reader to [5]. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $x \in V(G)$, we write $N_G(x)$ for the neighborhood of x in G. By $d_G(x)$, we denote the degree of x in G. For a subset $X \subseteq V(G)$, $N_G(X) = (\bigcup_{x \in X} N_G(x)) - X$ is the neighborhood of X in G, and $G[X]$ is the subgraph of G induced by X. Let G be a graph with connectivity k. For $T \subset V(G)$, if there are at least two components in $G - T$, then T is said to be a cut-set of G. If $|T| = k$, we call T a smallest cut-set or a

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k-cut-set of G. The set of all smallest cut-sets of G is denoted by $\mathcal{T}_G$. Let M be a nonempty subset of G and let $\overline{M} = V(G) - (M \cup N_G(M))$. The set M, or the subgraph induced by M, is called a fragment of G if $N_G(M)$ is a smallest cut-set of G and $\overline{M} \neq \emptyset$. Clearly, if M is a fragment of G, then $\overline{M}$ is also a fragment of G. An inclusion-minimal fragment is called an end. A fragment with minimum number of vertices is called an atom. Let T be a smallest cut-set of G and let M be the union of at least one but not all components of $G - T$. Clearly, M is a fragment of G, which is also called a T-fragment. For a subgraph G' of a graph G, when there is no ambiguity, we write simply G' for $V(G')$.

The distance between two vertices x and y in G is the length of a shortest path between x and y in G.

Let $k \geq 2$ be an integer. An edge e of a k-connected graph G is k-contractible or simply contractible if its contraction (i.e., deleting e, and identifying its two end vertices, finally, replacing each of the resulting pairs of double edges by a single edge) yields again a k-connected graph. An edge that is not k-contractible is said to be a non-contractible edge. Clearly, an edge of a non-complete k-connected graph is non-contractible if and only if its two end vertices are contained in some smallest cut-set. A k-connected non-complete graph G is contraction critical if G has no k-contractible edges. Egawa [6] proved the following Theorem A which is about the relation between contractible edges and the size of a fragment in a graph.

Theorem A [6] *Let $k \geq 2$ be an integer, and G be a k-connected graph not isomorphic to K_{k+1}. If G has no k-contractible edges, then G has a fragment of cardinality at most $\frac{k}{4}$.*

In the same paper, by Theorem A, Egawa [6] obtained the following Theorem B which gives us a minimum degree condition for a k-connected graph to have a k-contractible edge.

Theorem B [6] *Let $k \geq 2$ be an integer, and let G be a k-connected graph with minimum degree greater than or equal to $[\frac{5k}{4}]$. Then G has a k-contractible edge, unless $2 \leq k \leq 3$ and $G \cong K_{k+1}$.*

By Theorem A, we know that the connectivity of a contraction critical k-connected graph is at least 4.

For $k \in \{4, 5, 6, 7\}$, some results about the contraction critical k-connected graphs can be found in [1, 2, 3, 4, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17]. In this paper, we discuss the contraction critical 8-connected graphs. By Theorems A and B, the minimum degree of a contraction critical 8-connected graph is 8 or 9. Further, Egawa ([6]) gave examples to show that there are contraction critical 8-connected graphs with minimum degree 9. The following is the main result of the paper.

Theorem 1 *Let G be a contraction critical 8-connected graph which has no vertices of degree 8. Let x be an arbitrary vertex of G, then either x is adjacent to a vertex of degree 9, or there are at least 4 vertices of degree 9 such that every of them is at distance 2 from x.*

2 Some Definitions and Lemmas

Let G be a k-connected graph. Suppose that $\mathcal{S}$ is a nonempty set of some subsets of $V(G)$. A fragment M of G is said to be an $\mathcal{S}$-fragment if $N_G(M)$ contains an element of $\mathcal{S}$. Therefore, M is also said to be a fragment with respect to $\mathcal{S}$. If $E(G[N_G(M)])$ contains an edge e, then we call M a fragment with respect to e. An inclusion-minimal $\mathcal{S}$-fragment is called an $\mathcal{S}$-end. An $\mathcal{S}$-fragment with minimum number of vertices is called an $\mathcal{S}$-atom. Now we list some properties about the fragments.

Lemma 1 [12] *Let F, F' be two distinct fragments of G, $T = N_G(F)$ and $T' = N_G(F')$.*

(1) If $F \cap F' \neq \emptyset$, then $|F \cap T'| \geq |T \cap \overline{F'}|, |F' \cap T| \geq |\overline{F} \cap T'|$.

(2) If $F \cap F' \neq \emptyset \neq \overline{F} \cap \overline{F'}$, *then both* $F \cap F'$ *and* $\overline{F} \cap \overline{F'}$ *are fragments of* G, *and*
$N_G(F \cap F') = (T \cap T') \cup (F' \cap T) \cup (F \cap T'), N_G(\overline{F} \cap \overline{F'}) = (T \cap T') \cup (T \cap \overline{F'}) \cup (\overline{F} \cap T')$.
(3) If $F \cap F' \neq \emptyset$ *and* $F \cap F'$ *is not fragment, then* $\overline{F} \cap \overline{F'} = \emptyset$ *and* $|F \cap T'| > |T \cap \overline{F'}|, |F' \cap T| > |\overline{F} \cap T'|$.

