ON THE GENERALIZED GAUSSIAN FIBONACCI NUMBERS

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ABSTRACT. Many authors define certain generalizations of the usual Fibonacci, Pell and Lucas numbers by matrix methods and then obtain the Binet formulas and combinatorial representations of the generalizations of these number sequence. In this article firstly we define and study the generalized Gaussian Fibonacci numbers and then find the matrix representation of the Generalized Gaussian Fibonacci numbers and prove some theorems by these matrix representations.

1. INTRODUCTION

Matrix methods are major tools in solving certain problems stemming from linear recurrence relations. In this paper, the procedure will be illustrated by means of a sequence.

To begin with, we introduce the concept of the *resultant* of given polynomials [12]. Let $f(x) = \sum_{i=0}^{n} a_i x^{n-i}$ and $g(x) = \sum_{i=0}^{m} b_i x^{m-i}$ be polynomials, where $a_0 \neq 0$ and $b_0 \neq 0$. The presence of a common divisor for $f(x)$ and $g(x)$ is equivalent to the fact that there exist polynomials $p(x)$ and $q(x)$ such that $f(x)q(x) = g(x)p(x)$ where $\deg p(x) \leq n-1$ and $\deg q(x) \leq m-1$. Let $q(x) = u_0x^{m-1} + \cdots + u_{m-1}$ and $p(x) = v_0x^{n-1} + \cdots + v_{n-1}$. The equality $f(x)q(x) = g(x)p(x)$ can be expressed in the form of a system of equations

$$\begin{aligned} a_0u_0 &= b_0v_0 \\ a_1u_0 + a_0u_1 &= b_1v_0 + b_0v_1 \\ a_2u_0 + a_1u_1 + a_0u_2 &= b_2v_0 + b_1v_1 + b_0v_2 \\ &\vdots \end{aligned}$$

The polynomials $f(x)$ and $g(x)$ have a common root if and only if this system of equations has a nonzero solution $(u_0, u_1, \ldots, v_0, v_1, \ldots)$. If, for

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example, $m = 3$ and $n = 2$, then the determinant of this system is of the from

$$\begin{vmatrix} a_0 & 0 & 0 & -b_0 & 0 \\ a_1 & a_0 & 0 & -b_1 & -b_0 \\ a_2 & a_1 & a_0 & -b_2 & -b_1 \\ 0 & a_2 & a_1 & -b_3 & -b_2 \\ 0 & 0 & a_2 & 0 & -b_3 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & a_2 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix} = |S(f(x), g(x))|.$$

The matrix $S(f(x), g(x))$ is called the *Sylvester matrix* of polynomials $f(x)$ and $g(x)$. The determinant of $S(f(x), g(x))$ is called the *resultant* of $f(x)$ and $g(x)$ and is denoted by $R(f(x), g(x))$. It is clear that $R(f(x), g(x)) = 0$ if and only if the polynomials $f(x)$ and $g(x)$ have a common divisor, and hence, an equation $f(x) = 0$ has multiple roots if and only if $R(f(x), f'(x)) = 0$.

Now, we introduce the Fibonacci sequence. The *Fibonacci sequence*, $\{F_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$F_{n+1} = F_n + F_{n-1}$$

where $F_0 = 0$, $F_1 = 1$. There are many applications of Fibonacci numbers and Golden Section in every branches of mathematics. See [19], [20], [21], [22].

The *Gaussian Fibonacci sequence* in [17] is defined as $GF_0 = \mathrm{i}$, $GF_1 = 1$ and $GF_n = GF_{n-1} + GF_{n-2}$ for $n > 1$, where $\mathrm{i} = \sqrt{-1}$. One can see that

$$GF_n = F_n + \mathrm{i}F_{n-1}$$

where F_n is the usual nth Fibonacci number.

