Some Operations on Fuzzy Hypergraphs *

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Abstract Hypergraph is a useful tool to model complex systems and it could be considered as a natural generalizations of graphs. In this paper, we define some operation of fuzzy hypergraphs and strong fuzzy $r-$uniform hypergraphs, such as Cartesian product, strong product, normal product, lexicographic product, union, join and we proved if hypergraph H is formed by one of these operations, then this hypergraph is fuzzy hypergraph or strong fuzzy $r-$uniform hypergraph. Finally, we discuss an application of fuzzy hypergraphs.

Keywords : Fuzzy hypergraphs; Strong fuzzy hypergraphs; product; union; join

1 Introduction

In 1965, Zadeh [27] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, signal processing, multi-agent systems, pattern recognition, robotics, computer networks, expert systems, decision making and automata theory.

In 1975, Rosenfeld [21] introduced the concept of fuzzy graphs. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya [7] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [17]. Shan-

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non and Atanassov [22] introduced the concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs, and investigated some of their properties in [23]. Parvathi et al. defined operations on intuitionistic fuzzy graphs in [20]. Recently, the bipolar fuzzy graphs, interval-valued fuzzy graphs and strong intuitionistic fuzzy graphs have been discussed in [1, 2, 3, 4, 26].

Hypergraphs are natural generalizations of graphs in case of set of multi-ary relations, see [6]. It means the expansion of graph models for the modeling complex systems. In case of modeling systems with fuzzy binary and multi-ary relations between objects, transition to fuzzy hypergraphs, which combine advantages both fuzzy and graph models, is more natural. It allows to realize formal optimization and logical procedures. However, using of the fuzzy graphs and hypergraphs as the models of various systems (social, economic systems, communication networks and others) leads to difficulties. Lee-kwang and Lee [16] generalized and redefined the concept of fuzzy hypergraphs whose basic idea was given by Kaufmann [13]. Further the concept of fuzzy hypergraphs was also discussed in [25] and [10]. Chen [9] introduced the concept of interval-valued fuzzy hypergraphs, Parvathi et al. [20] defined intuitionistic fuzzy hypergraphs. Recently, Akram and Dudek in [5] apply the concept of intuitionistic fuzzy set theory to generalize results concerning hypergraphs. In this paper, we use the definition of fuzzy hypergraphs which is proposed by Yu Bin in [25] to define some operation of fuzzy hypergraphs, such as Cartesian product, union, join of two fuzzy hypergraphs, the strong product, normal product and lexicographic product of two strong fuzzy r-uniform hypergraphs and investigate some of their important properties. Finally, we discuss application of fuzzy hypergraphs.

2 Preliminaries

In this section, we first review some definitions of undirected graphs that are necessary for this paper.

A (crisp) hypergraph is a generalized form of a graph that can have edges containing any number of vertices. A hypergraph is illustrated with $H = (V, E)$ with V and E representing the vertices and edges of the hypergraph, respectively. A hypergraph $H = (V, E)$ is called simple if no edge is contained in any other edge. A hypergraph is trivial if $|V| = 1$.

H is called k-uniform if every edge in E contains exactly k vertices. If $k = 2$, then H is a graph. Two examples of simple hypergraphs are shown in Fig 1.

Definition 2.1 A fuzzy set A defined on a non empty set X is the family $A = \{(x, \mu_A(x))|\ x \in X\}$ where $\mu_A : X \to [0, 1]$ is the membership function such that $\mu_A = 0$ if x does not belong to A, $\mu_A = 1$ if x strictly belongs to A and any intermediate value represents the degree in which x could belong to A, where $\mu_A(x) < \mu_A(x')$ indicates that the degree of membership of x to A is lower than the degree of membership of x'.

Definition 2.2[25] Let V be a finite set and let E be a finite family of nonempty set V. The fuzzy hypergraphs with underlying set V is a pair (σ, μ), where σ is a fuzzy subset of V and μ is a fuzzy subset of E such that $\forall v \in V, \mu(e) \leq$

$\bigwedge_{v \in e} \sigma(v)$. For convenience, we use $\mathcal{H} = (\sigma, \mu)$ to denote a fuzzy hypergraph. The hypergraphs $H = (V, E)$ is called the elementary hypergraph of fuzzy hypergraph $\mathcal{H} = (\sigma, \mu)$.

