Proof of a conjecture on intersection graph of finite abelian groups

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Abstract

In this paper, we characterize all finite abelian groups with isomorphic intersection graphs. This solves a conjecture proposed by B. Zelinka.

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1 Introduction

Let G be a group. In [2], B. Csákány and G. Pollák defined the intersection graphs $\Gamma(G)$ of G, whose vertices are the proper non-trivial subgroups of G, and two vertices H_1 and H_2 are adjacent if and only if $H_1 \neq H_2$ and they have a non-trivial intersection. This work was inspired by the study of intersection graphs of nontrivial proper subsemigroups of semigroups due to J. Bosák [1]. In [3], B. Zelinka continued the work on intersection graphs of finite abelian groups and proposed the following conjecture.

Conjecture 1. *Two finite abelian groups with isomorphic intersection graphs are isomorphic.*

In this paper, each group G is a finite abelian group written additively with identity 0, and each subgroup H of G is assumed to be nontrivial and proper. The order of G is the number of elements in G and is denoted by $o(G)$. The order $o(a)$ of an element $a \in G$ is the smallest positive integer k such that $ka = 0$, and the exponent $e(G)$ of G is $\max_{a\in G}\{o(a)\}$. Let C_n denote the cyclic group of order n. A primary cyclic group is a cyclic group whose order is a power of a prime. Let G^* be the set of non-identity elements of G. Let K_n denote that complete graph of order n. Let Γ be a graph and x be a vertex of Γ, $N(x)$ is the set of vertices those are adjacent with x.

Definition 2. *Let $n = \prod_{i=1}^{r} p_i^{e_i}$ be the prime factorizations of an integer $n > 1$. We define the power set of n to be the multiset $\{e_1, e_2, \dots, e_r\}$.*

It is not hard to show that $\Gamma(C_n) \simeq \Gamma(C_m)$ if n and m have the same power set. In fact, We completely characterize finite abelian groups with isomorphic intersection graphs as follows.

Theorem 3. *Let $G_i = C_{n_i} \oplus M_i$, where C_{n_i} is the direct sum of all cyclic Sylow subgroups of G_i, and M_i is the direct sum of all non-cyclic Sylow subgroups of G_i, $i = 1, 2$. Then $\Gamma(G_1) \simeq \Gamma(G_2)$ if and only if n_1 and n_2 have the same power set and $M_1 \simeq M_2$.*

2 The case in p-groups

Recall that a subset X of the vertices of Γ is called an independent set of Γ if u and v are not adjacent for any $u, v \in X$. We need the following result.

Lemma 4. *([3]) A proper subgroup of G belongs to some independent set of $\Gamma(G)$ of maximal cardinality if and only if it is a primary cyclic group.*

Remark 5. *Let $\Gamma_P(G)$ be the subgraph of $\Gamma(G)$ induced by the vertex set consisting of all primary cyclic groups of G. Then $\Gamma_P(G)$ is a union of complete graphs. This subgraph plays an important role in the study of $\Gamma(G)$.*

Lemma 6. *Let p be a prime and G be a non-cyclic group with exponent p^e. Then*

$$\Gamma_P(G) \simeq \cup_{j=0}^{e-1} c_j K_{\sum_{i=0}^{j} \frac{m_i}{p^i}},$$

where $m_i = \sharp\{x \in G \mid p^i x = 0\}$ and $c_j = \frac{1}{p-1}(\frac{m_{j+1}}{m_j} - \frac{m_{j+2}}{m_{j+1}})$.

Proof. We define $\mathcal{F}^i(x) := \{y \in G \mid p^i y = x\}$ and $\mathcal{F}(x) := \cup_{i=0}^{\infty} \mathcal{F}^i(x)$ for any $x \in G$ and $i \geq 0$. Note that $|\mathcal{F}^i(x)| = m_i$ or 0. Let $A_i = \{x \in G \mid o(x) = p^i\}$ for $i \geq 0$. We decompose A_1 as follows: $A_1 = \cup_{i=0}^{e-1} B_i$, where $B_i = \{x \in A_1 \mid \mathcal{F}^i(x) \neq \emptyset, \mathcal{F}^{i+1}(x) = \emptyset\}$. Observe that $|A_{i+1}| = m_{i+1} - m_i$ and $A_{i+1} = \cup_{x \in A_1} \mathcal{F}^i(x)$, which is a disjoint union. Since for $x \in A_1$, $\mathcal{F}^i(x) \neq \emptyset$ if and only if $x \in B_j$ for some $j \geq i$, therefore,

$$m_{i+1} - m_i = (|A_1| - \sum_{j=0}^{i-1} |B_j|) m_i,$$

for $1 \leq i \leq e$. We obtain $|B_j| = \frac{m_{j+1}}{m_j} - \frac{m_{j+2}}{m_{j+1}}$.