The complex Fibonacci numbers and Gaussian Fibonacci numbers are studied by some other authors [14, 15]. Harman [14] gives a new approach toward the extension of Fibonacci numbers into the complex plane. Before this study there were two different methods for defining such numbers studied by Horadam [16] and Berzsenyi [5]. Harman [14] generalized both of the methods. Good [13] points out that the square root of the Golden Ratio is the real part of a simple periodic continued fraction but using (complex) Gaussian integers $a + ib$ instead of the natural integers. The authors in [3] defined and studied the Bivariate Gaussian Fibonacci and Bivariate Gaussian Lucas Polynomials. They gave generating function, Binet formula, explicit formula and partial derivation of these polynomials. By defining these bivariate polynomials for special cases $F_n(x, 1)$ is the Gaussian Fibonacci polynomials, $L_n(x, 1)$ is the Gaussian Lucas polynomials, $F_n(1, 1)$ is the Gaussian Fibonacci numbers and $L_n(1, 1)$ is the Gaussian Lucas numbers defined in [17]. Also the authors in [4] define the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. They give generating functions, Binet formulas, explicit formulas and Q matrix of these numbers. They also present explicit combinatorial and determinantal expressions, study negatively subscripted numbers and give various identities. Similar to the Jacobsthal and Jacobsthal Lucas numbers they give some interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers

Many authors [8], [9] define certain generalizations of the usual Fibonacci, Pell and Lucas numbers by matrix methods and then obtain the Binet formulas and combinatorial representations of the generalizations of these number sequence. Furthermore, using matrix methods for computing of properties of recurrence relations are very convenient to parallel algorithm in computer science. The authors in [23] construct the symmetric tridiagonal family of matrices $M_{-\alpha-\beta}(k)$, $k = 1, 2, ...$ whose determinants form any linear subsequence of the Fibonacci numbers.

Several Hessenberg matrices whose determinants are Fibonacci numbers have been studied up to now. It has been shown in [10] that the maximum determinant achieved by $n \times n$ Hessenberg $(0, 1)$-matrices is the nth Fibonacci number. In [1] the authors give some determinantal and permanental representations of k-generalized Fibonacci numbers and Lucas numbers. They also obtained the Binet's formula for these sequences by using their representations.

In this paper, we define and study the generalized Gaussian Fibonacci numbers.

2. Generalization of the Gaussian Fibonacci numbers

In this section, we consider the generalization of the Gaussian Fibonacci numbers.

We define *Generalized Gaussian Fibonacci numbers* (abbr. GGFNs) f_n as following; $f_0 = 0$, $f_1 = \mathbf{k}$, $f_2 = \mathbf{j}$, $f_3 = \mathbf{i}$, $f_4 = 1$, for $n \geq 5$, and

$$f_n = af_{n-1} + bf_{n-2} + cf_{n-3} + df_{n-4}, \tag{2.1}$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$ and a, b, c, d are complex numbers.

Our natural question now becomes what is an explicit expression for f_n in terms of $1, \mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, a, b, c, d? If $a = b = 1$ and $c = d = 0$, then the GGFNs are the Gaussian Fibonacci numbers, and, in [26], Rosenbaum gave the explicit expression for the case.

In this section, we give an explicit expression for

$$f_n = af_{n-1} + bf_{n-2} + cf_{n-3} + df_{n-4}, n \geq 5.$$

Let $G_n = (f_n, f_{n-1}, f_{n-2}, f_{n-3})^T$ for $n \geq 3$. The fundamental recurrence relation (2.1) can be defined by vector recurrence relation

$$G_{n+1} = QG_n$$

where

$$Q = \begin{bmatrix} a & b & c & d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Also, we have $G_{n+4} = Q^n G_4$. That is,

$$\begin{bmatrix} f_{n+4} \\ f_{n+3} \\ f_{n+2} \\ f_{n+1} \end{bmatrix} = Q^n \begin{bmatrix} f_4 \\ f_3 \\ f_2 \\ f_1 \end{bmatrix}. \tag{2.2}$$

And the characteristic equation of Q is

$$p(\lambda) = \lambda^4 - a\lambda^3 - b\lambda^2 - c\lambda - d = 0.$$

If $R(p(\lambda),\ p'(\lambda)) \neq 0$, then the equation $p(\lambda) = 0$ has distinct 4 roots.