If $H = (V, E)$ is graph G, then $\mathcal{H} = (\sigma, \mu)$ is the fuzzy graph.

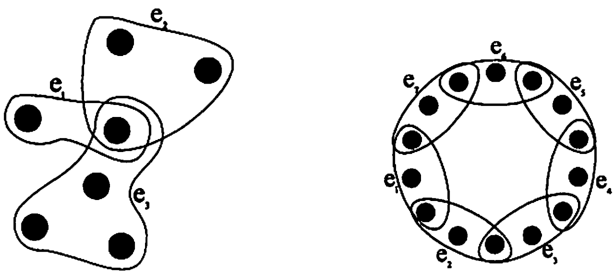

Figure 1: A simple hypergraph with 7 vertices and 3 edges and cyclic hypergraph with 14 vertices and 7 edges.

Example 2.3 Consider a hypergraph $H = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $E = \{e_1, e_2, e_3, e_4\}$, and $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_3, v_4\}$, $e_3 = \{v_4, v_5, v_6\}$, $e_4 = \{v_2, v_7\}$, such that $\sigma(v_1) = 0.7$, $\sigma(v_2) = 0.5$, $\sigma(v_3) = 0.3$, $\sigma(v_4) = 0.6$, $\sigma(v_5) = 0.5$, $\sigma(v_6) = 0.4$, $\sigma(v_7) = 0.8$, $\mu(e_1) = 0.1$, $\mu(e_2) = 0.2$, $\mu(e_3) = 0.3$, $\mu(e_4) = 0.5$, then $\mathcal{H} = (\sigma, \mu)$ is a fuzzy hypergraph. See Fig 2.

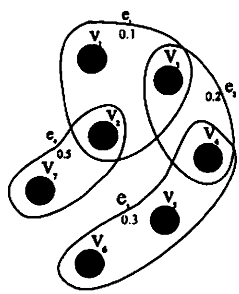

Figure 2: A fuzzy hypergraph with 7 vertices and 4 edges.

Definition 2.4 A fuzzy hypergraph is called strong fuzzy hypergraph if $\mu(e) = \bigwedge_{v \in e} \sigma(v)$.

Definition 2.5 Let $H = (V, E)$ be a hypergraph, and $\sigma : V \to [0, 1]$, $\mu : E \to [0, 1]$, suppose $0 < \lambda \leq 1$, then the λ-cut is defined by

$$\sigma_\lambda = \{v \in V \mid \sigma(v) \geq \lambda\}, \ \mu_\lambda = \{e \in E \mid \mu(e) \geq \lambda\}.$$

As shown in [11], it is possible to find several non-equivalent generalizations of the standard graph products to hypergraph products. In [12], Marc Hellmutha et al. define the Cartesian product $\Box$, the normal product $\breve{\boxtimes}$ and the strong product $\widehat{\boxtimes}$ in the following, where the latter two products can be considered as generalizations of the usual strong graph product.

In all of these three products, the vertex sets are the Cartesian products of the vertex sets of the factors:
$V(H_1 \Box H_2) = V(H_1 \breve{\boxtimes} H_2) = V(H_1 \widehat{\boxtimes} H_2) = V(H_1) \times V(H_2)$.

For an arbitrary Cartesian product $V = \times_{i=1}^{n} V_i$ of (finitely many) sets V_i, the projection $p_j : V \to V_j$ is defined by $v = (v_1, \ldots, v_n) \mapsto v_j$. We will call v_j the jth coordinate of $v \in V$. With this notation, the edge sets are defined as follows.

Cartesian product: $e \in E(H_1 \Box H_2)$ if and only if $p_i(e) \in E(H_i)$, $p_j(e) \in V(H_j)$ with $i, j \in \{1, 2\}$, $i \neq j$.

