The upper vertex detour monophonic number of a graph*

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Abstract

For any vertex x in a connected graph G of order $n \geq 2$, a set $S \subseteq V(G)$ is an x-detour monophonic set of G if each vertex $v \in V(G)$ lies on an x-y detour monophonic path for some element y in S. The minimum cardinality of an x-detour monophonic set of G is the x-detour monophonic number of G, denoted by $dm_x(G)$. An x-detour monophonic set S_x of G is called a minimal x-detour monophonic set if no proper subset of S_x is an x-detour monophonic set. The upper x-detour monophonic number of G, denoted by $dm_x^+(G)$, defined as the maximum cardinality of a minimal x-detour monophonic set of G. We determine bounds for it and find the same for some special classes of graphs. For positive integers r, d and k with $2 \leq r \leq d$ and $k \geq 2$, there exists a connected graph G with monophonic radius r, monophonic diameter d and upper x-detour monophonic number k for some vertex x in G. Also, it is shown that for positive integers j, k, l and n with $2 \leq j \leq k \leq l \leq n-7$, there exists a connected graph G of order n with $dm_x(G) = j, dm_x^+(G) = l$ and a minimal x-detour monophonic set of cardinality k.

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1 Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to [1, 2]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an x-y *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighborhood* of a vertex v is the set $N[v] = N(u) \cup \{v\}$. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete.

A *chord* of a path P is an edge joining any two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. A longest x-y monophonic path P is called an *x-y detour monophonic path.* For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest u-v monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, rad_mG of G is $rad_mG = min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_mG$ of G is $diam_mG = max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [3].

The concept of vertex monophonic number was introduced in [4]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-monophonic set* of G if each vertex v of G lies on an x-y monophonic path in G for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G and is denoted by $m_x(G)$. An x-monophonic set of cardinality $m_x(G)$ is called a *m_x-set* of G.

The concept of vertex detour monophonic number was introduced in [5]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-detour monophonic set* of G if each vertex v of G lies on an x-y detour monophonic path in G for some element y in S. The minimum cardinality of an x-detour monophonic set of G is defined as the *x-detour monophonic number* of G and is denoted by $dm_x(G)$. An x-detour monophonic set of cardinality $dm_x(G)$ is called a dm_x-set of G.

Theorem 1.1. [5] Let x be any vertex of a connected graph G.
(i) Every extreme vertex of G other than the vertex x (whether x is extreme or not) belongs to every x-detour monophonic set.
(ii) No cutvertex of G belongs to any dm_x-set.

Theorem 1.2. [5] (i) For any non-trivial tree T with k endvertices, $dm_x(T) = k$ or $k - 1$ according as x is a cutvertex or not.
(ii) For any vertex x in the complete graph K_n of order $n \geq 2$, $dm_x(K_n) = n - 1$.

Throughout this paper G denotes a connected graph with at least two vertices.

2 Upper vertex detour monophonic number

Definition 2.1. Let x be any vertex of a connected graph G. An x-detour monophonic set S_x is called a *minimal x-detour monophonic set* if no proper subset of S_x is an x-detour monophonic set. The *upper x-detour monophonic number* is the maximum cardinality of a minimal x-detour monophonic set of G and is denoted by $dm_x^+(G)$.

Example 2.2. For the graph G given in Figure 2.1, the minimum vertex detour monophonic sets, the minimum vertex detour monophonic numbers, the minimal vertex detour monophonic sets and the upper vertex detour monophonic numbers are given in Table 2.1.