Theorem 1. *Let $p(x)$ be the characteristic equation of the matrix Q. If*

$$R(p(x),\ p'(x)) \neq 0$$

then, for $n \geq 5$, $f_n = af_{n-1} + bf_{n-2} + cf_{n-3} + df_{n-4}$ has an explicit expression in terms of $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$.

Proof. If $R(p(x),\ p'(x)) \neq 0$, then the characteristic equation of Q have 4 distinct roots, say $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$. Since the matrix Q is diagonalizable, there exists a matrix Λ such that $\Lambda^{-1}Q\Lambda = \mathrm{diag}(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4)$. Then, by (2.2), we have

$$G_{n+4} = \Lambda \mathrm{diag}(\lambda_1^n,\ \lambda_2^n,\ \lambda_3^n,\ \lambda_4^n)\Lambda^{-1}G_4.$$

And hence we have

$$f_{n+4} = s_1\lambda_1^n + s_2\lambda_2^n + s_3\lambda_3^n + s_4\lambda_4^n,$$

where $s_1,\ s_2,\ s_3,\ s_4$ are complex numbers independent of 4, and we can determine the values of $s_1,\ s_2,\ s_3,\ s_4$ by Cramer's rule. That is, by setting $n = 3,\ 2,\ 1,\ 0$, we have

$$\begin{aligned} f_7 &= s_1\lambda_1^3 + s_2\lambda_2^3 + s_3\lambda_3^3 + s_4\lambda_4^3 \\ f_6 &= s_1\lambda_1^2 + s_2\lambda_2^2 + s_3\lambda_3^2 + s_4\lambda_4^2 \\ f_5 &= s_1\lambda_1 + s_2\lambda_2 + s_3\lambda_3 + s_4\lambda_4 \\ f_4 &= s_1 + s_2 + s_3 + s_4 \end{aligned}$$

Hence, for $\mathbf{s} = (s_1,\ s_2,\ s_3,\ s_4)^T$,

$$V\mathbf{s} = G_7, \tag{2.3}$$

where the matrix V is a Vandermonde matrix such that

$$V = \begin{bmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Since $G_7 = Q^3 G_4$, we have the desired result from (2.3).

The proof is completed. □

From [7], we know the (i,j) entry $q_{ij}^{(n)}$ in Q^n is given by the following formula:

$$q_{ij}^{(n)} = \sum_{(m_1,\ldots,m_4)} \frac{m_j + \cdots + m_4}{m_1 + \cdots + m_4} \times \binom{m_1 + \cdots + m_4}{m_1, \ldots, m_4} a^{m_1} b^{m_2} c^{m_3} d^{m_4}, \quad (2.4)$$

where the summation is over nonnegative integers satisfying

$$m_1 + 2m_2 + 3m_3 + 4m_4 = n - i + j$$

and the coefficient in (2.4) is defined to be 1 if $n = i - j$.

Corollary 1. *For GGFNs f_n, we have*

$$\sum_{k=1}^{n+4} f_k = \sum_{k=1}^{n} (q_{11}^{(k)} + q_{12}^{(k)}\mathbf{i} + q_{13}^{(k)}\mathbf{j} + q_{14}^{(k)}\mathbf{k}) + (1 + \mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Proof. Since $G_{n+4} = Q^n G_4$, we have

$$\sum_{k=5}^{n+4} G_k = \sum_{k=1}^{n} Q^k G_4.$$

That is,

$$\sum_{k=5}^{n+4} f_k = \sum_{k=1}^{n} (q_{11}^{(k)} + q_{12}^{(k)}\mathbf{i} + q_{13}^{(k)}\mathbf{j} + q_{14}^{(k)}\mathbf{k}).$$

And,

$$f_4 = 1, \ f_3 = \mathbf{i}, \ f_2 = \mathbf{j}, \ f_1 = \mathbf{k}.$$

Thus, the proof is completed. □

Applying the $G_{n+4} = Q^n G_4$ to (2.4), we have, for $n \geq 1$,

$$f_{n+1} = q_{41}^{(n)} + q_{42}^{(n)}\mathbf{i} + q_{43}^{(n)}\mathbf{j} + q_{44}^{(n)}\mathbf{k} \quad (2.5)$$

