A stronger relation between coloring number and the modification of Randić index*

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Abstract

The coloring number $col(G)$ of a graph G is the smallest number k for which there exists a linear ordering of the vertices of G such that each vertex is preceded by fewer than k of its neighbors. It is well known that $\chi(G) \leq col(G)$ for any graph G, where $\chi(G)$ denotes the chromatic number of G. The Randić index $R(G)$ of a graph G is defined as the sum of the weights $\frac{1}{\sqrt{d(u)d(v)}}$ of all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G. We show that $\chi(G) \leq col(G) \leq 2R'(G) \leq R(G)$ for any connected graph G with at least one edge, and $col(G) = 2R'(G)$ if and only if G is a complete graph with some pendent edges attaching to its same vertex, where $R'(G)$ is a modification of Randić index, defined as the sum of the weights $\frac{1}{\max\{d(u),d(v)\}}$ of all edges uv of G. This strengths a relation between Randić index and chromatic number by Hansen et al. [7], a relation between Randić index and coloring number by Wu et al. [17] and extends a theorem of Deng et al. [2].

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1 Introduction

All graphs considered in this paper are finite and simple. Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E. As usual, $N(v)$ denotes the set of neighbours of a vertex v in G, and $d(v) = |N(v)|$ is the

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degree of v. $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of G, respectively.

A k-coloring of G is a mapping $c: V \rightarrow C$, such that $c(u) \neq c(v)$ for any $uv \in E$, where C is a set of k colors. The chromatic number of G, denoted by $\chi(G)$, is the smallest integer k for which G has a k-coloring.

The chromatic number of G, denoted by $\chi(G)$, is the smallest number of colors needed to color all vertices of G such that no pair of adjacent vertices gets the same color.

The coloring number of G, denoted by $col(G)$, is the smallest integer k such that G has a vertex ordering in which each vertex is preceded by fewer than k of its neighbors.

The degeneracy of G, denoted by $deg(G)$, is defined as $deg(G) = \max\{\delta(F) : F \text{ is a subgraph of } G\}$. It is known in [8] that

$$col(G) = deg(G) + 1. \tag{1}$$

for any graph G.

As an extension of coloring of a graph, Vizing [16] and Erdös et al. [4] introduced a list coloring of a graph. For each vertex v of a graph G, let $L(v)$ denote a list of colors assigned to v. A list coloring is a coloring c of vertices of G such that $c(v) \in L(v)$ and $c(x) \neq c(y)$ for any $xy \in E$, where $v, x, y \in V$. A graph G is k-list-colorable if for any list assignment L to each vertex $v \in V$ with $|L(v)| \geq k$, there always exists a list coloring c of G. The list chromatic number of G, denoted by $\chi_l(G)$, is the minimum k for which G is k-list-colorable. It can be found in Tuza [15] that the well-known Brooks theorem on chromatic number was extended as follows

$$\chi(G) \leq \chi_l(G) \leq col(G) \leq \Delta(G) + 1. \tag{2}$$

The Randić index (or molecular connectivity index) of G is defined as $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}$. It was proposed by the chemist Milan Randić [14] for correlating macroscopic properties with graph theoretical invariants of the structural formula of the organic molecule. Many applications and mathematical properties of this descriptor have also been studied extensively in [6, 10, 11, 12, 13]. The harmonic index of G, denoted by $H(G)$, is defined in [5] as $H(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)}$. A modification $R'(G)$ of the Randić index was introduced by Dvořák et al. [3]. It is defined as $R'(G) = \sum_{uv \in E(G)} \frac{1}{\max\{d(u),d(v)\}}$.

Note that $\sqrt{d(u)d(v)} \leq \frac{d(u)+d(v)}{2} \leq \max\{d(u), d(v)\}$, we have

$$R'(G) \leq H(G) \leq R(G). \tag{3}$$

There are some relations relating the above invariants. First, Hansen et al. [7] proved that a relation between Randić index $R(G)$ and the chromatic number $\chi(G)$ conjectured by the system AutoGraphiX.

Theorem 1. *[7] Let G be a simple graph with the chromatic number $\chi(G)$ and the Randić index $R(G)$, then $\chi(G) \leq 2R(G)$ with equality if and only if G is a complete graph possibly with some additional isolated vertices.*

In [2] and [9], Deng et al. and Jiang et al. obtained the relations relating the harmonic index $H(G)$, the modified Randić index $R'(G)$ and the chromatic number $\chi(G)$, respectively. They strengthen Theorem 1 from the relation (3).

Theorem 2. *[2] Let G be a simple graph with the chromatic number $\chi(G)$ and the harmonic index $H(G)$, then $\chi(G) \leq 2H(G)$ with equality if and* *only if G is a complete graph possibly with some additional isolated vertices.*

Theorem 3. *[9] Let G be a simple graph with the chromatic number $\chi(G)$ and the modified Randić index $R'(G)$, then $\chi(G) \leq 2R'(G)$ with equality if and only if G is a complete graph possibly with some additional isolated vertices.*

Recently, Wu et al. [17] established a relation between the Randić index and the coloring number of a graph, and extended Theorem 1 from the relation (2).