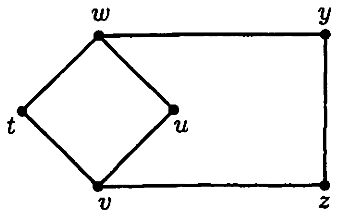

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Figure 2.1

vertex x	minimum x-detour monophonic sets	$dm_x(G)$	minimal x-detour monophonic sets	$dm_x^+(G)$
t	$\{u,y\}$, $\{u,z\}$	2	$\{u,y\}$, $\{u,z\}$	2
u	$\{t,y\}$, $\{t,z\}$	2	$\{t,y\}$, $\{t,z\}$	2
v	$\{w,y\}$, $\{z,y\}$	2	$\{w,y\}$, $\{z,y\}$, $\{w,t,u\}$	3
w	$\{z,y\}$, $\{z,v\}$	2	$\{z,y\}$, $\{z,v\}$, $\{v,t,u\}$	3
y	$\{v,z\}$, $\{v,t\}$, $\{v,u\}$	2	$\{v,z\}$, $\{v,t\}$, $\{v,u\}$, $\{t,u,w\}$	3
z	$\{w,y\}$, $\{w,t\}$, $\{w,u\}$	2	$\{w,y\}$, $\{w,t\}$, $\{w,u\}$, $\{v,t,u\}$	3

Table 2.1

For any vertex x in a connected graph G, every minimum x-detour monophonic set is a minimal x-detour monophonic set, but the converse is not true. For the graph G given in Figure 2.1, $\{w, t, u\}$ is a minimal v-detour monophonic set but it is not a minimum v-detour monophonic set. Also, note that x does not belong to any minimal x-detour monophonic set of G.

Theorem 2.3. Let x be any vertex of a connected graph G.
(i) Every extreme vertex of G other than the vertex x (whether x is an extreme vertex or not) belongs to every minimal x-detour monophonic set of G.
(ii) No cutvertex of G belongs to any minimal x-detour monophonic set of G.

Proof. (i) Let x be any vertex of G. Since x does not belong to any minimal x-detour monophonic set, let $v \neq x$ be an extreme vertex of G. Clearly v is not an internal vertex of any detour monophonic path so that v belongs to every minimal x-detor monophonic set of G.
(ii) Let $y \neq x$ be a cutvertex of G. Let U and W be two components of $G - \{y\}$. For any vertex x in G, let S_x be a minimal x-detour monophonic set of G. Suppose that $x \in U$. Now, suppose that $S_x \cap W = \emptyset$. Let $w_1 \in W$. Then $w_1 \notin S_x$. Since S_x is an x-detour monophonic set, there exists an element z in S_x such that w_1 lies in some x-z detour monophonic path $P : x = z_0, z_1, \ldots, w_1, \ldots, z_k = z$ in G. Since $S_x \cap W = \emptyset$ and y is a cutvertex of G, it follows that the x-w_1 subpath of P and the w_1-z subpath of P both contain y so that P is not a path in G. Hence $S_x \cap W \neq \emptyset$. Let $w_2 \in S_x \cap W$. Then $w_2 \neq y$ so that y is an internal vertex of an x-w_2 detour monophonic path. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on an x-y detour monophonic path also lies on an x-w_2 detour monophonic path. Hence it follows that S is an x-detour monophonic set of G, which is a contradiction to S_x a minimal x-detour monophonic set of G. Thus y does not belong to any minimal x-detour monophonic set of G. Similarly, if $x \in W$, then y does not belong to any minimal x-detour monophonic set of G. □

Since every endvertex is an extreme vertex, the following theorem is an easy consequence of the definition of the upper vertex detour monophonic number of a graph and Theorem 2.3.

Theorem 2.4. (i) For any non-trivial tree T with k endvertices, $dm_x^+(T) = k$ or $k - 1$ according as x is a cutvertex or not.
(ii) For any vertex x in the complete graph K_n of order $n \geq 2$, $dm_x^+(K_n) = n - 1$.

Theorem 2.5. For any vertex x in the cycle C_n of order $n \geq 4$,

$$dm_x^+(C_n) = \begin{cases} 1 & \text{if } n = 4 \\ 2 & \text{if } n > 4. \end{cases}$$

Proof. Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle of order $n \geq 4$. Let x be any vertex in C_n, say $x = u_1$. If $n = 4$, then $S_x = \{u_3\}$ is the unique minimal x-detour monophonic set of C_n and so $dm_x^+(C_n) = 1$. Now, assume that $n > 4$. If n is even, then $S_1 = \{u_{\frac{n}{2}+1}\}, S_2 = \{u_2, u_3\}, S_3 = \{u_{n-1}, u_n\}$ and $S_4 = \{u_i, u_j : 3 \leq i \leq \frac{n}{2}$ and $\frac{n}{2} + 2 \leq j \leq n-1\}$ are the minimal x-detour monophonic sets of C_n. If n is odd, then $S_1 = \{u_2, u_3\}, S_2 = \{u_{n-1}, u_n\}$ and $S_3 = \{u_i, u_j : 3 \leq i \leq \frac{n+1}{2}$ and $\frac{n+3}{2} \leq j \leq n-1\}$ are the minimal x-detour monophonic sets of C_n. Hence $dm_x^+(C_n) = 2$. □