Hence, from Theorem 1 and (2.5), for $n \geq 1$,

$$\begin{aligned} f_{n+1} &= q_{41}^{(n)} + q_{42}^{(n)}\mathbf{i} + q_{43}^{(n)}\mathbf{j} + q_{44}^{(n)}\mathbf{k} \\ &= s_1\lambda_1^n + s_2\lambda_2^n + s_3\lambda_3^n + s_4\lambda_4^n. \end{aligned}$$

The characteristic equation of Q is $p(\lambda) = \lambda^4 - a\lambda^3 - b\lambda^2 - c\lambda - d = 0$, and hence

$$\begin{aligned} R(p(\lambda), p'(\lambda)) &= \begin{vmatrix} 1 & -a & -b & -c & -d & 0 & 0 \\ 0 & 1 & -a & -b & -c & -d & 0 \\ 0 & 0 & 1 & -a & -b & -c & -d \\ 4 & -3a & -2b & -c & 0 & 0 & 0 \\ 0 & 4 & -3a & -2b & -c & 0 & 0 \\ 0 & 0 & 4 & -3a & -2b & -c & 0 \\ 0 & 0 & 0 & 4 & -3a & -2b & -c \end{vmatrix} \\ &= 144bc^2d - 256d^3 - 16b^4d - 18abc^3 - 192acd^2 \\ &\quad -27c^4 + 80ab^2cd + 18a^3bcd - 4a^3c^3 - 128b^2d^2 + 4b^3c^2 \\ &\quad -27a^4d^2 - 144a^2bd^2 - 4a^2b^3d + 6a^2c^2d + a^2b^2c^2. \end{aligned}$$

We know that, from the definition of resultant, the equation $p(\lambda) = 0$ has multiple roots if and only if $R(p(\lambda), p'(\lambda)) = 0$. That is, the equation $p(\lambda) = 0$ has 4 distinct roots if and only if $R(p(\lambda), p'(\lambda)) \neq 0$.

Suppose that $R(p(\lambda), p'(\lambda)) \neq 0$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the distinct roots of $p(\lambda) = 0$. Then we have

$$\begin{aligned} f_{n+1} &= s_1\lambda_1^n + s_2\lambda_2^n + s_3\lambda_3^n + s_4\lambda_4^n \\ &= s_1\alpha_1^n + s_2\alpha_2^n + s_3\alpha_3^n + s_4\alpha_4^n. \end{aligned} \tag{2.6}$$

Since $f_4 = 1$, $f_5 = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, $f_6 = (a^2 + b) + (ab + c)\mathbf{i} + (ac + d)\mathbf{j} + ad\mathbf{k}$, $f_7 = (a^3 + 2ab + c) + (a^2b + ac + b^2 + d)\mathbf{i} + (a^2c + ad + bc)\mathbf{j} + (a^2d + bd)\mathbf{k}$ and

$$\begin{aligned} s_1 + s_2 + s_3 + s_4 &= f_4, \\ s_1\alpha_1 + s_2\alpha_2 + s_3\alpha_3 + s_4\alpha_4 &= f_5, \\ s_1\alpha_1^2 + s_2\alpha_2^2 + s_3\alpha_3^2 + s_4\alpha_4^2 &= f_6, \\ s_1\alpha_1^3 + s_2\alpha_2^3 + s_3\alpha_3^3 + s_4\alpha_4^3 &= f_7, \end{aligned}$$

we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} \tag{2.7}$$

Set

$$v = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \end{bmatrix}$$

$$v_{\alpha_1} = \det\begin{bmatrix} f_4 & 1 & 1 & 1 \\ f_5 & \alpha_2 & \alpha_3 & \alpha_4 \\ f_6 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ f_7 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \end{bmatrix}, \; v_{\alpha_2} = \det\begin{bmatrix} 1 & f_4 & 1 & 1 \\ \alpha_1 & f_5 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & f_6 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & f_7 & \alpha_3^3 & \alpha_4^3 \end{bmatrix}$$

$$v_{\alpha_3} = \det\begin{bmatrix} 1 & 1 & f_4 & 1 \\ \alpha_1 & \alpha_2 & f_5 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & f_6 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & f_7 & \alpha_4^3 \end{bmatrix}, \; v_{\alpha_4} = \det\begin{bmatrix} 1 & 1 & 1 & f_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & f_5 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & f_6 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & f_7 \end{bmatrix}$$

Then we have the following theorem which is a generalization of Binet formula.

