

# On Interval Edge-Colorings of Outerplanar Graphs

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## Abstract

A proper edge-coloring of a graph  $G$  with colors  $1, \dots, t$  is called an interval  $t$ -coloring if the colors of edges incident to any vertex of  $G$  form an interval of integers. A graph  $G$  is interval colorable if it has an interval  $t$ -coloring for some positive integer  $t$ . For an interval colorable graph  $G$ , the least value of  $t$  for which  $G$  has an interval  $t$ -coloring is denoted by  $w(G)$ . A graph  $G$  is outerplanar if it can be embedded in the plane so that all its vertices lie on the same (unbounded) face. In this paper we show that if  $G$  is a 2-connected outerplanar graph with  $\Delta(G) = 3$ , then  $G$  is interval colorable and

$$w(G) = \begin{cases} 3, & \text{if } |V(G)| \text{ is even,} \\ 4, & \text{if } |V(G)| \text{ is odd.} \end{cases}$$

We also give a negative answer to the question of Axenovich on the outerplanar triangulations.

## 1 Introduction

In this paper we consider graphs which are finite, undirected, and have no loops or multiple edges. Let  $V(G)$  and  $E(G)$  denote the sets of vertices and edges of a graph  $G$ , respectively. The degree of a vertex  $v \in V(G)$  is denoted by  $d_G(v)$ , the maximum degree of  $G$  by  $\Delta(G)$ , and the chromatic index of  $G$  by  $\chi'(G)$ . A  $u, v$ -path in  $G$  is a walk without repeated vertices and with first vertex  $u$  and last vertex  $v$ . A graph  $G$  is outerplanar if it can be embedded in the plane so that all its vertices lie on the same (unbounded) face. An outerplanar

triangulation is an outerplanar graph in which every bounded face is a triangle. An edge of the outerplanar graph is internal if it does not belong to unbounded face. A separating triangle of the outerplanar graph is a triangular face in which every edge is internal. The terms and concepts that we do not define can be found in [10].

A proper edge-coloring of a graph  $G$  is a coloring of the edges of  $G$  such that no two adjacent edges receive the same color. If  $\alpha$  is a proper edge-coloring of  $G$  and  $v \in V(G)$ , then  $S(v, \alpha)$  denotes the set of colors of edges incident to  $v$ . A proper edge-coloring of a graph  $G$  with colors  $1, \dots, t$  is called an interval  $t$ -coloring if all colors are used, and for any vertex  $v$  of  $G$ , the set  $S(v, \alpha)$  is an interval of integers. A graph  $G$  is interval colorable if it has an interval  $t$ -coloring for some positive integer  $t$ . The set of all interval colorable graphs is denoted by  $\mathfrak{N}$ . For a graph  $G \in \mathfrak{N}$ , the least value of  $t$  for which  $G$  has an interval  $t$ -coloring is denoted by  $w(G)$ .

The concept of interval coloring of graphs was introduced by Asratian and Kamalian [1, 2]. In [1, 2], they proved that if  $G$  is interval colorable, then  $\chi'(G) = \Delta(G)$ . They also showed that if a triangle-free graph  $G$  has an interval  $t$ -coloring, then  $t \leq |V(G)| - 1$ . In [6], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph  $K_{m,n}$  has an interval  $t$ -coloring if and only if  $m+n - \gcd(m, n) \leq t \leq m+n-1$ , where  $\gcd(m, n)$  is the greatest common divisor of  $m$  and  $n$ . In [7], Petrosyan investigated interval colorings of complete graphs and  $n$ -dimensional cubes. In particular, he proved that if  $n \leq t \leq \frac{n(n+1)}{2}$ , then the  $n$ -dimensional cube  $Q_n$  has an interval  $t$ -coloring. Recently, Petrosyan, Khachatryan and Tananyan [8] showed that the  $n$ -dimensional cube  $Q_n$  has an interval  $t$ -coloring if and only if  $n \leq t \leq \frac{n(n+1)}{2}$ . In [9], Sevast'janov proved that it is an  $NP$ -complete problem to decide whether a bipartite graph has an interval coloring or not.