Theorem 2.6. Let $W_n = K_1 + C_{n-1} (n \geq 5)$ be the wheel.
(i) If $n = 5$, then $dm_x^+(W_n) = n - 1$ or 1 according as x is K_1 or x is in C_{n-1}.
(ii) If $n > 5$, then $dm_x^+(W_n) = n - 1$ or 3 according as x is K_1 or x is in C_{n-1}.

Proof. Let $C_{n-1} : u_1, u_2, \ldots, u_{n-1}, u_1$ be a cycle of order $n-1$ and let u be the vertex of K_1. If $x = u$, then no vertex of C_{n-1} is an internal vertex of any detour monophonic path starting from x. It follows that $V(C_{n-1})$ is the minimal x-detour monophonic set of W_n and so $dm_x^+(W_n) = n-1$. Let x be any vertex in C_{n-1}, say $x = u_1$. If $n = 5$, then $S_x = \{u_3\}$ is the unique minimal x-detour monophonic set of G and so $dm_x^+(W_n) = 1$. Now, assume that $n > 5$. If n is odd, then $S_1 = \{u, u_{\frac{n-1}{2}+1}\}, S_2 = \{u, u_2, u_3\}, S_3 = \{u, u_{n-2}, u_{n-1}\}$ and $S_4 = \{u, u_i, u_j : 3 \leq i \leq \frac{n-1}{2}$ and $\frac{n+3}{2} \leq j \leq n-2\}$ are the minimal x-detour monophonic sets of W_n. It follows that $dm_x^+(W_n) = 3$. If n is even, then $S_1 = \{u, u_2, u_3\}, S_2 = \{u, u_{n-2}, u_{n-1}\}$ and $S_3 = \{u, u_i, u_j : 3 \leq i \leq \frac{n}{2}$ and $\frac{n+2}{2} \leq j \leq n-2\}$ are the minimal x-detour monophonic sets of W_n and hence $dm_x^+(W_n) = 3$. □

Theorem 2.7. Let $G = K_{r,s} (2 \leq r \leq s)$ be the complete bipartite graph with bipartition (V_1, V_2). Then $dm_x^+(G) = \begin{cases} r-1 & \text{if } x \in V_1 \\ s-1 & \text{if } x \in V_2. \end{cases}$

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{w_1, w_2, \ldots, w_s\}$ be a partition of G. Let $x \in V_1$, say $x = u_1$. Since the vertex $u_i (2 \leq i \leq r)$ does not lie on any detour monophonic path starting from x and every vertex of V_2 lies on an x-u_2 detour monophonic path, $S_x = V_1 - \{x\}$ is the unique minimal x-detour monophonic set of G. Hence $dm_x^+(G) = |S_x| = r - 1$. Let $x \in V_2$. Then by a similar argument, we get $dm_x^+(G) = s - 1$. □

3 Bounds and realization results for $dm_x^+(G)$

Theorem 3.1. For any vertex x in a connected graph G of order $n \geq 2$, $1 \leq dm_x^+(G) \leq n - 1$.

Proof. It is clear that every minimal x-detour monophonic set contains at least one vertex so that $dm_x^+(G) \geq 1$. Since the vertex x does not belong to any minimal x-detour monophonic set of G, it follows that $dm_x^+(G) \leq n-1$. □

Remark 3.2. The bounds in Theorem 3.1 are sharp. For the path P_n $(n \geq 2)$, $dm_x^+(P_n) = 1$ for an endvertex x in P_n. Also, for the complete graph $K_n(n \geq 2)$, $dm_x^+(K_n) = n-1$.

