Periodicity of A Partition Function Related to Making Change Modulo Prime Powers

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Abstract

Let $p_c(n)$ be the number of ways to make change for n cents using pennies, nickels, dimes, and quarters. By manipulating the generating function for $p_c(n)$, we prove that the sequence $\{p_c(n) \pmod{\ell^j}\}$ is periodic for every prime power ℓ^j.

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1. Introduction

For a certain amount of money, how many ways are there to make change using pennies, nickels, dimes, and quarters? This problem was popularized by Polya [5] in 1956 where half-dollars are included. Let m be the number of money, p the number of pennies, n the number of nickels, d the number of dimes, and q the number of quarters. The problem is to find solutions to $m = p * 1 + n * 5 + d * 10 + q * 25$. Let $p_c(n)$ be the number of ways to make change for n cents using pennies, nickels, dimes, and quarters. Graham, Knuth, and Patashnik [3] showed that the problem can be solved by writing the generating function for $p_c(n)$ as a product of known closed formulas for other series. Recently, following this method, Costello and Osborne [2] established the generating function and a closed formula for $p_c(n)$. Moreover, based on a recurrence for $p_c(5n)$, they proved that the parity of the sequence $\{p_c(n)\}$ is periodic, and that the period length is 200.

The main purpose of the present paper is to study periodicity of $\{p_c(n)\}$ modulo powers of a prime. We will prove a simpler recurrent formula for $p_c(5n)$. For a prime power ℓ^j, we show that the sequence $\{p_c(n) \pmod{\ell^j}\}$ is periodic and $p_c(n) \pmod{\ell^j}$ are the coefficients in anti-reciprocal polynomials (see the last section for definitions). As consequences, we extend the results obtained by Costello and Osborne in [2]. We remark that, in contrast to the recurrences for $p_c(5n)$ used in [2], the generating function for $p_c(n)$ plays a crucial role in our proofs.

2. The generating function and recurrences for $p_c(n)$

The method to find the generating function for $p_c(n)$ was suggested by Graham, Knuth and Patashnik in [3]. Following this, Costello and Osborne proved in [2] that the generating function for $p_c(n)$ is

$$\sum_{n=0}^{\infty} p_c(n) z^n = \frac{1}{1-z} \cdot \frac{1}{1-z^5} \cdot \frac{1}{1-z^{10}} \cdot \frac{1}{1-z^{25}}, \tag{1}$$

where $|z| < 1$. We adopt the convention that $p_c(n) = 0$ if $n < 0$. Using Sage, we illustrate the first 60 values of $p_c(n)$ as follow:

Table 1. Values of $p_c(n)$ for $0 \le n \le 59$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p_c(n)$	1	1	1	1	1	2	2	2	2	2	4	4	4	4	4

15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
6	6	6	6	6	9	9	9	9	9	13	13	13	13	13

30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
18	18	18	18	18	24	24	24	24	24	31	31	31	31	31

45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
39	39	39	39	39	49	49	49	49	49	60	60	60	60	60

From Table 1, it is easy to find that for $0 \le n \le 11$,

$$p_c(5n) = p_c(5n+1) = p_c(5n+2) = p_c(5n+3) = p_c(5n+4). \tag{2}$$

Indeed, this is true for all $n \ge 0$. Because after making change for $5n$, you must use pennies for the remaining $1, 2, 3, 4$ cents. Note that the identity (2) is mentioned in [2] without mathematical proofs. Here, using the generating function for $p_c(n)$, we give two proofs of this fact.

The first proof is based on the following observation from (1):

$$\sum_{n=0}^{\infty}(p_c(n) - p_c(n-1))z^n = (1-z)\sum_{n=0}^{\infty} p_c(n) z^n = \frac{1}{1-z^5} \cdot \frac{1}{1-z^{10}} \cdot \frac{1}{1-z^{25}}.$$

Since the expansion of the right hand side of the identity has the form $\sum_n c_n z^{5n}$, we conclude that $p_c(n) - p_c(n-1) = 0$ for all n coprime to 5. This implies (2).