For the convenience of reading Lemma 1, we give the following figure.

	F	T	$\overline{F}$
F'	$F \cap F'$	$F' \cap T$	$\overline{F} \cap F'$
T'	$F \cap T'$	$T \cap T'$	$\overline{F} \cap T'$
$\overline{F'}$	$F \cap \overline{F'}$	$T \cap \overline{F'}$	$\overline{F} \cap \overline{F'}$

Fig. 1.

Lemma 2 [12] *Let* G *be a contraction critical* k*-connected graph. Let* x *be a vertex of* G. *Suppose that* $S_x := \{\{x, y\} \mid y \in N_G(x)\}$, *and* A *is an* S_x*-atom. Assume that there is a smallest cut-set* T *of* G *such that* $T \cap (A \cup N_G(A))$ *contains an element of* S_x. *If* $A \cap T \neq \emptyset$, *then* $A \subseteq T$, *and* $|A| \leq \frac{k-1}{2}$.

A k-connected graph G is called almost critical if for every fragment M of G, there is a smallest cut-set T such that $M \cap T \neq \emptyset$.

Lemma 3 [12] *Every almost critical, non-complete graph* G *has fragments* F_1, F_2, F_3, F_4 *such that* F_1, F_2, F_3 *and* $F_4 \cap (\bigcup T_G)$ *are disjoint.*

Lemma 4 [8] *Let* G *be a contraction critical* k*-connected graph, and let* A *be an atom of* G, *or a set consisting of a single vertex of* G, *or a set of vertices with* $|N_G(A)| \geq k$ *such that there is a pair* $(a', t') \in A \times N_G(A)$ *such that* a, t *are adjacent if* $(a, t) \in A \times N_G(A) - \{(a', t')\}$. *Then* $G - A$

is almost critical of connectivity $(k - |A|)$, $N_G(A) \subseteq (\bigcup \mathcal{T}_{G-A})$, *and every* T*-fragment of* $G - A$ *is a* $(T \cup A)$*-fragment of* G.

Lemma 5 [1] *Let* $k \geq 2$ *be an integer and let* G *be a* k*-connected graph. Let* S *be a* k*-cut-set of* G *and let* A *be a fragment of* $G - S$. *Let* T *be a cut-set of* G *such that* $A \subset T$. *Let* B *be a nonempty union of components of* $G - T$ *such that* $G - (T \cup B) \neq \emptyset$. *If* $|A| > |B \cap S|$ *and* $|T| = k$, *then* $B = B \cap S$.

3 Proof of Theorem 1

In this section, we are going to prove Theorem 1 by contradiction. Let G be a contraction critical 8-connected graph without vertices of degree 8. Suppose that there is a vertex x of G such that it is not adjacent to any vertex of degree 9, and there are at most 3 vertices of degree 9 such that every of them is at distance 2 from x. Let $\mathcal{S}_x := \{\{x, y\} \mid y \in N_G(x)\}$, and A be an $\mathcal{S}_x$-atom. Suppose that $S = N_G(A)$. Then S is an 8-cut-set of G such that $\{x, y\} \subseteq S$ for some $y \in N_G(x)$. Since G has no vertices of degree 8, we have $|A| \geq 2$. If $|A| = 2$ and $A = \{a, b\}$, then $d_G(a) = d_G(b) = 9$ and $ax, bx \in E(G)$, a contradiction. Therefore, $|A| \geq 3$. Since $x \in N_G(A)$, we have $N_G(x) \cap A \neq \emptyset$. Let $z \in N_G(x) \cap A$. Clearly, xz is a non-contractible edge of G. Then there is an 8-cut-set S' of G such that $\{x, z\} \subseteq S'$. This implies that $S' \cap A \neq \emptyset$. Noting that $S' \cap (A \cup S)$ contains an element of $\mathcal{S}_x$, then $|A| \leq \frac{8-1}{2}$ by Lemma 2. Hence, $|A| = 3$. Assume that $A = \{a, b, c\}$. Since x is not adjacent to any vertices of degree 9 and $x \in N_G(A)$, we have that at least one of $d_G(a), d_G(b)$ and $d_G(c)$ is 10.

Now, we prove that exactly one of $d_G(a), d_G(b)$ and $d_G(c)$ is 10. Suppose that at least two of $d_G(a), d_G(b)$ and $d_G(c)$ is 10. Then $G - A$ is an almost critical 5-connected graph by Lemma 4. By Lemmas 3 and 4, there are 4 fragments F_1, F_2, F_3, F_4 in $G - A$ such that $F_1 \cap S, F_2 \cap S, F_3 \cap S$ and