Proper edge-colorings of outerplanar graphs were investigated by Fiorini in [4]. In [4], he proved that if  $G$  is an outerplanar graph, then  $\chi'(G) = \Delta(G)$  if and only if  $G$  is not an odd cycle. Interval edge-colorings of outerplanar graphs were first considered by Axenovich in [3]. In [3], she proved that all outerplanar triangulations with more than three vertices and without separating triangles are interval colorable. Later, interval edge-colorings of outerplanar graphs were investigated by Giaro and Kubale in [5], where they proved that all outerplanar bipartite graphs are interval colorable.

In the present paper we show that if  $G$  is a 2-connected outerplanar graph with  $\Delta(G) = 3$ , then  $G$  is interval colorable and

$$w(G) = \begin{cases} 3, & \text{if } |V(G)| \text{ is even,} \\ 4, & \text{if } |V(G)| \text{ is odd.} \end{cases}$$

We also give a negative answer to the question of Axenovich on the outerplanar triangulations.

## 2 Interval edge-colorings of subcubic outerplanar graphs

First we need the following lemma which was proved in [4].

**Lemma 1** *If  $G$  is a 2-connected outerplanar graph with  $\Delta(G) = 3$ , then  $G$  has either (1)  $u$  and  $v$  adjacent vertices such that  $d_G(u) = d_G(v) = 2$ , or (2)  $u, v$  and  $w$  mutually adjacent vertices such that  $d_G(u) = d_G(w) = 3$  and  $d_G(v) = 2$ .*

Now we can prove our first result.

**Theorem 2** *If  $G$  is a 2-connected outerplanar graph  $G$  with  $\Delta(G) \leq 3$  and  $G$  is not an odd cycle, then  $G \in \mathfrak{N}$  and  $w(G) \leq 4$ .*

**Proof.** For the proof, it suffices to show that if  $G$  is a 2-connected outerplanar graph  $G$  with  $\Delta(G) \leq 3$  and  $G$  is not an odd cycle, then  $G$  has an interval coloring with no more than four colors.

We show it by induction on  $|E(G)|$ . The statement is trivial for the case  $|E(G)| \leq 5$ . Assume that  $|E(G)| \geq 6$ , and the statement is true for all 2-connected outerplanar graphs  $G'$  with  $\Delta(G') \leq 3$  which are not odd cycles and  $|E(G')| < |E(G)|$ .

Let us consider a 2-connected outerplanar graph  $G$  with  $\Delta(G) \leq 3$  which is not an odd cycle. If  $\Delta(G) = 2$ , then  $G \in \mathfrak{N}$  and  $w(G) \leq 2$ . Now suppose that  $\Delta(G) = 3$ . By Lemma 1,  $G$  has either  $u$  and  $v$  adjacent vertices such that  $d_G(u) = d_G(v) = 2$ , or  $u, v$  and  $w$  mutually adjacent vertices such that  $d_G(u) = d_G(w) = 3$  and  $d_G(v) = 2$ . We consider two cases.

Case 1:  $uv \in E(G)$  and  $d_G(u) = d_G(v) = 2$ .

Clearly, in this case there are vertices  $x, y$  ( $x \neq y$ ) in  $G$  such that  $ux \in E(G)$  and  $vy \in E(G)$ .

Case 1.1:  $xy \notin E(G)$ .

In this case let us consider a 2-connected outerplanar graph  $G' = (G - u - v) + xy$ . By induction hypothesis,  $G'$  has an interval coloring  $\alpha$  with no more than four colors.

If  $\alpha(xy) = 1$ , then we delete the edge  $xy$  and color the edges  $ux$  and  $vy$  with color 1 and the edge  $uv$  with color 2. If  $\alpha(xy) \geq 2$ , then we delete the edge  $xy$  and color the edges  $ux$  and  $vy$  with color  $\alpha(xy)$  and the edge  $uv$  with color  $\alpha(xy) - 1$ . It is not difficult to see that the obtained coloring is an interval coloring of the graph  $G$  with no more than four colors.