For the second proof, we define an operator $U_5 : \mathbb{Z}[[z]] \to \mathbb{Z}[[z]]$ as follows:

$$\left(\sum_{n=0}^{\infty} a_n z^n\right) |U_5 := \sum_{n=0}^{\infty} a_{5n} z^n.$$

It is easy to see that

$$\left(\frac{1}{1-z}\right) |U_5 = \left(\sum_{n=0}^{\infty} z^n\right) |U_5 = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

For $i = 1, 2, 3, 4, 5$, we observe that

$$\left(\frac{z^i}{1-z}\right) |U_5 = \left(\sum_{n=i}^{\infty} z^n\right) |U_5 = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

By (1.1) of [1], we get

$$\left(\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^{5n}\right) |U_5 = \sum_{n=0}^{\infty} a_{5n} z^n \cdot \sum_{n=0}^{\infty} b_n z^n.$$

Now multiplying z^i on both sides of (1) and using U_5, we have

$$\left(\frac{z^i}{1-z} \cdot \frac{1}{1-z^5} \cdot \frac{1}{1-z^{10}} \cdot \frac{1}{1-z^{25}}\right) |U_5 = \frac{z}{1-z} \cdot \frac{1}{1-z} \cdot \frac{1}{1-z^2} \cdot \frac{1}{1-z^5}.$$

On the other hand,

$$\left(\sum_{n=0}^{\infty} p_c(n) z^{n+i}\right) |U_5 = \sum_{n=1}^{\infty} p_c(5n-i) z^n.$$

It follows that for $i = 1, 2, 3, 4, 5$,

$$\sum_{n=1}^{\infty} p_c(5n-i) z^n = \frac{z}{1-z} \cdot \frac{1}{1-z} \cdot \frac{1}{1-z^2} \cdot \frac{1}{1-z^5}. \tag{3}$$

This proves (2).

In view of (2), to determine the values of $p_c(n)$, it suffices to compute $p_c(5n)$. In section 5 of [2], Costello and Osborne proved a recurrence for $p_c(5n)$:

$$\begin{aligned} p_c(5n) &= 2p_c(5n-5) - 2p_c(5n-15) + p_c(5n-20) + p_c(5n-25) \\ &\quad - 2p_c(5n-30) + 2p_c(5n-40) - p_c(5n-45). \end{aligned}$$

This recurrence is crucial for their proof of the periodicity of the parity of $p_c(n)$. Here we give a shorter recurrence for $p_c(5n)$. We remark that, following the arguments of section 7 of [2], we can also prove $p_c(n)$ modulo 2 is periodic. In section 3, we will give a simple proof of the periodicity of the parity of $p_c(n)$ independent on this recurrence. Our recurrence states that

$$(4)\quad p_c(5n) = n + 1 + p_c(5n-10) + p_c(5n-25) - p_c(5n-35), \quad n \geq 0.$$

To prove this, we take $i = 5$ in (3) and obtain

$$\sum_{n=1}^{\infty} p_c(5n-5)z^n = \frac{z}{(1-z)^2(1-z^2)(1-z^5)}.$$

Multiplying by $(1-z^2)(1-z^5)$ on both sides, we get

$$\sum_{n=1}^{\infty}(p_c(5n-5) - p_c(5n-15) - p_c(5n-30) + p_c(5n-40)z^n = \frac{z}{(1-z)^2}.$$

Since

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n,$$

we deduce that for $n \geq 1$,

$$p_c(5n-5) - p_c(5n-15) - p_c(5n-30) + p_c(5n-40) = n.$$

This yields the desired recurrence (4).

We remark that the recurrence (4) holds for all $n \geq 0$, and with the aid of (2), all values of $p_c(n)$ are determined.

3. Periodicity of $p_c(n)$ modulo 2

The periodicity of the sequence $\{p_c(n) \pmod 2\}$ was proved in [2] by recurrence. In this section, employing the generating function, we give a simple proof. Moreover, we find that $p_c(n) \pmod 2$ possesses certain symmetrical properties.