Theorem 2. *Let $p(\lambda)$ be the characteristic equation of the matrix Q and let α_1, α_2, α_3, α_4 be the roots of $p(\lambda) = 0$. If $R(p(x),\, p'(x)) \neq 0$, then, for GGFN f_{n+1},*

$$f_{n+1} = \frac{v_{\alpha_1}\alpha_1^n + v_{\alpha_2}\alpha_2^n + v_{\alpha_3}\alpha_3^n + v_{\alpha_4}\alpha_4^n}{v}.$$

Proof. From (2.6), we have

$$f_{n+1} = s_1\alpha_1^n + s_2\alpha_2^n + s_3\alpha_3^n + s_4\alpha_4^n,$$

where

$$s_1 = \frac{v_{\alpha_1}}{v}, \quad s_2 = \frac{v_{\alpha_2}}{v}, \quad s_3 = \frac{v_{\alpha_3}}{v}, \quad s_4 = \frac{v_{\alpha_4}}{v}.$$

The proof is completed. □

From Theorem 2, in particular, if $a = b = c = d = 1$ then $R(p(\lambda),\, p'(\lambda)) = -563 \neq 0$. In this case, the equation $p(\lambda) = 0$ have 4 distinct roots and hence we have the following corollary.

Corollary 2. *For GGFN $f_{n+1} = af_n + bf_{n-1} + cf_{n-2} + df_{n-3}$, if $a = b = c = d = 1$, then we have*

$$f_{n+1} = \frac{v_{\alpha_{11}}\alpha_{11}^n + v_{\alpha_{21}}\alpha_{21}^n + v_{\alpha_{31}}\alpha_{31}^n + v_{\alpha_{41}}\alpha_{41}^n}{v_1},$$

where α_{11}, α_{21}, α_{31}, α_{41} are the roots of $p_1(\lambda) = \lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1 = 0$ and

$$v_1 = \det\begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{11}^2 & \alpha_{21}^2 & \alpha_{31}^2 & \alpha_{41}^2 \\ \alpha_{11}^3 & \alpha_{21}^3 & \alpha_{31}^3 & \alpha_{41}^3 \end{bmatrix} = \prod_{1 \le i < j \le 4} (\alpha_{j1} - \alpha_{i1})$$

Proof. Since $f_{n+1} = f_n + f_{n-1} + f_{n-2} + f_{n-3}$, $R(p_1(\lambda), p_1'(\lambda)) = -563 \neq 0$. That is, the equation $p_1(\lambda) = 0$ have 4 distinct roots α_{11}, α_{21}, α_{31}, α_{41}. From (2.7), we have $f_4 = 1$, $f_5 = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$, $f_6 = 2 + 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $f_7 = 4 + 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and

$$\begin{aligned} s_1 + s_2 + s_3 + s_4 &= f_4, \\ s_1\alpha_{11} + s_2\alpha_{21} + s_3\alpha_{31} + s_4\alpha_{41} &= f_5, \\ s_1\alpha_{11}^2 + s_2\alpha_{21}^2 + s_3\alpha_{31}^2 + s_4\alpha_{41}^2 &= f_6, \\ s_1\alpha_{11}^3 + s_2\alpha_{21}^3 + s_3\alpha_{31}^3 + s_4\alpha_{41}^3 &= f_7, \end{aligned}$$

That is,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{11}^2 & \alpha_{21}^2 & \alpha_{31}^2 & \alpha_{41}^2 \\ \alpha_{11}^3 & \alpha_{21}^3 & \alpha_{31}^3 & \alpha_{41}^3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} \\ 2+2\mathbf{i}+2\mathbf{j}+\mathbf{k} \\ 4+4\mathbf{i}+3\mathbf{j}+2\mathbf{k} \end{bmatrix}.$$

