The number of boundary H-points of H-triangles *

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Abstract

An H-triangle is a triangle with corners in the set of vertices of a tiling of $\mathbb{R}^2$ by regular hexagons of unit edge. Let $b(\Delta)$ be the number of the boundary H-points of an H-triangle Δ. In [3] we made a conjecture that for any H-triangle with k interior H-points, we have $b(\Delta) \in \{3, 4, ..., 3k+4, 3k+5, 3k+7\}$. In this note, we prove the conjecture is true for $k = 4$, but not true for $k = 5$ because $b(\Delta)$ can not equal 15.

1 Introduction and Notation

We start with some basic definitions as in [1] [2]. Let H be the set of vertices of a tiling of $\mathbb{R}^2$ by regular hexagons of unit edge. A point of H is called an H-point. H can be considered as the union of two disjoint triangular lattices denoted by H^+, H^-, such that for any two points in H^+ (H^-) there exists a translation of the plane which maps one of the two points to the other and H to H. A point of H^+ (H^-) is called an H^+-point (H^--point). Two points x and y are said to be equivalent if $x, y \in H^+$ or $x, y \in H^-$. Otherwise we say that x, y are non-equivalent.

Let A denote the set of all centers of the hexagonal tiles which determines H. A point of A is called an A-point. Clearly, $H^+ \cup H^- \cup A$ forms a triangular lattice with the area of each triangular tile $\frac{\sqrt{3}}{4}$. We will denote this triangular lattice by $T = H^+ \cup H^- \cup A$, and a point of T is called a T-point. A segment with endpoints in T (H, A) is called a T-segment (an H-segment, an A-segment). A simple polygon in $\mathbb{R}^2$ whose corners lie in H (T, A) is called an H-polygon

*This research was supported by National Natural Science Foundation of China (11471095), Natural Science Foundation of Hebei Province (A2014208095).

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(a T-polygon, an A-polygon). Clearly an H-polygon or an A-polygon is also a T-polygon. For the related researches see [4], [5], [6].

For a planar H-polygon P, let ∂P, $intP$ be the boundary, interior of P respectively, and denote $b(P) = |H \cap \partial P|$ and $i(P) = |H \cap intP|$. It is known that for an H-triangle Δ with exactly one interior H-point, $b(\Delta) \in \{3,4,5,6,7,8,10\}$ (see [1]), for an H-triangle Δ with exactly three interior H-points, $b(\Delta) \in \{3,4,5,6,7,8,9,10,11,12,13,14,16\}$ (see [2]), and for any H-triangle with exactly k interior H-points can have at most $3k+7$ boundary H-points and can not have $3k+6$ boundary H-points (see [3]). Moreover, in [3] we gave the following Conjecture.

Conjecture 1. [3] If Δ is an H-triangle with k interior H-points, then $b(\Delta) \in \{3, 4, ..., 3k+4, 3k+5, 3k+7\}$.

In this note, we prove the Conjecture is true for $k = 4$, but not true for $k = 5$ because $b(\Delta)$ can not equal 15.

2 Basic facts and related lemmas

As in [1][3] we will use the notion of *level* of a T-triangle $\Delta = \Delta xyz$, here the corners x, y, z of Δ are labeled in such a way that yz has the largest number of points form $T = H \cup A$. The line containing yz is denoted by l_0. From l_0 to x draw all lines $l_1, l_2, ..., l_s$ passing through T-points that are parallel to l_0 and intersect Δ. Clearly $x \in l_s$, and the distance between l_j and l_{j+1} is the same for every j. We say that Δ has s levels $l_1, l_2, ..., l_s$. It is obvious that every H-triangle with one interior H-point has at least two levels.

Let m_j be the relative length of $\Delta \cap l_j$, that is , the length with the unit length being the distance between two consecutive T-points on l_j. Denote $t_j = |l_j \cap int\Delta \cap T|$. Obviously for $j > 0$ we have $\lfloor m_j \rfloor - 1 \le t_j \le \lfloor m_j \rfloor + 1$, where $\lfloor . \rfloor$ denotes the greatest integer function. Notice that if $t_j = \lfloor m_j \rfloor - 1$, then m_j is an integer, and $m_j = t_j + 1$.

Lemma 2. *[1] If an H-triangle Δ has s levels, then for $0 \le j < s$ we have $m_j = m_0(1 - \frac{j}{s})$.*

Lemma 3. *[3] There exists no H-triangle Δ with k interior H-points and $2k+2$ interior A-points.*

As in [1][3] let $\Delta = \Delta xyz$ be an H-triangle with k interior H-points, and consider the number of T-points on each side of the H-triangle Δ. Denote by α, β, γ the relative lengths of the sides of Δ. We may assume that $\alpha \ge \beta \ge \gamma$.