Firstly, we show $\{p_c(n) \pmod 2\}$ is periodic with period length 200. By (1) we find that

$$(5)\quad \frac{1-z^{200}}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} = \sum_{n=0}^{\infty}(p_c(n) - p_c(n-200))z^n.$$

It is easy to verify that $(1-z^{25})^8 \equiv (1-z^{200}) \pmod 2$ by the binomial theorem. Hence

(6)
$$\frac{1-z^{200}}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} \equiv \frac{(1-z^{25})^8}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} \pmod 2.$$

Since the right hand side of (6) is a polynomial of degree 159, it follows by (5) that for $n > 159$,

(7)
$$p_c(n) - p_c(n-200) \equiv 0 \pmod 2.$$

This shows that $p_c(n) \pmod 2$ has period 200.

Using the recurrence (4), we compute $p_c(5n) \pmod 2$ for $0 \le n \le 39$ to observe an additional interesting property mod 2.

Table 2. Values of $p_c(5n) \pmod 2$ for $0 \le n \le 39$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
$p_c(5n) \pmod 2$	1	0	0	0	1	1	0	0	1	1	1	0	1	1	1	1	1	1	1	1	0	1	1	1	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0

Observing the entries in Table 2, we find that the values of $p_c(5n) \pmod 2$ indicate an interesting symmetrical property. In particular,

(8)
$$p_c(5(31-n)) \equiv p_c(5n) \pmod 2, \quad 0 \le n \le 31.$$

[Note that because of the zeroes from 32 to 39 and (2), we have $p_c(n) \pmod 2$ for all $160 \le n \le 199$.] To prove the congruence (8), we denote by $h(z)$ the right hand side of (6). Then $h(z)$ is a polynomial of degree 159, and we may write

(9)
$$h(z) := \frac{(1-z^{25})^8}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} = \sum_{i=0}^{159} a_i z^i.$$

Now we have, on the one hand,

$$\begin{aligned} z^{159}h\left(\frac{1}{z}\right) &= \frac{z^{159}(1-\frac{1}{z^{25}})^8}{(1-\frac{1}{z})(1-\frac{1}{z^5})(1-\frac{1}{z^{10}})(1-\frac{1}{z^{25}})} \\ &= \frac{(z^{25}(1-\frac{1}{z^{25}}))^8}{z(1-\frac{1}{z})\cdot z^5(1-\frac{1}{z^5})\cdot z^{10}(1-\frac{1}{z^{10}})\cdot z^{25}(1-\frac{1}{z^{25}})} \\ &= -\frac{1-z^{200}}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} \\ &= -h(z). \end{aligned}$$

On the other hand,

$$z^{159}h\left(\frac{1}{z}\right) = \sum_{i=0}^{159} a_{159-i}z^i.$$

Thus for $0 \le n \le 159$,

$$a_n = -a_{159-n}.$$

Since $p_c(n) = 0$ for $n < 0$, by (5), (6) and (9), we get

$$p_c(n) \equiv a_n \pmod 2.$$

Hence we conclude that for $0 \le n \le 159$,

$$p_c(n) \equiv p_c(159-n) \pmod 2.$$

In particular, by (2) we establish the observation (8).

Combining (7), we deduce that for $0 \le n \le 79$ and any integer $k \ge 0$,

$$p_c(n) \equiv p_c(159-n) \equiv p_c(n+200k) \equiv p_c(159-n+200k) \pmod 2.$$

4. Periodicity of $\{p_c(n)\}$ modulo powers of a prime

In this section, ℓ is denoted by a prime. We shall show that $\{p_c(n)\}$ is periodic modulo any powers of ℓ.

Lemma 4.1. *The sequence $\{p_c(n) \pmod \ell\}$ is periodic. Moreover, let $L(\ell)$ be the period length. Then $L(2)=200$, $L(3)=450$ and $L(\ell) = 50\ell$ for $\ell \ge 5$.*

Proof. We have

$$\frac{1-z^d}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} = \sum_{n=0}^{\infty}(p_c(n)-p_c(n-d))z^n.$$

It follows that $\{p_c(n) \pmod{\ell}\}$ is periodic if and only if the left hand side is a polynomial in z, and the smallest d is $L(\ell)$.