Case 1.2:  $xy \in E(G)$ .

In this case let us consider a 2-connected outerplanar graph  $G' = G - u - v$ . If  $G'$  is an odd cycle, then we color the edges  $xy$  with color 3 and the edges of the  $x, y$ -path in  $G' - xy$  alternately with colors 1 and 2. Next, we color the edge

$ux$  with color 2, the edge  $uv$  with color 3 and the edge  $vy$  with color 4. Clearly, the obtained coloring is an interval 4-coloring of the graph  $G$ .

Now we can suppose that  $G'$  is a 2-connected outerplanar graph with  $\Delta(G') \leq 3$  which is not an odd cycle. By induction hypothesis,  $G'$  has an interval coloring  $\alpha$  with no more than four colors. Since  $G$  is 2-connected, we have  $d_G(x) = d_G(y) = 3$ .

Suppose that  $S(x, \alpha) = S(y, \alpha) = \{c, c + 1\}$ . If  $c = 1$ , then we color the edges  $ux$  and  $vy$  with color 3 and the edge  $uv$  with color 2; otherwise we color the edges  $ux$  and  $vy$  with color  $c - 1$  and the edge  $uv$  with color  $c$ . If  $S(x, \alpha) \cup S(y, \alpha) = \{c, c + 1, c + 2\}$ , then without loss of generality we may assume that  $S(x, \alpha) = \{c, c + 1\}$  and  $S(y, \alpha) = \{c + 1, c + 2\}$ . We color the edge  $ux$  with color  $c + 2$ , the edge  $uv$  with color  $c + 1$  and the edge  $vy$  with color  $c$ . Clearly, the obtained coloring is an interval coloring of the graph  $G$  with no more than four colors.

Case 2:  $uv, vw, uw \in E(G)$  and  $d_G(u) = d_G(w) = 3$  and  $d_G(v) = 2$ .

In this case by contracting the  $uvw$  triangle to a single vertex  $v^*$ , we obtain a 2-connected outerplanar graph  $G'$  with  $\Delta(G') \leq 3$ . If  $G'$  is an odd cycle, then we color the edge  $uw$  with color 3, the first edge of the  $u, w$ -path in  $G$  with color 4 and the remaining edges alternately with colors 3 and 2. Next, we color the edge  $uv$  with color 2 and the edge  $vw$  with color 1. Clearly, the obtained coloring is an interval 4-coloring of the graph  $G$ .

Now we can suppose that  $G'$  is a 2-connected outerplanar graph with  $\Delta(G') \leq 3$  which is not an odd cycle. By induction hypothesis,  $G'$  has an interval coloring  $\alpha$  with no more than four colors.

Let  $S(v^*, \alpha) = \{c, c + 1\}$ . If  $c = 1$ , then we color the edge  $uw$  with color 3 and the edges  $uv$  and  $vw$  with colors 1 and 2 or 2 and 1 depending on the colors of the colored edges incident to vertices  $u$  and  $w$ ; otherwise we color the edge  $uw$  with color  $c - 1$  and the edges  $uv$  and  $vw$  with colors  $c$  and  $c + 1$  or  $c + 1$  and  $c$  depending on the colors of the colored edges incident to vertices  $u$  and  $w$ . Clearly, the obtained coloring is an interval coloring of the graph  $G$  with no more than four colors.  $\square$

Next we prove the following result:

**Theorem 3** *If  $G$  is a 2-connected outerplanar graph  $G$  with  $\Delta(G) = 3$ , then  $G \in \mathfrak{N}$  and*

$$w(G) = \begin{cases} 3, & \text{if } |V(G)| \text{ is even,} \\ 4, & \text{if } |V(G)| \text{ is odd.} \end{cases}$$

**Proof.** By Theorem 2, we have  $G \in \mathfrak{N}$  and  $w(G) \leq 4$ . On the other hand, since  $\Delta(G) = 3$ , we obtain  $3 \leq w(G) \leq 4$ . We consider two cases.