Set

$$v_1 = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{11}^2 & \alpha_{21}^2 & \alpha_{31}^2 & \alpha_{41}^2 \\ \alpha_{11}^3 & \alpha_{21}^3 & \alpha_{31}^3 & \alpha_{41}^3 \end{bmatrix}$$

$$v_{\alpha_{11}} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ 2+2\mathbf{i}+2\mathbf{j}+\mathbf{k} & \alpha_{21}^2 & \alpha_{31}^2 & \alpha_{41}^2 \\ 4+4\mathbf{i}+3\mathbf{j}+2\mathbf{k} & \alpha_{21}^3 & \alpha_{31}^3 & \alpha_{41}^3 \end{bmatrix}$$

$$v_{\alpha_{21}} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{11} & 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & \alpha_{31} & \alpha_{41} \\ \alpha_{11}^2 & 2+2\mathbf{i}+2\mathbf{j}+\mathbf{k} & \alpha_{31}^2 & \alpha_{41}^2 \\ \alpha_{11}^3 & 4+4\mathbf{i}+3\mathbf{j}+2\mathbf{k} & \alpha_{31}^3 & \alpha_{41}^3 \end{bmatrix}$$

$$v_{\alpha_{31}} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{11} & \alpha_{21} & 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & \alpha_{41} \\ \alpha_{11}^2 & \alpha_{21}^2 & 2+2\mathbf{i}+2\mathbf{j}+\mathbf{k} & \alpha_{41}^2 \\ \alpha_{11}^3 & \alpha_{21}^3 & 4+4\mathbf{i}+3\mathbf{j}+2\mathbf{k} & \alpha_{41}^3 \end{bmatrix}$$

$$v_{\alpha_{41}} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & 1+\mathbf{i}+\mathbf{j}+\mathbf{k} \\ \alpha_{11}^2 & \alpha_{21}^2 & \alpha_{31}^2 & 2+2\mathbf{i}+2\mathbf{j}+\mathbf{k} \\ \alpha_{11}^3 & \alpha_{21}^3 & \alpha_{31}^3 & 4+4\mathbf{i}+3\mathbf{j}+2\mathbf{k} \end{bmatrix}$$

Thus, we have

$$f_{n+1} = \frac{v_{\alpha_{11}}\alpha_{11}^n + v_{\alpha_{21}}\alpha_{21}^n + v_{\alpha_{31}}\alpha_{31}^n + v_{\alpha_{41}}\alpha_{41}^n}{v_1}.$$

where

$$v_1 = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{11}^2 & \alpha_{21}^2 & \alpha_{31}^2 & \alpha_{41}^2 \\ \alpha_{11}^3 & \alpha_{21}^3 & \alpha_{31}^3 & \alpha_{41}^3 \end{bmatrix} = \prod_{1 \le i < j \le 4} (\alpha_{j1} - \alpha_{i1})$$

The proof is completed. □

For GGFNs $f_{n+1} = af_n + bf_{n-1} + cf_{n-2} + df_{n-3}$, if $a = b = 1$ and $c = d = 0$, then we have $R(p(\lambda), p'(\lambda)) = 0$. So, the equation $p(\lambda) = 0$ have multiple roots, i.e., $\lambda = 0$ is the multiple root of multiplicity 2. And hence, from the Theorem 2 , we have the Binet formula for the Gaussian Fibonacci number GF_n.