$F_4 \cap S$ are pairwise disjoint, and every $F_i (i = 1, 2, 3, 4)$ is a T_i-fragment of G for some smallest cut-set T_i of G, where $T_i \supset A$. In this case, we claim that $F_i \cap S \neq \emptyset$ for $i = 1, 2, 3, 4$. If there is an F_i, saying F_1, such that $F_1 \cap S = \emptyset$, then $|F_1 \cap S| = 0 < |A \cap T_1| = |A|$. Since F_1 is a T_1-fragment of G, $A \subset T_1$ and $|T_1| = 8$, we have $F_1 = F_1 \cap S = \emptyset$ by Lemma 5, a contradiction. Hence, $F_i \cap S \neq \emptyset$ for $i = 1, 2, 3, 4$. Clearly, we have

$$4 \leq \sum_{i=1}^{4} |F_i \cap S| \leq |S| = 8.$$

Now, we claim that $|F_i \cap S| = 2$ for $i = 1, 2, 3, 4$. On the contrary, assume that there is an F_i, saying F_2, such that $|F_2 \cap S| \neq 2$. First suppose that $|F_2 \cap S| = 1$. Since $T_2 \supset A$, we have $1 = |F_2 \cap S| < |A \cap T_2| = |A| = 3$. Noting that F_2 is a T_2-fragment of G and $|T_2| = 8$, we have $F_2 = F_2 \cap S$ by Lemma 5. Thus $|F_2| = |F_2 \cap S| = 1$. So, the only vertex of F_2 must have degree 8, i.e., G has a vertex of degree 8, a contradiction. Next suppose that $|F_2 \cap S| \geq 3$. Then there is an $F_l (l \in \{1, 3, 4\})$ such that $|F_l \cap S| = 1$. By the same arguments as above, we also have that G has a vertex of degree 8, a contradiction. Therefore, $|F_i \cap S| = 2$ hold for $i = 1, 2, 3, 4$.

It is easy to see that $2 = |F_i \cap S| < |A| = |A \cap T_i| = 3 (i = 1, 2, 3, 4)$. Since $|T_i| = 8$, we have $F_i = F_i \cap S (i = 1, 2, 3, 4)$ by Lemma 5. Thus, we have $|F_i| = |F_i \cap S| = 2 (i = 1, 2, 3, 4)$. Hence, each F_i consists of exactly two adjacent vertices of degree 9, $i = 1, 2, 3, 4$. Since $|S| = 8$ and that $F_1 \cap S, F_2 \cap S, F_3 \cap S$ and $F_4 \cap S$ are pairwise disjoint, we have $\bigcup_{i=1}^{4} F_i = \bigcup_{i=1}^{4} F_i \cap S = S$. Therefore, every vertex of S is a vertex of degree 9. Hence, $d_G(x) = d_G(y) = 9$. By $y \in N_G(x)$, we have that x is adjacent to a vertex of degree 9, a contradiction. In addition, if every vertex of $S - \{x, y\}$ is not adjacent to x, then the 6 vertices of degree 9 in $S - \{x, y\}$ are at distance 2 from x, also a contradiction. Hence, exactly one of $d_G(a), d_G(b)$ and $d_G(c)$ is 10.

Therefore, in the work below, we always assume that two of $d_G(a)$, $d_G(b), d_G(c)$ are 9, the third one is 10. Without loss of generality, we may assume that $d_G(a) = d_G(c) = 9, d_G(b) = 10$. Then $xb \in E(G), ax, cx \notin E(G)$, i.e., $N_G(x) \cap A = \{b\}$. Notice that $S = N_G(A)$ and $\{x, y\} \subset S$ (i.e., $xy \in E(G[S])$). So $ac \in E(G)$. For otherwise, if $ac \notin E(G)$, then we can easily get that $S \subseteq N_G(a) \cap N_G(c)$. Thus $xa \in E(G), xc \in E(G)$ and $d_G(a) = d_G(b) = 9$, a contradiction. Therefore, $ab, ac, bc \in E(G)$, i.e., $abca$ is a triangle. So, we have that $S - \{x\} \subseteq N_G(a) \cap N_G(c), S \subseteq N_G(b)$. Hence, y, b are two adjacent vertices in the neighborhood of x. Clearly, y and b have two common neighbors a and c such that each of a and c has degree 9.

Since $xb \in E(G)$, xb is a non-contractible edge of G. Then there is an 8-cut-set T of G such that $\{x, b\} \subset T$. Suppose that B is a fragment of $G - T$, $\overline{B} = G - (T \cup B)$. Clearly, $\{x, b\} \in \mathcal{S}_x$. Therefore, each of B and $\overline{B}$ is an $\mathcal{S}_x$-fragment. It is obvious that $T \cap (A \cup S)$ contains an element of $\mathcal{S}_x$ and $A \cap T \neq \emptyset$. By Lemma 2, we have $A \subset T$.

Claim 1 $3 \le |B \cap S| \le 4, 3 \le |\overline{B} \cap S| \le 4$ *and* $1 \le |T \cap S| \le 2$.

Proof Assume that $|B \cap S| \le 2$. By $A \subset T$, we have $|A \cap T| = |A| = 3$. Thus $|B \cap S| \le 2 < 3 = |A \cap T| = |A|$. Noting that $|T| = 8$, we have $B = B \cap S$ by Lemma 5. Then $|B| = |B \cap S| \le 2$. By $B \neq \emptyset$, we have $B \cap S \neq \emptyset$. Since G has no vertices of degree 8, $|B| = |B \cap S| = 2$. Thus, $2 = |B| < |A| = 3$. This contradicts that A is an $\mathcal{S}_x$-atom because of that B is also an $\mathcal{S}_x$-fragment. Hence, $|B \cap S| \ge 3$. Similarly, we have $|\overline{B} \cap S| \ge 3$. Since $x \in S \cap T$, we have $|S \cap T| \ge 1$. If $|B \cap S| > 4$, then $|\overline{B} \cap S| \le 2$, a contradiction. Therefore, $3 \le |B \cap S| \le 4$. Similarly, we have $3 \le |\overline{B} \cap S| \le 4$. By the arguments as above, we have $1 \le |S \cap T| \le 2$. This completes the proof of Claim 1. □