Case 1:  $|V(G)|$  is even.

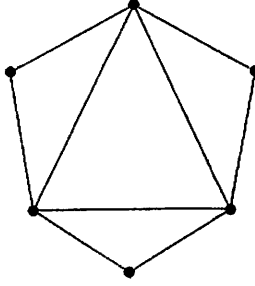


Figure 1: A 2-connected outerplanar graph  $G$  with  $\Delta(G) = 4$  which has no interval coloring.

Since  $G$  is 2-connected and outerplanar, it is clear that  $G$  is Hamiltonian. Let  $C$  be a Hamiltonian cycle in  $G$ . Since  $|V(G)|$  is even, clearly  $|C|$  is even, too. Now we construct an interval 3-coloring of the graph  $G$ . First we color the edges of the cycle  $C$  alternately with colors 1 and 2. Next we color the edges from the set  $E(G) \setminus E(C)$  by color 3. Since  $E(G) \setminus E(C)$  is a matching in  $G$ , the obtained coloring is an interval 3-coloring of the graph  $G$  and thus  $w(G) = 3$ .

Case 2:  $|V(G)|$  is odd.

For the proof, it suffices to show that  $w(G) \geq 4$ .

Suppose, to the contrary, that  $G$  has an interval 3-coloring  $\alpha$ . In this case we consider the set  $S(v, \alpha)$  for every  $v \in V(G)$ . Since  $G$  is 2-connected and outerplanar, we have  $|S(v, \alpha)| \geq 2$  for every  $v \in V(G)$ . This implies that  $2 \in S(v, \alpha)$ , and the edges with color 2 form a perfect matching in  $G$ , but this contradicts the fact that  $|V(G)|$  is odd. Thus,  $w(G) = 4$ .  $\square$

On the other hand, there are 2-connected outerplanar graphs  $G$  with  $\Delta(G) = 4$  which are not interval colorable. For example, the graph  $G$  shown in Fig. 1 has no interval coloring. Now we show a more general result. For that we define a triangle graph  $T_{k,l,m}$  ( $k, l, m \in \mathbb{N}$ ) as follows:

$$V(T_{k,l,m}) = \{x, y, z, u_1, \dots, u_{2k-1}, v_1, \dots, v_{2l-1}, w_1, \dots, w_{2m-1}\} \text{ and}$$

$$E(T_{k,l,m}) = \{xy, xu_1, u_{2k-1}y, yz, yv_1, v_{2l-1}z, zx, xw_1, w_{2m-1}z\} \cup$$

$$\{u_i u_{i+1} : 1 \leq i \leq 2k-2\} \cup \{v_i v_{i+1} : 1 \leq i \leq 2l-2\} \cup \{w_i w_{i+1} : 1 \leq i \leq 2m-2\}.$$

Clearly,  $T_{k,l,m}$  is a 2-connected outerplanar graph with  $\Delta(T_{k,l,m}) = 4$ .

**Theorem 4** For any  $k, l, m \in \mathbb{N}$ , we have  $T_{k,l,m} \notin \mathfrak{N}$ .

**Proof.** Suppose, to the contrary, that the graph  $T_{k,l,m}$  has an interval  $t$ -coloring  $\alpha$  for some  $t \geq 4$ .

Since all degrees of vertices of the graph  $T_{k,l,m}$  are even, the sets  $S(x, \alpha)$ ,  $S(y, \alpha)$  and  $S(z, \alpha)$  contain two even colors and two odd colors, and the sets  $S(u_i, \alpha)$ ,  $S(v_j, \alpha)$  and  $S(w_p, \alpha)$  contain one even color and one odd color for  $i = 1, \dots, 2k - 1$ ,  $j = 1, \dots, 2l - 1$ ,  $p = 1, \dots, 2m - 1$ .