Theorem 3.3. For any two integers k and n with $1 \leq k \leq n-1$ and $n \geq 2$, there exists a connected graph G of order n and $dm_x^+(G) = k$ for some vertex x in G.

Proof. Let G be the graph obtained from the path $P_{n-k} : u_1, u_2, \ldots, u_{n-k}$ of order $n-k \geq 1$ by adding k new vertices $w_1, w_2, \ldots, w_k$ and joining each $w_i (1 \leq i \leq k)$ with u_{n-k} in P_{n-k}. The graph G is a tree of order n and is shown in Figure 3.1.

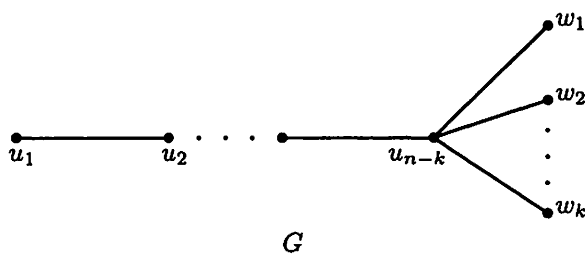

Figure 3.1

Let $x = u_1$. If $n-k=1$, then the tree G has k endvertices and x is the cutvertex. If $n-k > 1$, then the tree G has $k+1$ endvertices and x is an endvertex. In both cases, by Theorem 2.4 (i), $dm_x^+(G) = k$. □

For every connected graph G, $rad_m G \leq diam_m G$. It is showed in [3] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. It can be extended so that the upper vertex detour monophonic number can be prescribed under some conditions.

Theorem 3.4. For integers r, d and k with $2 \leq r \leq d$ and $k \geq 2$, there exists a connected graph G with $rad_m G = r$, $diam_m G = d$ and $dm_x^+(G) = k$ for some vertex x in G.

Proof. Case 1. $r = d = 2$. Let $C_4 : v_1, v_2, v_3, v_4, v_1$ be the cycle of order 4. Let G be the graph obtained from C_4 by adding $k-1$ new vertices

$w_1, w_2, \ldots, w_{k-1}$ and joining each $w_i (1 \le i \le k-1)$ with v_2 and v_4 in C_4. The graph G is shown in Figure 3.2. It is easily verified that $rad_m G = diam_m G = 2$. Now, let $x = v_1$. Since v_3 and $w_i (1 \le i \le k-1)$ are not internal vertices of any detour monophonic path starting from x, it follows that v_3 and each $w_i (1 \le i \le k-1)$ must belong to every minimal x-detour monophonic set of G. Let $S = \{v_3, w_1, w_2, \ldots, w_{k-1}\}$. Clearly S is the unique minimal x-detour monophonic set of G and so $dm_x^+(G) = |S| = k$.

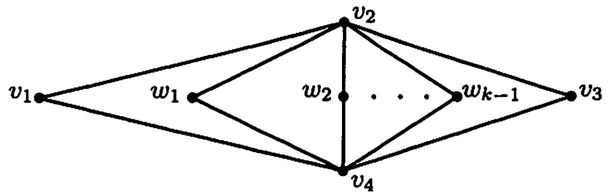

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Figure 3.2

Case 2. $2 < r = d$ or $2 \le r < d$. Let H be a graph obtained from a cycle $C_{r+2} : v_1, v_2, \ldots, v_{r+2}, v_1$ of order $r+2$ and a path $P_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ of order $d-r+1$ by identifying the vertex v_{r+1} in C_{r+2} and u_0 in P_{d-r+1}; also join each vertex $u_i (1 \le i \le d-r)$ in P_{d-r+1} with v_{r+2} in C_{r+2}. Now, let G be the graph obtained from H by adding $k-2$ new vertices $w_1, w_2, \ldots, w_{k-2}$ and join each $w_i (1 \le i \le k-2)$ with v_2 and v_{r+2} in H. The graph G is shown in Figure 3.3.