Let f be a linear transformation, which maps T into $\mathbb{Z}^2$ and Δ into a lattice triangle Δ'. Notice that the number of T-points on the sides of Δ and the number of lattice points on the corresponding sides of Δ' are the same. Denote by b', i' the number of boundary, interior lattice points of Δ', respectively. Then $b' = \alpha + \beta + \gamma \geq b(\Delta)$, clearly $\alpha \geq \frac{b'}{3}$. And from [3] we know that $b' + 2i_A + 2(k-1)$ is divisible by α, β, γ, $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$, where i_A denotes the number of interior A-points in H-triangle $\Delta = \Delta xyz$, then $i_A \in \{0, 1, ..., 2k+1\}$ by Lemma 7.

Lemma 4. *[3] There exists no H-triangle with k interior H-points and triple (α, β, γ), where $\alpha > 3(k+1)$.*

3 H-triangle with 4 interior H-points

The Conjecture 1 is valid for $k \leq 3$. In the following theorem we find the Conjecture is also true for $k = 4$.

Theorem 5. *If Δ is an H-triangle with 4 interior H-points, then* $b(\Delta) \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19\}$.

Proof. From Theorem 10 and Theorem 11 of [3] we know that $b(\Delta) \leq 3k + 7$ and $b(\Delta) \neq 3k + 6$ for an H-triangle with k interior H-points. Hence if Δ is an H-triangle with 4 interior H-points, then $b(\Delta) \leq 19$ and $b(\Delta) \neq 18$. So we only need to consider the cases $3 \leq b(\Delta) \leq 17$ and $b(\Delta) = 19$. Figure 1-3 provide H-triangles with 4 interior points and $3 \leq b(\Delta) \leq 17$, and 19 boundary H-points. The proof is complete. □

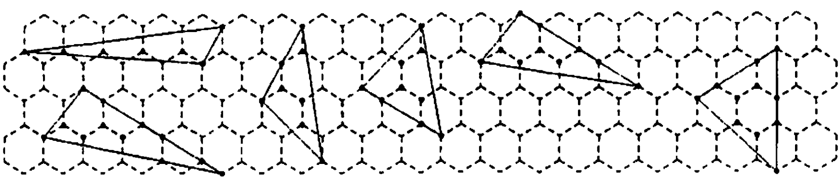

Figure 1: H-triangles Δ with $i(\Delta)$=4 and $b(\Delta) = 3, 4, 5, 6, 7, 8$.

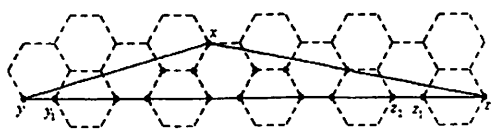

Figure 2: H-triangles Δ with $i(\Delta)$=4, $b(\Delta xyz) = 13$, $b(\Delta xy_1z) = 12$, $b(\Delta xyz_1) = 11$, $b(\Delta xy_1z_1) = 10$, $b(\Delta xy_1z_2) = 9$.

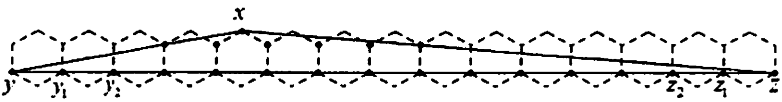

Figure 3: H-triangles Δ with $i(\Delta)$=4, $b(\Delta xyz) = 19$, $b(\Delta xy_1z) = 17$, $b(\Delta xy_2z) = 16$, $b(\Delta xy_1z_1) = 15$, $b(\Delta xy_1z_2) = 14$, $b(\Delta xy_2z_2) = 13$.

4 H-triangle with 5 interior H-points

Theorem 6. *There exists no H-triangle Δ with 5 interior H-points and 15 boundary H-points.*

Proof. Let (α, β, γ) be the triple of an H-triangle Δ with 5 interior H-points. Recalled that $b' = \alpha+\beta+\gamma \geq b(\Delta)$. To prove the theorem, we need to examine every triple (α, β, γ) for which $\alpha + \beta + \gamma \geq 15$. From [3] we know that when $\alpha + \beta + \gamma \geq 3k + 8$ there exists only one realizable triple $(3k + 3, 3, 3)$ which only for $b(\Delta) = 3k + 7$. Therefore we only need to examine triples (α, β, γ) for $15 \leq \alpha + \beta + \gamma \leq 22$. We give the detailed proof for $\alpha + \beta + \gamma = 22$, 21. The proofs for other cases see [7].