By Claim 1, we see that at least one of $|B \cap S| = 3$ and $|\overline{B} \cap S| = 3$ holds. Therefore, in the work below, we always assume that $|B \cap S| = 3$. Let $V(B \cap S) = \{u, v, w\}$. Take a vertex of $B \cap S$ saying u (possibly $u = y$). It is obvious that $u \in S - \{x\} \subseteq N_G(a)$. Hence, $au \in E(G)$ and au is a non-contractible edge. Let F be a fragment with respect to au. Then $Z = N_G(F)$ is an 8-cut-set of G such that $\{a, u\} \subset Z$. Denote $\overline{F} = G - (Z \cup F)$.

Claim 2 $A \subset Z$.

Proof By contradiction. Suppose that $A \not\subset Z$. Thus, we have $A \cap F \neq \emptyset$ or $A \cap \overline{F} \neq \emptyset$. Since $abca$ is a triangle and $a \in A \cap Z$, we only need to consider that one of $A \cap F$ and $A \cap \overline{F}$ is nonempty. Without loss of generality, we assume that $A \cap F \neq \emptyset$ and $A \cap \overline{F} = \emptyset$. This implies that $\{b, c\} \subseteq A - \overline{F}$. If $b \in A \cap F$, then $S \subseteq S \cap (F \cup Z)$ by $S \subseteq N_G(b)$. Thus $\overline{F} \cap S = \emptyset$. Therefore, $|(\overline{F} \cap S) \cup (S \cap Z) \cup (\overline{A} \cap Z)| = |\overline{F} \cap S| + |(S \cap Z) \cup (\overline{A} \cap Z)| = |(S \cap Z) \cup (\overline{A} \cap Z)| = |Z| - |A \cap Z| \leq 7$(since $a \in A \cap Z$). Since G is 8-connected, we have $\overline{A} \cap \overline{F} = \emptyset$. Hence, $\overline{F} = \emptyset$, a contradiction. Therefore, $b \notin A \cap F$. So, $b \in A \cap Z$. Hence, by $A \cap F \neq \emptyset$, we have that $A \cap Z = \{a, b\}$ and $c \in A \cap F$. Thus $|A \cap Z| = 2$. By $S - \{x\} \subseteq N_G(c)$, we have $S - \{x\} \subseteq S \cap (F \cup Z)$. It follows that $|\overline{F} \cap S| \leq 1$. Therefore, $|(\overline{F} \cap S) \cup (Z \cap S) \cup (\overline{A} \cap Z)| = |\overline{F} \cap S| + |(Z \cap S) \cup (\overline{A} \cap Z)| = |\overline{F} \cap S| + |Z - (A \cap Z)| = |\overline{F} \cap S| + |Z| - |A \cap Z| \leq 1 + 8 - 2 = 7$. Since G is 8-connected, we have $\overline{A} \cap \overline{F} = \emptyset$. Hence, $\overline{F} = \overline{F} \cap S$ and $|\overline{F}| = |\overline{F} \cap S| \leq 1$. By $\overline{F} \neq \emptyset$, we have $|\overline{F}| = |\overline{F} \cap S| = 1$. This implies that $\overline{F} = \overline{F} \cap S = \{x\}$. Therefore, $d_G(x) = 8$, a contradiction. So $A \subset Z$. □

Claim 3 $x \notin Z$.

Proof By contradiction. Suppose that $x \in Z$. Hence, we have $x \in Z \cap S \cap T$ since $x \in T \cap S$. In addition, we have $\{b, x\} \subset Z$ since $b \in A$ and $A \subset Z$

(by Claim 2). Therefore, each of F and $\overline{F}$ is an S_x-fragment. By using the same way as in the proof of Claim 1, we easily have that $3 \leq |F \cap S| \leq 4$, $3 \leq |\overline{F} \cap S| \leq 4$ and $1 \leq |Z \cap S| \leq 2$. Since $\{u, x\} \subseteq Z \cap S$, $|Z \cap S| \geq 2$. Thus, we have that $|Z \cap S| = 2$ and $|F \cap S| = 3 = |\overline{F} \cap S|$.

By Claim 2, we have $A \subset Z$. Thus $A \subseteq Z \cap T$ since $A \subset T$. Therefore, $A \cup \{x\} \subseteq Z \cap T$ by $x \in Z \cap S \cap T$. Thus $|Z \cap T| \geq |A \cup \{x\}| = 4$ (since $x \notin A$).