For any integers $\alpha \geq 0$ and $\beta \geq 1$, the binomial theorem gives

$$(1-z^\beta)^{\ell^\alpha} = (1+(-1)^{\ell^\alpha} z^{\beta\ell^\alpha}) + \sum_{i=1}^{\ell^\alpha - 1} \binom{\ell^\alpha}{i} (-z^\beta)^i.$$

Note that $\binom{\ell^\alpha}{i} \equiv 0 \pmod{\ell}$ for $1 \leq i \leq \ell^\alpha - 1$. We obtain

$$(1-z^\beta)^{\ell^\alpha} \equiv (1-z^{\beta\ell^\alpha}) \pmod{\ell}.$$

Thus

$$\frac{1-z^{50\ell^\alpha}}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} \equiv \frac{(1-z^{50})^{\ell^\alpha}}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} \pmod{\ell}.$$

Since $(1-z^i)|(1-z^{50})$ for $i = 1, 5, 10$ and 25, we deduce that if $\ell^\alpha \geq 4$, then the right hand side of the identity above is a polynomial. If α_0 is the smallest α such that $\ell^\alpha \geq 4$, then it is clear that $L(\ell) = 50\ell^{\alpha_0}$. Lemma 4.1 follows immediately.

The next lemma allows us to obtain the periodicity of $\{p_c(n)\}$ modulo powers of ℓ.

Lemma 4.2. *Let $d \geq 1$, $j \geq 1$ be integers and $f(z)$ be a polynomial. If $(1-z^d)/f(z)$ is a polynomial modulo ℓ, then $(1-z^{d\ell^j})/f(z)$ is a polynomial modulo ℓ^{j+1}.*

Proof. Let $g(z)$ be a polynomial such that

$$\frac{1-z^d}{f(z)} \equiv g(z) \pmod{\ell}. \tag{10}$$

We have

$$\frac{(1-z^d)^\ell}{f^\ell(z)} \equiv g^\ell(z) \pmod{\ell^2}.$$

Therefore

$$\frac{(1-z^d)^\ell}{f(z)} \equiv f^{\ell-1}(z) g^\ell(z) \pmod{\ell^2}. \tag{11}$$

Note that

$$(1-z^d)^\ell - (1-z^{d\ell}) = \sum_{i=1}^{\ell-1} \binom{\ell}{i} (-z^d)^i \equiv 0 \pmod{\ell(1-z^d)}.$$

Hence we can find a polynomial $h(z)$ such that

$$(1-z^d)^\ell - (1-z^{d\ell}) = \ell(1-z^d)h(z).$$

It follows from (10) and (11) that

$$\begin{aligned}\frac{1-z^{d\ell}}{f(z)} &= \frac{(1-z^d)^\ell}{f(z)} - \frac{\ell(1-z^d)h(z)}{f(z)} \\ &\equiv f(z)^{\ell-1}g^\ell(z) - \ell g(z)h(z) \pmod{\ell^2}.\end{aligned}$$

Since the right hand side is a polynomial, Lemma 4.2 follows by induction.

Now taking $d = L(\ell)$ and $f(z) = (1-z)(1-z^5)(1-z^{10})(1-z^{50})$ in Lemma 4.2, we find that $\{p_c(n) \pmod{\ell^{j+1}}\}$ is periodic. Following the arguments in section 3 for $\ell = 2, j = 0$, one can easily prove that for $0 \le n \le L(\ell)\ell^j - 41$,

$$p_c(n) \equiv -p_c(L(\ell)\ell^j - 41 - n) \pmod{\ell^{j+1}}.$$

We omit the details here. According to [4], a polynomial $P(z) = a_n z^n + a_{n-1}z^{z-1} + \cdots + a_1 z + a_0$ of degree n is called anti-reciprocal if for each $0 \le i \le n$, $a_i = -a_{n-i}$. Hence the $p_c(n) \pmod{\ell^{j+1}}$ values are coefficients in anti-reciprocal polynomials. In conclusion, we establish the following

Main theorem. *For any prime powers ℓ^j, the sequence $\{p_c(n) \pmod{\ell^j}\}$ is periodic and the $p_c(n) \pmod{\ell^j}$ values are coefficients in anti-reciprocal polynomials.*

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