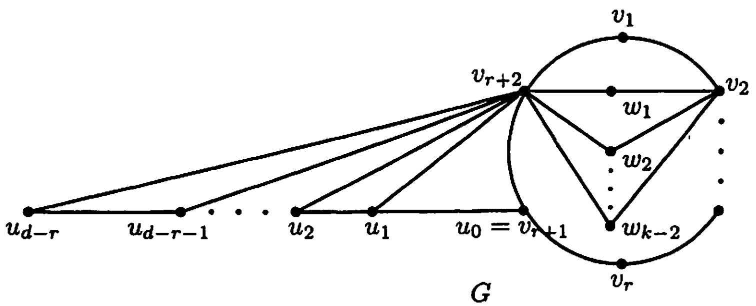

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Figure 3.3

It is easily verified that $r \le e_m(x) \le d$ for any vertex x in G. Also $e_m(v_{r+2}) = r$ and $e_m(v_1) = d$. It follows that $rad_m G = r$ and $diam_m G = d$. Now, let $x = u_{d-r}$ and let $S = \{v_1, v_{r+2}, w_1, w_2, \ldots, w_{k-2}\}$. Since every vertex of G lies on an x-y, where $y \in S$, detour monophonic path, S is an x-detour monophonic set of G. Suppose that S_1 is a proper subset of S such that S_1 is an x-detour monophonic set of G. Then there exists a vertex z in

S such that $z \notin S_1$. It is clear that z may be v_1 or v_{r+2} or w_i $(1 \le i \le k-2)$. In all cases, z does not lie on any x-u, where $u \in S_1$, detour monophonic path, it follows that S_1 is not an x-detour monophonic set of G. This shows that S is a minimal x-detour monophonic set of G and so $dm_x^+(G) \ge k$. Also, it is clear that any minimal x-detour monophonic set contains at most k elements and hence $dm_x^+(G) \le k$. Therefore, $dm_x^+(G) = k$. □

Since every minimum x-detour monophonic set is a minimal x-detour monophonic set, we have $1 \le dm_x(G) \le dm_x^+(G) \le n-1$. In view of this we have the following theorems.

Theorem 3.5. Let x be any vertex in a connected graph G of order $n \ge 3$. If $dm_x(G) = 1$, then $dm_x^+(G) \le n-2$.

Proof. Let $S_x = \{y\}$ be an x-detour monophonic set of G and let T_x be a minimal x-detour monophonic set of G. Then $y \ne x$. If $y \in T_x$, then $T_x = S_x$ and so $dm_x^+(G) = 1 \le n-2$. If $y \notin T_x$, then $dm_x^+(G) = |T_x| \le n-2$. □

Theorem 3.6. Let x be any vertex in a connected graph G. Then $dm_x(G) = n-1$ if and only if $dm_x^+(G) = n-1$.

Proof. Let $dm_x(G) = n-1$. Since $dm_x(G) \le dm_x^+(G) \le n-1$, we have $dm_x^+(G) = n-1$. Conversely, let $dm_x^+(G) = n-1$. Then $T = V(G) - \{x\}$ is the minimal x-detour monophonic set of G. Now, claim that $dm_x(G) = n-1$. If not, then G has a minimum x-detour monophonic set T_1 with $|T_1| \le n-2$. Since x is not in any minimum x-detour monophonic set, T_1 is a proper subset of T and so T is not a minimal x-detour monophonic set of G, which is a contradiction. □

Theorem 3.7. For any three positive integers j, k and l with $2 \le j \le k \le l \le n-7$, there exists a connected graph G of order n with $dm_x(G) = j$, $dm_x^+(G) = l$ and a minimal x-detour monophonic set of cardinality k.

Proof. Case 1. $2 \le j = k = l \le n-7$. Let G be a tree of order $n \ge 9$ with k endvertices. Then for any cutvertex x in G, by Theorems 1.2 and 2.4, $dm_x(G) = dm_x^+(G) = k$ and the set of all endvertices in G is a minimal x-detour monophonic set with cardinality k by Theorem 2.3.
Case 2. $2 \le j = k < l \le n-7$. Let G be the graph obtained from the cycle $C_{n-l+1} : v_1, v_2, \ldots, v_{n-l+1}, v_1$ of order $n-l+1$ by adding $l-1$ new vertices $w_1, w_2, \ldots, w_{j-1}, u_1, u_2, \ldots u_{l-j}$ and joining each w_i $(1 \le i \le j-1)$ with v_{n-l}; and also join every u_i $(1 \le i \le l-j)$ with v_1 and v_{n-l}. Then the graph G has order n and is shown in Figure 3.4.