Subclaim 3.1 *If $B \cap F \neq \emptyset$, then $|B \cap Z| > |\overline{F} \cap T|$, $|F \cap T| > |\overline{B} \cap Z|$ and $\overline{B} \cap \overline{F} = \emptyset$.*

Proof We first prove that $|B \cap Z| > |\overline{F} \cap T|$. If $B \cap F \neq \emptyset$, then we have $|B \cap Z| \geq |\overline{F} \cap T|$ by Lemma 1. Meanwhile, by $B \cap F \neq \emptyset$, we know that $(B \cap Z) \cup (Z \cap T) \cup (F \cap T)$ is a cut-set of G. Now, suppose that $|B \cap Z| = |\overline{F} \cap T|$. Then, we have $|(B \cap Z) \cup (Z \cap T) \cup (F \cap T)| = |B \cap Z| + |(Z \cap T) \cup (F \cap T)| = |\overline{F} \cap T| + |(Z \cap T) \cup (F \cap T| = |T| = 8$. Thus, by the definition of fragment, we easily have that $B \cap F$ is a fragment of G and $N_G(B \cap F) = (B \cap Z) \cup (Z \cap T) \cup (F \cap T)$ is an 8-cut-set of G. By $A \cup \{x\} \subseteq Z \cap T$, we have $\{x, b\} \subseteq Z \cap T$. So, $\{x, b\} \subset N_G(B \cap F)$. Thus $B \cap F$ is an S_x-fragment. In addition, we know that $u \notin F$ since $u \in Z$. Hence, $u \in B \cap S - F$. Noting that $|B \cap S| = 3$, we have $|(B \cap F) \cap S| = |B \cap (F \cap S)| = |B \cap (S \cap F)| = |(B \cap S) \cap F| \leq 2$. Since $A \subseteq Z \cap T$, we have $A \subset N_G(B \cap F)$. Noting that $|(B \cap F) \cap S| \leq 2 < |A| = 3$ and $|N_G(B \cap F)| = 8$, we have $B \cap F = (B \cap F) \cap S$ by Lemma 5. Thus, $|B \cap F| = |(B \cap F) \cap S| \leq 2 < 3 = |A|$. This contradicts that A is an S_x-atom. Therefore $|B \cap Z| > |\overline{F} \cap T|$. Similarly, we have $|F \cap T| > |\overline{B} \cap Z|$. By $|B \cap Z| > |\overline{F} \cap T|$, we have $|(\overline{F} \cap T) \cup (Z \cap T) \cup (\overline{B} \cap Z)| = |\overline{F} \cap T| + |(Z \cap T) \cup (\overline{B} \cap Z)| < |B \cap Z| + |(Z \cap T) \cup (\overline{B} \cap Z)| = |Z| = 8$. Since G is 8-connected, we have $\overline{B} \cap \overline{F} = \emptyset$. This completes the proof of Subclaim 3.1. □

Subclaim 3.2 *If $B \cap \overline{F} \neq \emptyset$, then $|\overline{F} \cap T| > |\overline{B} \cap Z|$, $|B \cap Z| > |F \cap T|$ and $\overline{B} \cap F = \emptyset$.*

Proof Notice that we also have $u \notin \overline{F}$ by $u \in Z$. Thus $u \in B \cap S - \overline{F}$. Since $|\overline{F} \cap S| = 3 = |B \cap S|$, we have $|(B \cap \overline{F}) \cap S| \leq 2$. Next, by using the same way as in the proof of Subclaim 3.1, we can prove Subclaim 3.2. □

Now, we continue the proof of Claim 3 by discussing the following two cases.

Case 1 $B \cap F \neq \emptyset$.

By $B \cap F \neq \emptyset$ and Subclaim 3.1, we have $|B \cap Z| > |\overline{F} \cap T|$ and $\overline{B} \cap \overline{F} = \emptyset$.

Subcase 1.1 $B \cap \overline{F} \neq \emptyset$.

By $B \cap \overline{F} \neq \emptyset$ and Subclaim 3.2, we have $|\overline{F} \cap T| > |\overline{B} \cap Z|$ and $\overline{B} \cap F = \emptyset$. Therefore, $\overline{B} \cap \overline{F} = \emptyset = \overline{B} \cap F$. So, $\overline{B} = \overline{B} \cap Z$. By Claim 1, we have $|\overline{B}| \geq |\overline{B} \cap S| \geq 3$. Therefore, $|\overline{B} \cap Z| = |\overline{B}| \geq 3$. Since $|Z \cap T| \geq 4$, we have $|B \cap Z| \leq 1$. By $u \in B \cap Z$, we have $|B \cap Z| = 1$. But by Subclaims 3.1 and 3.2, we can easily get that $1 = |B \cap Z| > |\overline{F} \cap T| > |\overline{B} \cap Z| \geq 3$, a contradiction. This contradiction shows that Subcase 1.1 does not occur.

Subcase 1.2 $B \cap \overline{F} = \emptyset$.