Clearly, there are $n = \frac{m_1 - 1}{p-1}$ distinct cyclic subgroups of G of order p, namely, $H_1, H_2, \ldots, H_n$. It deduce that there are n connected components of $\Gamma(G)$, each one containing exactly one H_i. Let $H = \{0, a_1, a_2, \ldots, a_{p-1}\}$ be a subgroup of order p of G such that $H^* \subseteq B_j$. For any primary cyclic subgroup F of G, F and H are adjacent if and only if $F^* \subseteq \cup_{1 \leq i \leq p-1} \mathcal{F}(a_i)$. Suppose that there are s_i distinct primary cyclic subgroups of G of order p^i which are adjacent with H. Then there are exactly $s_i(p^i - p^{i-1})$ elements of order p^i contained in $\cup_{1 \leq i \leq p-1} \mathcal{F}(a_i)$. Hence, $s_i(p^i - p^{i-1}) = \sum_{j=1}^{p-1} |\mathcal{F}^{i-1}(a_j)| = (p-1)m_{i-1}$, $s_i = \frac{m_{i-1}}{p^{i-1}}$. So H is contained in a connected component of $\Gamma_P(G)$ whose size is $\sum_{i=1}^{j+1} s_i$. This completes the proof. □

Definition 7. *We define an equivalent relation $\sim$ on the vertex set of $\Gamma(G)$ by the rule that $H_1 \sim H_2$ if and only if $\{H_1\} \cup N(H_1) = \{H_2\} \cup N(H_2)$. Let $[H]$ be the equivalent class containing H.*

Lemma 8. *Let H_1, H_2 be two primary cyclic subgroups of G. Then $H_1 \sim H_2$ in $\Gamma(G)$ if and only if $H_1 \cap H_2$ is non-trivial.*

Proof. It follows immediately from Definition 7. □

Theorem 9. *Let p_1, p_2 be two primes and G_i be a p_i-group. Then $\Gamma(G_1) \simeq \Gamma(G_2)$ if and only if at least one of the following conditions holds.*

(i) $G_1 \simeq G_2$;

(ii) $G_1 \simeq C_{p_1^e}$ and $G_2 \simeq C_{p_2^e}$ for a positive integer e.

Proof. We only need to prove the necessity. If G_1 is a cyclic group, then $\Gamma(G_1)$ is a complete graph. So G_2 contains exactly one subgroup of order p_2 and G_2 is also a cyclic group. Condition (ii) is satisfied by comparing the number of vertices. Assume that neither G_1 nor G_2 is a cyclic group and $\varphi : \Gamma(G_1) \longrightarrow \Gamma(G_2)$ is an isomorphism.

We claim that $p_1 = p_2$. Let $\Gamma(G_i) = \Gamma_P(G_i) \cup Y_i$ be disjoint union of the vertex set. Then $Y_i \neq \emptyset$ and each $H \in Y_i$ contains a subgroup isomorphic to $C_{p_i} \oplus C_{p_i}$. Let $k(H) = \sharp\{[F] \mid F \in N(H) \cap \Gamma_P(G_i)\}$ for any subgroup H of G_i. Then $\min_{H \in Y_i}\{k(H)\} = k(C_{p_i} \oplus C_{p_i}) = p_i + 1$, since $C_{p_i} \oplus C_{p_i}$ contains exactly $p_i + 1$ distinct subgroups of order p_i. However, by Lemma 4, $\varphi(\Gamma_P(G_1)) = \Gamma_P(G_2)$ and $\varphi(Y_1) = Y_2$. So $\varphi(N(H) \cap \Gamma_P(G_1)) = N(\varphi(H)) \cap \Gamma_P(G_2)$ for any subgroup H of G_1. Moreover, $F_1 \sim F_2$ if and only if $\varphi(F_1) \sim \varphi(F_2)$. Therefore, $\min_{H \in Y_1}\{k(H)\} = \min_{H \in Y_2}\{k(H)\}$, $p_1 = p_2 = p$.

Suppose $G_1 \simeq \oplus_{i=1}^r (C_{p^i})^{k_i}$, $G_2 \simeq \oplus_{i=1}^r (C_{p^i})^{l_i}$, $k_i \geq 0$, $l_i \geq 0$, where $(C_n)^m$ denotes the direct sum of m copies of C_n. Let $m_i = \sharp\{x \in G_1 \mid p^i x = 0\}$, $n_i = \sharp\{x \in G_2 \mid p^i x = 0\}$, $c_i = \frac{1}{p-1}(\frac{m_{i+1}}{m_i} - \frac{m_{i+2}}{m_{i+1}})$ and $d_i = \frac{1}{p-1}(\frac{n_{i+1}}{n_i} -$