3. Generalization of the Gaussian Lucas numbers

In [19], the author introduced a generalized Lucas number from generalized Fibonacci number. The k-Fibonacci sequence $\{g_n^{(k)}\}$ is defined as

$$g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \quad g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and for $n > k \ge 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

From the definition of the generalization, the author define the k-Lucas sequence $\{l_n^{(k)}\}$ as follows;

$$l_n^{(k)} = g_{n-1}^{(k)} + g_{n+k-1}^{(k)}. \tag{3.1}$$

In this section, we also consider the generalization of the Gaussian Lucas numbers. We define *Generalized Gaussian Lucas numbers* (abbr. GGLNs) l_n as following; for $n \ge 1$,

$$l_n = f_{n+3} + f_{n-1}. \tag{3.2}$$

From the definition of GGLNs, we have, for $n \ge 6$,

$$\begin{aligned} l_n &= f_{n+3} + f_{n-1} \\ &= af_{n+2} + bf_{n+1} + cf_n + df_{n-1} + af_{n-2} + bf_{n-3} + cf_{n-4} + df_{n-5} \\ &= a(f_{n+2} + f_{n-2}) + b(f_{n+1} + f_{n-3}) + c(f_n + f_{n-4}) + d(f_{n-1} + f_{n-5}). \end{aligned}$$

Since $l_n = f_{n+3} + f_{n-1}$ for $n \ge 1$, we have, for $n \ge 6$,

$$l_n = al_{n-1} + bl_{n-2} + cl_{n-3} + dl_{n-4}. \tag{3.3}$$

From the Theorem 2 and (3.2), we have the following theorem which is the Binet formula for the GGLNs.

Theorem 3. *Let $p(\lambda)$ be the characteristic equation of the matrix Q and let α_1, α_2, α_3, α_4 be the roots of $p(\lambda) = 0$. If $R(p(x), p'(x)) \neq 0$, then, for GGLN l_n,*

$$l_n = \frac{v_{\alpha_1}\alpha_1^{n-2}(\alpha_1^4+1)+v_{\alpha_2}\alpha_2^{n-2}(\alpha_2^4+1)+v_{\alpha_3}\alpha_3^{n-2}(\alpha_3^4+1)+v_{\alpha_4}\alpha_4^{n-2}(\alpha_4^4+1)}{v}.$$

Proof. From (3.2) and Theorem 2, we have

$$\begin{aligned} l_n &= f_{n+3} + f_{n-1} \\ &= \frac{v_{\alpha_1}\alpha_1^{n+2}+v_{\alpha_2}\alpha_2^{n+2}+v_{\alpha_3}\alpha_3^{n+2}+v_{\alpha_4}\alpha_4^{n+2}}{v} \\ &\quad + \frac{v_{\alpha_1}\alpha_1^{n-2}+v_{\alpha_2}\alpha_2^{n-2}+v_{\alpha_3}\alpha_3^{n-2}+v_{\alpha_4}\alpha_4^{n-2}}{v} \\ &= \frac{v_{\alpha_1}\alpha_1^{n-2}(\alpha_1^4+1)+v_{\alpha_2}\alpha_2^{n-2}(\alpha_2^4+1)+v_{\alpha_3}\alpha_3^{n-2}(\alpha_3^4+1)+v_{\alpha_4}\alpha_4^{n-2}(\alpha_4^4+1)}{v}. \end{aligned}$$

The proof is completed. □

Also, from the Corollary 2 and Theorem 3, we have the following corollary.

Corollary 3. *For GGFN $l_n = al_{n-1} + bl_{n-2} + cl_{n-3} + dl_{n-4}$, $n \geq 6$, if $a = b = c = d = 1$, then*

$$l_n = \frac{v_{\alpha_{11}}\alpha_{11}^{n-2}(\alpha_{11}^4+1)+v_{\alpha_{21}}\alpha_{21}^{n-2}(\alpha_{21}^4+1)+v_{\alpha_{31}}\alpha_{31}^{n-2}(\alpha_{31}^4+1)+v_{\alpha_{41}}\alpha_{41}^{n-2}(\alpha_{41}^4+1)}{v},$$

where α_{11}, α_{21}, α_{31}, α_{41} are the roots of $p_1(\lambda) = \lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1 = 0$ and

$$v_1 = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{11}^2 & \alpha_{21}^2 & \alpha_{31}^2 & \alpha_{41}^2 \\ \alpha_{11}^3 & \alpha_{21}^3 & \alpha_{31}^3 & \alpha_{41}^3 \end{bmatrix} = \prod_{1 \leq i < j \leq 4} (\alpha_{j1} - \alpha_{i1})$$

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