Let $S = \{w_1, w_2, \ldots, w_{j-1}\}$ be the set of all extreme vertices of G and let $x = v_1$. First, we show that $dm_x(G) = j$. By Theorem 1.1, every minimum x-detour monophonic set of G contains S. Since S is not an x-detour monophonic set of G, $S_1 = S \cup \{v_3\}$ is a minimum x-detour monophonic set of G so that $dm_x(G) = |S_1| = j$. Also, S_1 is a minimal x-detour monophonic set of cardinality $k = j$.

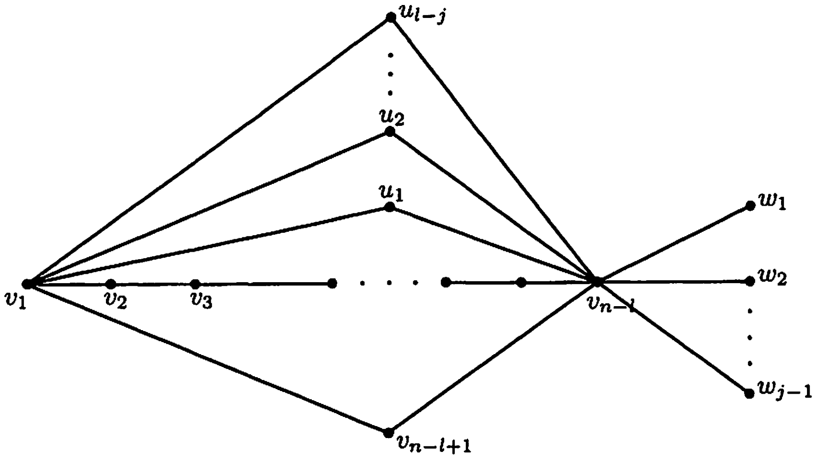

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Figure 3.4

Next, we show that $dm_x^+(G) = l$. It is clear that $M = \{w_1, w_2, \ldots, w_{j-1}, v_{n-l+1}, u_1, u_2, \ldots, u_{l-j}\}$ is a minimal x-detour monophonic set of G with maximum cardinality and so $dm_x^+(G) = |M| = l$.

Case 3. $2 \leq j < k = l \leq n - 7$. For the graph G given in Figure 3.4, $dm_x(G) = j$, $dm_x^+(G) = l$ and $M = \{w_1, w_2, \ldots, w_{j-1}, v_{n-l+1}, u_1, u_2, \ldots, u_{l-j}\}$ is a minimal x-detour monophonic set of cardinality $k = l$.

Case 4. $2 \leq j < k < l \leq n - 7$. Let $C_6 : y_1, y_2, y_3, y_4, y_5, y_6, y_1$ be the cycle of order 6 and $P_{n-l-4} : v_1, v_2, \ldots, v_{n-l-4}$ be the path of order $n - l - 4 \geq 3$. Let H be the graph obtained from C_6 and P_{n-l-4} by joining y_1 in C_6 with v_1 in P_{n-l-4}. Let G be the graph obtained from H by adding $l - 2$ new vertices $w_1, w_2, \ldots, w_{j-2}, u_1, u_2, \ldots, u_{k-j+1}, z_1, z_2, \ldots, z_{l-k-1}$ and joining each $w_i (1 \leq i \leq j - 2)$ with y_1 in H; also join each $u_i (1 \leq i \leq k - j + 1)$ with y_1 and v_{n-l-4} in H; and join each $z_i (1 \leq i \leq l - k - 1)$ with y_1 and y_3 in H. Then the graph G has order n and is shown in Figure 3.5.

Let $S = \{w_1, w_2, \ldots, w_{j-2}\}$ be the set of all extreme vertices of G and let $x = v_{n-l-4}$. Then by Theorem 1.1, every x-detour monophonic set of G contains S and also for any vertex $y \in V(G) - S$, $S \cup \{y\}$ is not an x-detour monophonic set of G. It is clear that $S_1 = S \cup \{y_4, v_{n-l-6}\}$ is a minimum x-detour monophonic set of G and so $dm_x(G) = |S_1| = j$.