$\frac{n_{i+2}}{n_{i+1}}$). By Lemma 6 and a direct computation $c_i = \frac{p^{\sum_{j=i+2}^{r} k_j}(p^{k_{i+1}}-1)}{p-1}$ and $d_i = \frac{p^{\sum_{j=i+2}^{r} l_j}(p^{l_{i+1}}-1)}{p-1}$. Thus, $k_i = l_i$ and $G_1 \simeq G_2$. This finishes the proof. □

Lemma 10. *Let G_i, M_i be groups, $1 \le i \le r$. Suppose that $\Gamma(G_i) \simeq \Gamma(M_i)$ for each i and $\gcd(o(G_i), o(G_j)) = \gcd(o(M_i), o(M_j)) = 1$ for any $i \ne j$. Then $\Gamma(\oplus_{i=1}^{r} G_i) \simeq \Gamma(\oplus_{i=1}^{r} M_i)$.*

Proof. Let $G = \oplus_{i=1}^{r} G_i$ and $M = \oplus_{i=1}^{r} M_i$. Let $\varphi_i : \Gamma(G_i) \longrightarrow \Gamma(M_i)$ be an isomorphism of graphs. We set $\varphi_i(0) = 0$ and $\varphi_i(G_i) = M_i$ and define $\varphi : \Gamma(G) \longrightarrow \Gamma(M)$ as follows. For any subgroup $H = \oplus_{i=1}^{r} H_i$ of G,

$$\varphi(H_1 \oplus H_2 \oplus \ldots \oplus H_r) = \varphi_1(H_1) \oplus \varphi_2(H_2) \oplus \ldots \oplus \varphi_r(H_r).$$

It is straightforward to show that φ is also an isomorphism. □

3 Proof of Theorem 3

Clearly, $G_1 \simeq G_2$ implies $\Gamma(G_1) \simeq \Gamma(G_2)$. So the sufficiency follows from Lemma 10. Assume that $\Gamma(G_1) \simeq \Gamma(G_2)$ and let $\varphi : \Gamma(G_1) \longrightarrow \Gamma(G_2)$ be an isomorphism of graphs. Let $\Gamma(G_i) = \Gamma_P(G_i) \cup Y_i$ be disjoint union of the vertex set. Then $\varphi(\Gamma_P(G_1)) = \Gamma_P(G_2)$ and $\varphi(Y_1) = \varphi(Y_2)$.

Let $H_1, H_2 \in \Gamma_P(G_1)$ such that $o(H_1)$ and $o(H_2)$ are powers of a same prime p. Suppose $F_i = \varphi(H_i) \in \Gamma_P(G_2)$ and $o(F_i)$ is a power of p_i. We will show $p_1 = p_2$. There are two cases.

Case 1: H_1 and H_2 are adjacent. So F_1 and F_2 are also adjacent. Hence, $p_1 = p_2$.

Case 2: H_1 and H_2 are not adjacent. If $p_1 \ne p_2$, let L_i be the cyclic subgroup of F_i of order p_i and $F = L_1 + L_2$. Then $F \simeq C_{p_1 p_2}$ and $\{[L] \mid L \in N(F) \cap \Gamma_P(G_2)\} = \{[L_1], [L_2]\}$. Both H_1 and H_2 are adjacent with $\varphi^{-1}(F)$, $\varphi^{-1}(F)$ contains a subgroup isomorphic to $C_p \oplus C_p$ since $H_1 \cap H_2 = \{0\}$. Hence, $\sharp\{[L] \mid L \in N(F) \cap \Gamma_P(G_2)\} = \sharp\{[L] \mid L \in N(\varphi^{-1}(F)) \cap \Gamma_P(G_1)\} \ge p + 1$. This is a contradiction. So $p_1 = p_2$.

Let $G_1 = \oplus_{i=1}^{r_1} A_i$ and $G_2 = \oplus_{i=1}^{r_2} B_i$ such that each A_i (resp. B_i) is a Sylow subgroup of G_1 (resp. G_2). By the above discussion $r_1 = r_2 = r$ and $\varphi(\Gamma_P(A_i)) = \Gamma_P(B_i)$ after a permutation of indices. Thus, each pair A_i, B_i satisfies one condition of Theorem 9. Without loss of generality we can assume $A_i \simeq C_{a_i^{e_i}}$, $B_i \simeq C_{b_i^{e_i}}$ for $1 \le i \le s$, and neither A_i and B_i is cyclic for $s+1 \le i \le r$. Let $n_1 = \prod_{i=1}^{s} a_i^{e_i}$, $n_2 = \prod_{i=1}^{s} b_i^{e_i}$, $M_1 = \oplus_{i=s+1}^{r} A_i$ and $M_2 = \oplus_{i=s+1}^{r} B_i$. Then $G_i \simeq C_{n_i} \oplus M_i$, where n_1 and n_2 have the same power set and $M_1 \simeq M_2$.

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