Consider the triangle  $xyz$ . Clearly, there is a vertex of the triangle for which the colors of two incident edges of the triangle have the same parity. Without loss of generality, we may assume that this vertex is  $x$  and  $\alpha(xy)$  and  $\alpha(xz)$  have the same parity. If  $\alpha(xy)$  and  $\alpha(xz)$  are even colors, then  $\alpha(xu_1)$  and  $\alpha(xw_1)$  are odd colors, and thus  $\alpha(u_{2k-1}y)$  and  $\alpha(w_{2m-1}z)$  are even colors. This implies that  $\alpha(yz)$ ,  $\alpha(yv_1)$  and  $\alpha(v_{2l-1}z)$  are odd colors. On the other hand, since  $\alpha(yz)$ ,  $\alpha(yv_1)$  are odd colors, we obtain  $\alpha(v_{2l-1}z)$  is an even color, which is a contradiction. Similarly, if  $\alpha(xy)$  and  $\alpha(xz)$  are odd colors, then  $\alpha(xu_1)$  and  $\alpha(xw_1)$  are even colors, and thus  $\alpha(u_{2k-1}y)$  and  $\alpha(w_{2m-1}z)$  are odd colors. This implies that  $\alpha(yz)$ ,  $\alpha(yv_1)$  and  $\alpha(v_{2l-1}z)$  are even colors. On the other hand, since  $\alpha(yz)$ ,  $\alpha(yv_1)$  are even colors, we obtain  $\alpha(v_{2l-1}z)$  is an odd color, which is a contradiction.  $\square$

### 3 Interval edge-colorings of outerplanar triangulations

In [3], Axenovich showed that all outerplanar triangulations with more than three vertices and without separating triangles are interval colorable, and also she posed the following

**Question 1** *Is it true that an outerplanar triangulation has an interval coloring if and only if it does not have a separating triangle?*

In this section we give a negative answer to the question. For that we define a triangular fan graph  $TF_n$  ( $n \geq 3$ ) as follows:

$$V(TF_n) = \{u, v_1, \dots, v_{n-1}, w_1, \dots, w_{n-2}\} \text{ and} \\ E(TF_n) = \{uv_i : 1 \leq i \leq n-1\} \cup \{v_i w_i, w_i v_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-2\}.$$

Clearly,  $TF_n$  is an outerplanar triangulation.

**Theorem 5** *For any  $n \geq 3$ ,  $TF_n$  has an interval  $\Delta(TF_n)$ -coloring.*

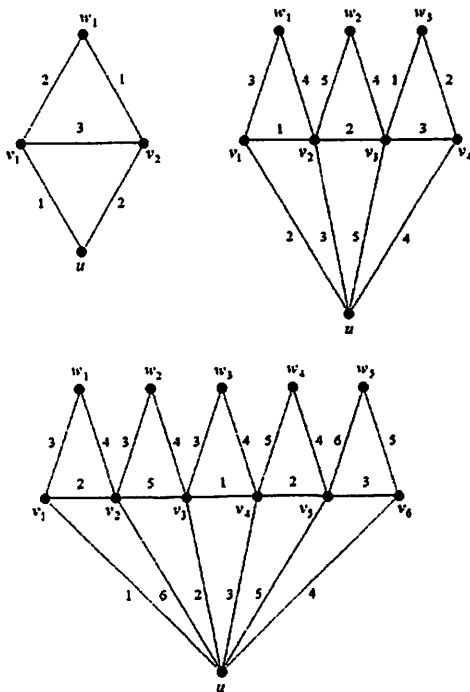


Figure 2: An interval 3-coloring of  $TF_3$ , an interval 5-coloring of  $TF_5$  and an interval 6-coloring of  $TF_7$ .

**Proof.** We consider two cases.

Case 1:  $n$  is odd.