Since $B \cap \overline{F} = \emptyset$ and $\overline{B} \cap \overline{F} = \emptyset$, we have $\overline{F} = \overline{F} \cap T$. Since $|\overline{F}| \geq |\overline{F} \cap S| = 3$, we have that $|\overline{F}| = |\overline{F} \cap T| \geq 3$. By $|Z \cap T| \geq 4$, we have $3 \leq |\overline{F} \cap T| \leq 4$. Since $1 \leq |S \cap T| \leq 2$ and $|A| = 3$, we have $4 \leq |T \cap (A \cup S)| \leq 5$ and $3 \leq |T \cap \overline{A}| \leq 4$. By $A \subset T$, we have $T = (A \cap T) \cup (T \cap S) \cup (\overline{A} \cap T) = A \cup \{x\} \cup ((T \cap S) - \{x\}) \cup (\overline{A} \cap T)$. Notice that $(A \cup \{x\}) \cap \overline{F} = \emptyset$ (since $(A \cup \{x\}) \subseteq Z \cap T$). If $|\overline{F}| = |\overline{F} \cap T| = 3$, then $\overline{F}$ has at least two vertices t_1 and t_2 such that $\{t_1, t_2\} \subseteq \overline{A} \cap T$. Thus each of t_1 and t_2 is not adjacent to any vertex of A. By $A \subset Z = N_G(\overline{F})$, we have $d_G(t_1) \leq 7$ and $d_G(t_2) \leq 7$. This contradicts that G is an 8-connected graph. Hence, $|\overline{F}| = |\overline{F} \cap T| = 4$. Similarly, $\overline{F}$ has at least 3 vertices t'_1, t'_2 and t'_3 such that $\{t'_1, t'_2, t'_3\} \subseteq \overline{A} \cap T$. Thus each of t'_1, t'_2 and t'_3 is not adjacent to any vertex of A. By $A \subset Z = N_G(\overline{F})$, we have $d_G(t'_1) \leq 8, d_G(t'_2) \leq 8$ and $d_G(t'_3) \leq 8$. Since G is 8-connected, we have $d_G(t'_1) = d_G(t'_2) = d_G(t'_3) = 8$, a contradiction. This contradiction shows that Subcase 1.2 also does not occur and the proof of Case 1 is completed.

Case 2 $B \cap F = \emptyset$.

Subcase 2.1 $B \cap \overline{F} \neq \emptyset$.

Since $B \cap \overline{F} \neq \emptyset$, we have $\overline{B} \cap F = \emptyset$ by Subclaim 3.2. Thus $B \cap F = \emptyset = \overline{B} \cap F$. Therefore, $F = F \cap T$. By using the same way as in the proof of Subcase 1.2 of Case 1, we can prove that Subcase 2.1 does not occur.

Subcase 2.2 $B \cap \overline{F} = \emptyset$.

In this case, $B \cap F = \emptyset = B \cap \overline{F}$. Therefore $B = B \cap Z$. By Claim 1, we have $|B| \geq |B \cap S| = 3$. It follows that $|B \cap Z| = |B| \geq 3$. If $|F \cap T| \leq 2$, Then $|F \cap T| \leq 2 < |B \cap Z| = |B|$. Since $B \subset Z$ and $|Z| = 8$, we have $F = F \cap T$ by Lemma 5. Thus $|F| = |F \cap T| \leq 2$. This contradicts that $|F| \geq |F \cap S| = 3$. Hence $|F \cap T| \geq 3$. Similarly, we have $|\overline{F} \cap T| \geq 3$. Since $|Z \cap T| \geq 4$, we have, $8 = |T| = |F \cap T| + |Z \cap T| + |\overline{F} \cap T| \geq 3 + 4 + 3 = 10$, a contradiction. This completes the proof of Claim 3. □

By Claim 3, we have $x \notin Z$. Hence, $x \in F$ or $x \in \overline{F}$. Without loss of generality, we suppose that $x \in F$ (and so $x \in F \cap T$). Therefore, by symmetry, we may always suppose that $x \in F \cap T$ in the work below.

Claim 4 *If $B \cap F \neq \emptyset$, then $|F \cap T| > |\overline{B} \cap Z|$, $|B \cap Z| > |\overline{F} \cap T|$ and $\overline{B} \cap \overline{F} = \emptyset$.*

Proof If $B \cap F \neq \emptyset$, then $(B \cap Z) \cup (Z \cap T) \cup (F \cap T)$ is a cut-set of G. By

Lemma 1, we have $|F \cap T| \geq |\overline{B} \cap Z|$ and $|B \cap Z| \geq |\overline{F} \cap T|$. If $|F \cap T| = |\overline{B} \cap Z|$ or $|B \cap Z| = |\overline{F} \cap T|$, then we can easily get that $|(B \cap Z) \cup (Z \cap T) \cup (F \cap T)| = 8$. Therefore, by the definition of fragment, we easily have that $B \cap F$ is a fragment of G and $N_G(B \cap F) = (B \cap Z) \cup (Z \cap T) \cup (F \cap T)$ is an 8-cut-set of G. By $A \subset T$ and Claim 2, we have $A \subseteq Z \cap T$. Since $x \in F \cap T$, we have $\{x, b\} \subseteq T \cap (F \cup Z)$. Hence, $\{x, b\} \subset N_G(B \cap F)$.Therefore, $B \cap F$ is an S_x-fragment. Clearly, we also have $|(B \cap F) \cap S| \leq 2$. Now, by using the same way as in the proof of Subclaim 3.1 of Claim 3, we can prove Claim 4. □

Claim 5 *If $B \cap \overline{F} \neq \emptyset$, then $|B \cap Z| > |F \cap T|, |\overline{F} \cap T| > |\overline{B} \cap Z|$ and $\overline{B} \cap F = \emptyset$.*