For every case, by lemma 4, we only need to consider the case $\alpha \leq 18$. We know that $\alpha\beta$ divides $b' + 2i_A + 2(k-1)$, where $i_A \in \{0, 1, ..., 11\}$ by lemma 3, and $b' + 2i_A + 2(k-1) \in \{b' + 8, b' + 10, ..., b' + 30\}$. Since $\alpha\beta$ divides $b' + 2i_A + 2(k-1) \leq b' + 30$, and hence $\alpha\beta \leq b' + 30$.

Moreover, in the same column (row) of every Table, $\alpha\beta$ increases as β increases.

Table 1: All possible solutions for $\alpha + \beta + \gamma = 22$

(18,3,1)	(18,2,2)					
(17,4,1)	(17,3,2)					
(16,5,1)	(16,4,2)	(16,3,3)				
(15,6,1)	(15,5,2)	(15,4,3)				
(14,7,1)	(14,6,2)	(14,5,3)	(14,4,4)			
(13,8,1)	(13,7,2)	(13,6,3)	(13,5,4)			
(12,9,1)	(12,8,2)	(12,7,3)	(12,6,4)	(12,5,5)		
(11,10,1)	(11,9,2)	(11,8,3)	(11,7,4)	(11,6,5)		
	(10,10,2)	(10,9,3)	(10,8,4)	(10,7,5)	(10,6,6)	
			(9,9,4)	(9,8,5)	(9,7,6)	
					(8,8,6)	(8,7,7)

Case 1: $b' = \alpha + \beta + \gamma = 22$. Table 1 provides all possible solutions for $\alpha + \beta + \gamma = 22$.

It is easy to check that there are only the following triple (α, β, γ) for which $\alpha\beta$ can divide $b' + 2i_A + 2(k-1)$ and $\alpha\beta \leq 52$: $(18, 2, 2)$.

If l_0 contains only H-points, then $b(\Delta) \geq 19 + 1 = 20$. So we can assume that l_0 contains both H-points and A-points. If Δ has two levels, then by lemma 2, $m_1 = 9$. So either $|l_1 \cap H \cap \partial\Delta| = 2$ or $|l_1 \cap A \cap \partial\Delta| = 2$, that is to say $b(\Delta) = 13 + 2 + 1 = 16$ or $b(\Delta) = 13 + 1 = 14$. If Δ has at least three levels, then by lemma 2, $m_1 \geq 12$. Then by inequality $t_j \geq \lfloor m_j \rfloor - 1$, we know that $t_1 \geq 11$, so $|l_1 \cap int\Delta \cap H| \geq 7$, a contradiction.

Table 2: All possible solutions for $\alpha + \beta + \gamma = 21$

(18,2,1)						
(17,3,1)	(17,2,2)					
(16,4,1)	(16,3,2)					
(15,5,1)	(15,4,2)	(15,3,3)				
(14,6,1)	(14,5,2)	(14,4,3)				
(13,7,1)	(13,6,2)	(13,5,3)	(13,4,4)			
(12,8,1)	(12,7,2)	(12,6,3)	(12,5,4)			
(11,9,1)	(11,8,2)	(11,7,3)	(11,6,4)	(11,5,5)		
(10,10,1)	(10,9,2)	(10,8,3)	(10,7,4)	(10,6,5)		
		(9,9,3)	(9,8,4)	(9,7,5)	(9,6,6)	
				(8,8,5)	(8,7,6)	
						(7,7,7)

Case 2: $b' = \alpha + \beta + \gamma = 21$. Table 2 provides all possible solutions for $\alpha + \beta + \gamma = 21$.

It is easy to check that there are only the following triples (α, β, γ) for which $\alpha\beta$ can divide $b' + 2i_A + 2(k-1)$ and $\alpha\beta \leq 51$: $(17, 3, 1)$, $(15, 3, 3)$, $(7, 7, 7)$.

Case 2.1: $(17, 3, 1)$ (for triple $(15, 3, 3)$, the discussion is similar).

Clearly Δ with such a triple has at least 3 levels, then by lemma 2, $m_1 \geq m_0(1 - \frac{1}{3}) = \frac{34}{3}$, $m_2 \geq m_0(1 - \frac{2}{3}) = \frac{17}{3}$. Then by inequality $t_j \geq \lfloor m_j \rfloor - 1$, we know that $t_1 \geq 11$ and $t_2 \geq 5$. If l_0 contains both H-points and A-points, $|l_1 \cap int\Delta \cap H| \geq 7$, a contradiction. If l_0 contains only H-points, then $b(\Delta) \geq 19 \neq 15$.