Theorem 4. *[17] If G is a simple graph with at least one edge, then $col(G) \leq 2R(G)$, with equality if and only if G is a complete graph, possibly with some additional isolated vertices.*

Moreover, Deng et al. [1] obtained a relation between the coloring number and the harmonic index of a graph and extended Theorem 4.

Theorem 5. *[1] If G is a simple graph with at least one edge, then $col(G) \leq 2H(G)$, with equality if and only if G is a complete graph, possibly with some additional isolated vertices.*

The aim of this paper is to extend further Theorems 1-5 as follows.

Theorem 6. *If G is a simple graph with at least one edge, then $col(G) \leq 2R'(G)$, with equality if and only if G is a complete graph K_r with t pendent edges attaching to the same vertex u of K_r, possibly some additional isolated vertices.*

Note that $\sqrt{d(u)d(v)} < \frac{d(u)+d(v)}{2} < \max\{d(u), d(v)\}$ if $d(u) \neq d(v)$, we have $R'(G) < H(G) < R(G)$ for any graph except the graph in which each component is regular. So, Theorem 6 is stronger than Theorems 1-5. And also, the following corollary can be easily deduced from Theorem 6 and the relation (2).

Corollary 7. *If G is a simple graph with at least one edge, then*

$$\chi_l(G) \leq 2R'(G)$$

with equality if and only if G is a complete graph K_r with t pendent edges attaching to the same vertex u of K_r, possibly some additional isolated vertices.

2 The proof of Theorem 6

The proof of Theorem 6 is based on the following result. For completeness, we rewrite its proof.

Lemma 8. *[9] Let G be a simple graph with the modified Randić index $R'(G)$ and the minimum degree $\delta \geq 1$. If v_0 is a vertex of G with degree equal to δ, then*

$$R'(G) - R'(G - v_0) \geq 0.$$

Proof. Let $N(v_0) = \{v_1, v_2, \cdots, v_\delta\}$ and d_i the degree of vertex v_i. m_i is the number of vertices in $N(v_i) - \{v_0\}$ with degree less than d_i, where

$0 \le m_i \le d_i - 1$. We have

$$\begin{aligned} R'(G) - R'(G - v_0) = & \sum_{i=1}^{\delta} \tfrac{1}{d_i} + \sum_{i=1}^{\delta} (\tfrac{m_i}{d_i} - \tfrac{m_i}{d_i - 1}) \\ = & \sum_{i=1}^{\delta} (\tfrac{m_i + 1}{d_i} - \tfrac{m_i}{d_i - 1}) = \sum_{i=1}^{\delta} \tfrac{d_i - m_i - 1}{d_i(d_i - 1)} \ge 0. \end{aligned}$$

□

Proof of Theorem 6. First, if G is a complete graph K_r with t pendent edges attaching to the same vertex u of K_r, possibly some additional isolated vertices, then $col(G) = deg(G) + 1 = r$ and $R'(G) = \frac{r}{2}$ for $r \ge 2$, and $col(G) = deg(G) + 1 = 2$ and $R'(G) = 1$ for $r = 1, t > 0$, otherwise $G = K_1$. So, we have $col(G) = 2R'(G)$.

Let n be the order of G. Since $deg(G) = max\{\delta(F) : F$ is a subgraph of $G\} \ge \delta(G)$, we consider two cases.

Case 1. $deg(G) = \delta(G)$.

In this case, $deg(G) = \delta(G)$ and $col(G) = \delta(G) + 1$ by inequality (1). Since $|E(G)| \ge \frac{\Delta(G) + (n-1)\delta(G)}{2}$,

$$\begin{aligned} R'(G) = & \sum_{uv \in E(G)} \frac{1}{\max\{d(u), d(v)\}} \ge \sum_{uv \in E(G)} \frac{1}{\Delta(G)} \\ \ge & \frac{\Delta(G) + (n-1)\delta(G)}{2} \times \frac{1}{\Delta(G)} \end{aligned} \tag{4}$$

and thus

$$2R'(G) \ge \frac{\Delta(G) + (n-1)\delta(G)}{\Delta(G)} = \frac{n-1}{\Delta(G)}\delta(G) + 1 \ge \delta(G) + 1 = col(G)$$

with equality if and only if G is Δ-regular by (4) and $\Delta = n - 1$ by the last inequality of the above displayed formula. Therefore, G is the complete graph of order n.

Case 2. $deg(G) > \delta(G)$. We prove by induction on n in this case.

If u is an isolated vertex of G, then $R'(G-u) = R'(G)$ and $col(G-u) = col(G)$. By the the inductive assumption, $col(G-u) \leq 2R'(G-u)$ with equality if and only if $G-u$ is a complete graph with some pendent edges attaching to the same vertex, possibly some additional isolated vertices. So, $col(G) \leq 2R'(G)$ with equality if and only if G is a complete graph with some pendent edges attaching to the same vertex, possibly some additional isolated vertices.