Proof First prove that $|B \cap Z| > |F \cap T|$. If $B \cap \overline{F} \neq \emptyset$, then $(B \cap Z) \cup (Z \cap T) \cup (\overline{F} \cap T)$ is a cut-set of G. By Lemma 1, we have $|B \cap Z| \geq |F \cap T|$. Suppose that $|B \cap Z| = |F \cap T|$. Then $|(B \cap Z) \cup (Z \cap T) \cup (\overline{F} \cap T)| = |B \cap Z| + |(Z \cap T) \cup (\overline{F} \cap T)| = |F \cap T| + |(Z \cap T) \cup (\overline{F} \cap T)| = |T| = 8$. Thus, by the definition of fragment, we easily have that $B \cap \overline{F}$ is a fragment of G and $N_G(B \cap \overline{F}) = (B \cap Z) \cup (Z \cap T) \cup (\overline{F} \cap T)$ is an 8-cut-set of G. By $A \subset T$ and Claim 2, we have $A \subseteq Z \cap T$. Thus $A \subset N_G(B \cap \overline{F})$. Since $u \in B \cap S, u \notin \overline{F}$ and $|B \cap S| = 3$, we have $|(B \cap \overline{F}) \cap S| \leq 2$. Therefore, $|(B \cap \overline{F}) \cap S| \leq 2 < |A \cap N_G(B \cap \overline{F})| = |A| = 3$. Since $|N_G(B \cap \overline{F})| = 8$, we have $B \cap \overline{F} = (B \cap \overline{F}) \cap S$ by Lemma 5. Hence, $|B \cap \overline{F}| = |(B \cap \overline{F}) \cap S| \leq 2$. Since G has no vertices of degree 8, $|B \cap \overline{F}| = |(B \cap \overline{F}) \cap S| = 2$. In this situation, by $u \in B \cap S$ and $u \notin \overline{F}$, we may assume that $(B \cap \overline{F}) \cap S = \{v, w\}$, i.e., $B \cap \overline{F} = \{v, w\}$. Thus, $d_G(v) = d_G(w) = 9$ and $v, w \notin N_G(x)$. It is obvious that $N_G(B \cap \overline{F}) \subseteq N_G(v) \cap N_G(w)$. Since $A \subset N_G(B \cap \overline{F})$ and $b \in A$, we have $bv, bw \in E(G)$. On the other hand, we have that $ba, bc, ya, yc \in E(G)$ and $d_G(a) = d_G(c) = 9$. Obviously, a, c, v and w are 4 distinct vertices of degree 9 which are not adjacent to x. By $b, y \in N_G(x)$, we have that each of a, c, v and w is at distance 2 from x. Namely, there are at least 4 vertices of degree 9 such that every of them is at distance 2 from x, a contradiction. Therefore, we have $|B \cap Z| > |F \cap T|$. Similarly, we have $|\overline{F} \cap T| > |\overline{B} \cap Z|$. By using the same way as in the proof of Subclaim 3.1 of Claim 3, we have $\overline{B} \cap F = \emptyset$. This completes the proof of Claim 5. □

Now, we continue the proof of Theorem 1 by discussing the following 4 cases.

Case 1 $B \cap F = \emptyset = B \cap \overline{F}$.

By $B \cap F = \emptyset = B \cap \overline{F}$, we have $B \subset Z$. Thus $B = B \cap Z$. By Claim 1, we have $|B| \geq |B \cap S| = 3$. Then $|B \cap Z| = |B| \geq 3$. By $A \subset T$ and Claim 2, we have $A \subseteq T \cap Z$. Thus $|T \cap Z| \geq |A| = 3$ and $|\overline{B} \cap Z| \leq 2$. Since $x \in F \cap T$, we have $|F \cap T| \geq 3$. Otherwise, if $|F \cap T| \leq 2 < 3 \leq |B \cap Z| = |B|$, then $F = F \cap T$ and $|F| = |F \cap T| \leq 2$ by $|Z| = 8$ and Lemma 5. Since G has no vertices of degree 8 and $F \neq \emptyset$, $|F| = |F \cap T| = 2$. Therefore, two vertices of F are adjacent vertices of degree 9. Noting that $x \in F \cap T$, we find that x is adjacent to a vertex of degree 9, a contradiction. Thus, $|F \cap T| \geq 3$. So, $|\overline{B} \cap Z| \leq 2 < 3 \leq |F \cap T|$. Hence, $|(\overline{B} \cap Z) \cup (Z \cap T) \cup (\overline{F} \cap T)| = |\overline{B} \cap Z| + |(Z \cap T) \cup (\overline{F} \cap T)| < |F \cap T| + |(Z \cap T) \cup (\overline{F} \cap T)| = |T| = 8$. Thus $\overline{B} \cap \overline{F} = \emptyset$ by G being an 8-connected graph. Therefore, $\overline{F} = \overline{F} \cap T$. Since $|F \cap T| \geq 3$ and $A \subseteq Z \cap T$, we have $|\overline{F}| = |\overline{F} \cap T| \leq 2$. Since G contains no vertices of degree 8, $|\overline{F}| = |\overline{F} \cap T| = 2$. By $1 \leq |S \cap T| \leq 2$, we have $|T \cap (A \cup S)| \leq 5$ and $|T \cap \overline{A}| \geq 3$. By $A \subset T$, we have $T = (A \cap T) \cup (T \cap S) \cup (\overline{A} \cap T) = A \cup \{x\} \cup ((T \cap S) - \{x\}) \cup (\overline{A} \cap T)$. Noting that $A \subseteq T \cap Z$ and $x \in F \cap T$, we have $(A \cup \{x\}) \cap \overline{F} = \emptyset$. Thus $\overline{F}$ has at least one vertex c_1 such that $c_1 \in V(\overline{A} \cap T)$. Hence, c_1 is not adjacent to any vertex of $A \cap Z = A$. However, $A \subset Z \subsetneq N_G(\overline{F})$. Therefore, $d_G(c_1) \leq 6$, a contradiction. This contradiction shows that Case 1 does not occur.