Next, we show that there is a minimal x-detour monophonic set of cardinality k. Let $T = S \cup \{y_4, u_1, u_2, \ldots, u_{k-j+1}\}$. It is clear that T is an x-detour monophonic set of G. We claim that T is a minimal x-detour monophonic set of G. Assume that T is not a minimal x-detour monophonic set of G. Then there is a proper subset T_1 of T such that T_1 is an x-detour monophonic set of G. Let $t \in T$ and $t \notin T_1$. By Theorem 2.3, clearly $t = y_4$ or $t = u_i$ for some $i = 1, 2, \ldots, k - j + 1$. If $t = y_4$, then each $y_i (2 \leq i \leq 6)$ and $z_i (1 \leq i \leq l - k - 1)$ does not lie on any x-y detour monophonic path for some $y \in T_1$, which is a contradiction. If $t = u_i$ for some $i = 1, 2, \ldots, k - j + 1$, then u_i does not lie on any x-y detour monophonic path for some $y \in T_1$, it follows that T_1 is not an x-detour monophonic set of G, which is a contradiction. Thus T is a minimal x-detour monophonic set of G with cardinality k.

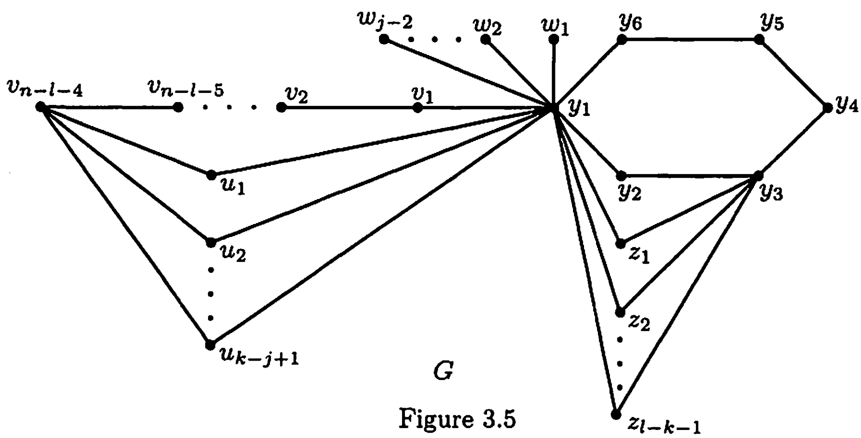

Figure 3.5

Finally, we claim that $dm_x^+(G) = l$. let $W = S \cup \{u_1, u_2, \ldots, u_{k-j+1}, y_2, y_3, z_1, z_2, \ldots, z_{l-k-1}\}$. It is clear that W is an x-detour monophonic set of G. We claim that W is a minimal x-detour monophonic set of G. Assume that W is not a minimal x-detour monophonic set of G. Then there exists a proper subset W_1 of W such that W_1 is an x-detour monophonic set of G. Let $w \in W$ and $w \notin W_1$. By Theorem 2.3, $w \neq w_i$ for all $i = 1, 2, \ldots, j-2$. If $w = y_3$, then y_i $(3 \leq i \leq 6)$ does not lie on any x-z detour monophonic path for some $z \in W_1$, which is a contradiction. If $w = u_i$ for some $i = 1, 2, \ldots, k - j + 1$, then for convenience, let $w = u_1$. Since u_1 does not lie on any x-z detour monophonic path for some $z \in W_1$, it follows that W_1 is not an x-detour monophonic set of G, which is a contradiction. If $w = y_2$ or $w = z_i$ for some $i = 1, 2, \ldots, l-k-1$, then similar to the above argument, W_1 is not an x-detour monophonic set of G, which is a contradiction. Thus W is a minimal x-detour monophonic set of G and so $dm_x^+(G) \geq |W| = l$.

Also, it is clear that every minimal x-detour monophonic set contains at most l elements and hence $dm_x^+(G) \leq l$. Therefore, $dm_x^+(G) = l$. □

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