Now, we assume that G has no isolated vertices. Let $v \in V(G)$ with $d(v) = \delta(G) \geq 1$. Since $deg(G) > \delta(G)$, no subgraph G' attaining $\delta(G') = deg(G)$ can contain v, and

$$\begin{aligned} deg(G) =& max\,\{\delta(F) : F \text{ is a subgraph of } G\} \\ =& max\,\{\delta(F) : F \text{ is a subgraph of } G-v\} \\ =& deg(G-v). \end{aligned}$$

By the equation (1), we have $col(G) = deg(G) + 1 = deg(G-v) + 1 = col(G-v)$. By Lemma 8, we have $R'(G-v) \leq R'(G)$. By the inductive assumption, $col(G-v) \leq 2R'(G-v)$. So,

$$col(G) = col(G-v) \leq 2R'(G-v) \leq 2R'(G). \tag{5}$$

If $col(G-v) < 2R'(G-v)$, then we have $col(G) < 2R'(G)$.

If $col(G-v) = 2R'(G-v)$, then, by the inductive assumption, $G-v$ is a complete graph K_r with t pendent edges $uu_1, uu_2, \cdots, uu_t$ attaching to the same vertex u of K_r, possibly some additional isolated vertices.

Subcase 1. If $t = 0$, i.e., $G-v$ is a complete graph K_r with $n-r-1$ isolated vertices, where $1 \leq r \leq n-1$, then G is obtained from $G-v$ by connecting $\delta(G)$ edges from v to K_{n-1} or $G = K_{n-2} \cup K_2$, since G has no isolated vertices and v is a vertex with the minimum degree of G.

(i) If G is a graph obtained by connecting $\delta(G)$ edges from v to K_{n-1}, then

$$\begin{aligned}
R'(G) =& \frac{(n-1-x)(n-2-x)}{2} \times \frac{1}{n-2} + x(n-1-x) \times \frac{1}{n-1} \\
&+ \frac{x(x-1)}{2} \times \frac{1}{n-1} + x \times \frac{1}{n-1} \\
=& \frac{(n-1-x)(n-2-x)}{2n-4} + \frac{2xn - x^2 - x}{2n-2}
\end{aligned}$$

where $x = d(v) = \delta(G)$, $1 \leq x \leq n-2$ since $deg(G) > \delta(G)$. And

$$\begin{aligned}
R'(G) - R'(G-v) = R'(G) - R'(K_{n-1}) &= R'(G) - \frac{n-1}{2} \\
&= \frac{x(x-1)}{2(n-2)(n-1)}.
\end{aligned}$$

If $1 < x \leq n-2$, then $R'(G) > R'(G-v)$ and $col(G) < 2R'(G)$ from (5).

If $x = 1$, then G is obtained by attaching a pendent edge to the complete graph K_{n-1}, and it is easy to check that $col(G) = n-1 = 2R'(G)$.

(ii) If $G = K_{n-2} \cup K_2$, then $R'(G) = R'(K_{n-2}) + R'(K_2) > R'(K_{n-2}) = R'(G-v)$. So, $col(G) < 2R'(G)$ by the equation (5).

Subcase 2. If $t \geq 1$, i.e., $G - v$ is a complete graph K_r with t pendent edges $uu_1, uu_2, \cdots, uu_t$ attaching to the same vertex u of K_r, and $n-r-t$ isolated vertices, then $r + t \geq 3$. Note that G has no isolated vertices and v is a vertex with the minimum degree of G, then $d(v) = \delta(G) = 1$ or 2 since $t \geq 1$, and G is one of the seven graphs are described in Figure 1.

If $G = G_1$, then G is a complete graph K_r with $t+1$ pendent edges attaching to the same vertex u, and $col(G) = 2R'(G) = r$.

If $G = G_7$ and $r = 2$, then G is a complete graph K_3 with one pendent edge attaching to the vertex u, and $col(G) = 2R'(G) = 3$; If $G = G_7$ and

$r \geq 3$, we have $R'(G_7) - R'(G_7 - v) = \frac{1}{2}$, and $col(G_7) < 2R'(G_7)$ by the equation (5).

For $2 \leq i \leq 6$, we have $R'(G_i) - R'(G_i - v) > 0$, and $col(G_i) < 2R'(G_i)$ by the equation (5).

The proof of the theorem is completed. □

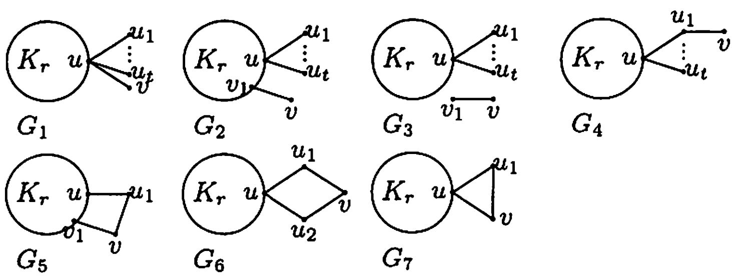

Figure 1. Graphs in the proof of Theorem 6.

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