Case 2 $B \cap F = \emptyset$ and $B \cap \overline{F} \neq \emptyset$.

By $B \cap \overline{F} \neq \emptyset$ and Claim 5, we have $|B \cap Z| > |F \cap T|$ and $\overline{B} \cap F = \emptyset$. Hence, $B \cap F = \emptyset = \overline{B} \cap F$. So we have that $F = F \cap T$. Since $x \in F \cap T$, we can easily get that $|F \cap T| = |F| \geq 3$. Thus, $|B \cap Z| > |F \cap T| \geq 3$. This implies that $|B \cap Z| \geq 4$. Because $A \subseteq T \cap Z$, we have $|T \cap Z| \geq |A| = 3$. Therefore, $|\overline{B} \cap Z| \leq 1 < |F \cap T| = |F|$. Since $F \subset T$ and $|T| = 8$, we have $\overline{B} = \overline{B} \cap Z$ by Lemma 5. So $|\overline{B}| = |\overline{B} \cap Z| \leq 1$. This contradicts that $|\overline{B}| \geq |\overline{B} \cap S| \geq 3$ (by Claim 1). This shows that Case 2 does not occur.

Case 3 $B \cap F \neq \emptyset$ and $B \cap \overline{F} = \emptyset$.

Since $B \cap F \neq \emptyset$, we have $\overline{B} \cap \overline{F} = \emptyset$ by Claim 4. Thus, $B \cap \overline{F} = \emptyset = \overline{B} \cap \overline{F}$. Hence, we have $\overline{F} = \overline{F} \cap T$. Since G has no vertices of degree 8, $|\overline{F}| = |\overline{F} \cap T| \geq 2$. By $x \in F \cap T$ and $A \subseteq Z \cap T$, we have $|\overline{F}| = |\overline{F} \cap T| \leq 4$. Thus $2 \leq |\overline{F}| \leq 4$. Similar to Case 1, by $A \subseteq Z \cap T$, and $x \in F \cap T$, we also have that $(A \cup \{x\}) \cap \overline{F} = \emptyset$. If $|\overline{F}| = |\overline{F} \cap T| = 2$, then $\overline{F}$ has at least one vertex c_1' such that $c_1' \in V(\overline{A} \cap T)$. Therefore, c_1' is not adjacent to any vertex of A. However, $A \subset Z = N_G(\overline{F})$. Thus $d_G(c_1') \leq 6$. This contradicts that G is an 8-connected graph. If $|\overline{F}| = |\overline{F} \cap T| = 3$, then by the same arguments as above, $\overline{F}$ has at least two vertices c_1'' and c_2'' such that $\{c_1'', c_2''\} \subseteq V(\overline{A} \cap T)$. Then each of c_1'' and c_2'' is not adjacent to any vertices of A. By $A \subset Z = N_G(\overline{F})$, we have $d_G(c_1'') \leq 7$ and $d_G(c_2'') \leq 7$, also a contradiction. Thus, $|\overline{F}| = |\overline{F} \cap T| = 4$. Again by the same arguments as above, $\overline{F}$ contains at least 3 vertices w_1, w_2 and w_3 such that $\{w_1, w_2, w_3\} \subseteq V(\overline{A} \cap T)$. Then each of w_1, w_2 and w_3 is not adjacent to any vertices of A. By $A \subset Z = N_G(\overline{F})$, we have $d_G(w_1) \leq 8, d_G(w_2) \leq 8$ and $d_G(w_3) \leq 8$. Since G is an 8-connected graph, $d_G(w_1) = d_G(w_2) =$

$d_G(w_3) = 8$. This contradicts the hypothesis that G has no vertices of degree 8. This shows that Case 3 does not occur.

Case 4 $B \cap F \neq \emptyset \neq B \cap \overline{F}$.

Since $B \cap F \neq \emptyset \neq B \cap \overline{F}$, we have $|F \cap T| > |\overline{B} \cap Z|$, $|\overline{F} \cap T| > |\overline{B} \cap Z|$ and $\overline{B} \cap \overline{F} = \emptyset = F \cap \overline{B}$ by Claims 4 and 5. Therefore $\overline{B} = \overline{B} \cap Z$. By Claim 1, we have $|\overline{B}| \geq |\overline{B} \cap S| \geq 3$. Thus $|\overline{B} \cap Z| = |\overline{B}| \geq 3$. Hence, we have $|F \cap T| > |\overline{B} \cap Z| \geq 3$ and $|\overline{F} \cap T| > |\overline{B} \cap Z| \geq 3$. This implies that $|F \cap T| \geq 4$ and $|\overline{F} \cap T| \geq 4$. By $A \subseteq Z \cap T$, we have $|Z \cap T| \geq |A| = 3$. Therefore, $8 = |T| = |F \cap T| + |Z \cap T| + |\overline{F} \cap T| \geq 4 + 3 + 4 = 11$, a contradiction. This completes the proof of Theorem 1. □

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