Case 2.2: $(7, 7, 7)$

Clearly Δ with such a triple has at least 7 levels, then by lemma 2, $m_1 \geq 6$, $m_2 \geq 5$, $m_3 \geq 4$, $m_4 \geq 3$, $m_5 \geq 2$, $m_6 \geq 1$. Then by inequality $t_j \geq \lfloor m_j \rfloor - 1$, we know that $t_1 \geq 5$, $t_2 \geq 4$, $t_3 \geq 3$, $t_4 \geq 2$ and $t_5 \geq 1$. If l_0 contains both H-points and A-points, $|l_1 \cap int\Delta \cap H| \geq 3$, $|l_2 \cap int\Delta \cap H| \geq 2$, $|l_3 \cap int\Delta \cap H| \geq 2$, $|l_4 \cap int\Delta \cap H| \geq 1$, then there are at least 8 H-points in $int\Delta$, a contradiction. If l_0 contains only H^--points (H^+-points), then on l_1 and l_4 there are only A-points, on l_2 and l_5 only H^+-points (H^--points) and on l_3 only H^--points (H^+-points), since otherwise there are at least 10 H-points in $int\Delta$, a contradiction. But now $t_2 + t_3 + t_5 \geq 8$, a contradiction. □

Theorem 7. *If Δ is an H-triangle with* 5 *interior H-points, then* $b(\Delta) \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 22\}$.

Proof. From Theorem 10 and Theorem 11 of [3] we know that $b(\Delta) \leq 3k + 7$ and $b(\Delta) \neq 3k + 6$ for an H-triangle with k interior H-points. Hence if Δ is an

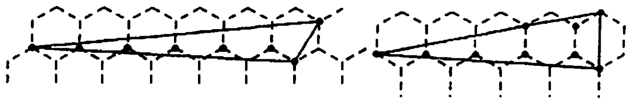

Figure 4: H-triangles Δ with $i(\Delta) = 5$ and $b(\Delta) = 3, 4$.

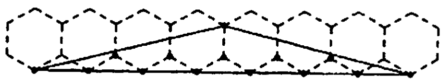

Figure 5: H-triangles Δ with $i(\Delta) = 5$ and $b(\Delta) = 9$.

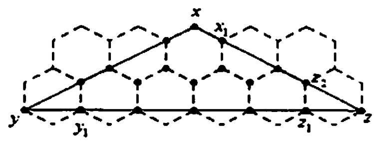

Figure 6: H-triangles Δ with $i(\Delta)$=5, $b(\Delta xyz) = 14$, $b(\Delta xy_1z) = 10$, $b(\Delta xyz_2) = 8$, $b(\Delta x_1yz_1) = 7$, $b(\Delta xy_1z_1) = 6$, $b(\Delta xy_1z_2) = 5$.

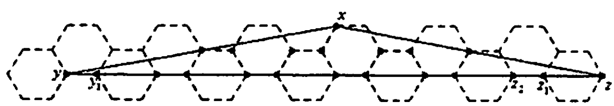

Figure 7: H-triangles Δ with $i(\Delta)$=5, $b(\Delta xyz) = 16$, $b(\Delta xy_1z) = 14$, $b(\Delta xyz_2) = 13$, $b(\Delta xy_1z_1) = 12$, $b(\Delta xy_1z_2) = 11$.

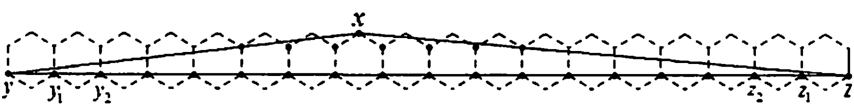

Figure 8: H-triangles Δ with $i(\Delta)$=5, $b(\Delta xyz) = 22$, $b(\Delta xy_1z) = 20$, $b(\Delta xy_2z) = 19$, $b(\Delta xy_1z_1) = 18$, $b(\Delta xy_1z_2) = 17$, $b(\Delta xy_2z_2) = 16$.

H-triangle with 5 interior H-points, then $b(\triangle) \leq 22$ and $b(\triangle) \neq 21$. So we only need to consider the cases $3 \leq b(\triangle) \leq 20$ and $b(\triangle) = 22$. Figures 4-8 provide H-triangles with 5 interior H-points and $3 \leq b(\triangle) \leq 14$, $16 \leq b(\triangle) \leq 20$, and 22 boundary H-points. Moreover, from Theorem 6 we know that $b(\triangle) \neq 15$.

The proof is complete. □

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