Fig. 2 gives an interval 3-coloring of  $TF_3$ , an interval 5-coloring of  $TF_5$  and an interval 6-coloring of  $TF_7$ . Let  $\alpha$  be an interval 6-coloring of  $TF_7$  shown in Fig. 2. Now we define an edge-coloring  $\beta$  of the graph  $TF_n$  as follows:

- (1) for every  $e \in E(TF_7)$ , let  $\beta(e) = \alpha(e)$ ;
- (2) for  $i = 3, \dots, \frac{n-3}{2}$ , let

$$\begin{aligned} \beta(uv_{2i+1}) &= 2i + 2 \text{ and } \beta(uv_{2i+2}) = 2i + 1, \\ \beta(v_{2i+1}w_{2i}) &= \beta(w_{2i+1}v_{2i+2}) = 2i - 1, \beta(v_{2i+1}v_{2i+2}) = \beta(v_{2i}w_{2i}) = 2i, \\ \beta(v_{2i}v_{2i+1}) &= 2i + 1 \text{ and } \beta(v_{2i+1}w_{2i+1}) = 2i - 2. \end{aligned}$$

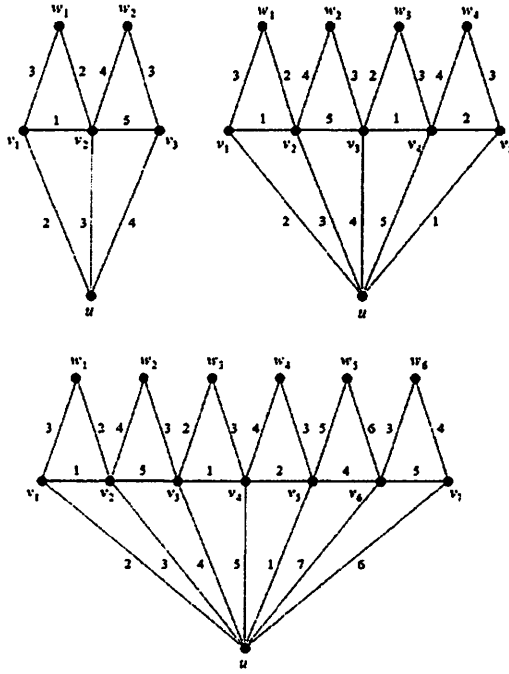


Figure 3: Interval 5-colorings of  $TF_4$  and  $TF_6$ , and an interval 7-coloring of  $TF_8$ .

It is not difficult to see that  $\beta$  is an interval  $\Delta(TF_n)$ -coloring of  $TF_n$  for odd  $n$ .

Case 2:  $n$  is even.

Fig. 3 gives interval 5-colorings of  $TF_4$  and  $TF_6$ , and an interval 7-coloring of  $TF_8$ . Let  $\alpha$  be an interval 7-coloring of  $TF_8$  shown in Fig. 3. Now we define an edge-coloring  $\beta$  of the graph  $TF_n$  as follows:

- (1) for every  $e \in E(TF_8)$ , let  $\beta(e) = \alpha(e)$ ;
- (2) for  $i = 3, \dots, \frac{n-4}{2}$ , let

$$\beta(uv_{2i+2}) = 2i + 3 \text{ and } \beta(uv_{2i+3}) = 2i + 2,$$

$$\beta(v_{2i+2}w_{2i+1}) = \beta(w_{2i+2}v_{2i+3}) = 2i,$$



$$\beta(v_{2i+2}v_{2i+3}) = \beta(v_{2i+1}w_{2i+1}) = 2i + 1,$$

$$\beta(v_{2i+1}v_{2i+2}) = 2i + 2 \text{ and } \beta(v_{2i+2}w_{2i+2}) = 2i - 1.$$

It is not difficult to see that  $\beta$  is an interval  $\Delta(TF_n)$ -coloring of  $TF_n$  for even  $n$ .  $\square$

**Corollary 6** *For any  $n \geq 3$ , we have  $TF_n \in \mathfrak{N}$  and  $w(TF_n) = \Delta(TF_n)$ .*

Clearly,  $uv_i v_{i+1}$  is a separating triangle in  $TF_n$  for  $i = 2, \dots, n - 3$ . Thus, for  $n \geq 5$ ,  $TF_n$  has  $n - 4$  separating triangles.

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