Strong product: $e \in E(H_1 \widehat{\boxtimes} H_2)$ if and only if (i) $e \in E(H_1 \Box H_2)$ or (ii) $p_i(e) \in E(H_i)$, for $i = 1, 2$ and $|e| = \max_{i=1,2}\{|p_i(e)|\}$.

Normal product: $e \in E(H_1 \breve{\boxtimes} H_2)$ if and only if (i) $e \in E(H_1 \Box H_2)$ or (ii) $p_i(e) \subseteq e_i \in E(H_i)$, for $i = 1, 2$ and $|e| = |p_i(e)| = \min_{j=1,2}\{|e_j|\}$.

In a hypergraph without defined adjacency functions, in a simple way it is considered that all the vertices of an edge will be adjacent to each other. In literature [8, 14, 18], the *Cartesian product*, *Strong product*, *Normal product* and *lexicographic product* of hypergraphs is defined in the following form that can be considered as a numeral definition rather than an algebraic one.

Definition 2.6 Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. The *Cartesian product* of H_1 and H_2 is the hypergraph $H_1 \Box H_2$ with set of vertices $V_1 \times V_2$ and set of edges:
$E_1 \Box E_2 = \{\{v_1\} \times e_2 : v_1 \in V_1, e_2 \in E_2\} \cup \{e_1 \times \{v_2\} : e_1 \in E_1, v_2 \in V_2\}$.

For $r-$uniform hypergraphs, the *Strong product* of two $r-$uniform hypergraphs could be defined as follows.

Definition 2.7 Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two $r-$uniform hypergraphs. The *Strong product* of H_1 and H_2 is the hypergraph $H_1 \widehat{\boxtimes} H_2$ with set of vertices $V_1 \times V_2$. For two edges $e_1 \in E_1$ and $e_2 \in E_2$, the edge set is defined as:

$E_1 \widehat{\boxtimes} E_2 = E_1 \Box E_2 \cup \{e \in e_1 \times e_2 | e_i \in E_i \text{ and } p_i(e) = e_i, i = 1, 2\}$.

The edge of $E_1 \widehat{\boxtimes} E_2$ is consisted with the *Cartesian product* edge and the non-*Cartesian product*. In other words, a subset which is the non-*Cartesian product* edge $e = \{(v_{11}, v_{12}), (v_{12}, v_{22}), \ldots, (v_{1r}, v_{2r})\}$ of $V_1 \times V_2$ is an edge in $H_1 \widehat{\boxtimes} H_2$ if and only if $\{v_{11}, v_{12}, \ldots, v_{1r}\}$ is an edge in H_1 and $\{v_{21}, v_{22}, \ldots, v_{2r}\}$ is an edge in H_2.

Definition 2.8 Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two $r-$uniform hypergraphs. The *Normal product* of H_1 and H_2 is the same as *Strong product* of H_1 and H_2.

Definition 2.9 Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two $r-$uniform hypergraphs. The *lexicographic product* $H = H_1 \circ H_2$ has vertex set $V(H) = V_1 \times V_2$ and edge set

$E(H) = \{e_1 \times e_2 | e_1 \in E_1, p(e_2) \subseteq V_2, |p_2(e)| \leq | e|\} \cup \{\{x\} \times e | x \in V_1, e_2 \in E_2\}$.

Since $|p_1(e)| = |e|$ there are $|e|$ vertices of e that have pairwise different first coordinates.

Definition 2.10 Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. The *union* of H_1 and H_2 is the hypergraph $H = H_1 \cup H_2$ with set of vertices $V_1 \cup V_2$ and set of edges $E_1 \cup E_2$.

Definition 2.11 Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. The *join* of H_1 and H_2 is the hypergraph $H = H_1 + H_2$ with set of vertices $V_1 \cup V_2$ and set of edges:

$E(H) = \{e | e \in E(H_1)$, or $e \in E(H_2)$, or $|e \cap V(H_1)| \geq 1$ and $|e \cap V(H_2)| \geq 1$ and $e \notin E(H_1)$ and $e \notin E(H_2)\}$.

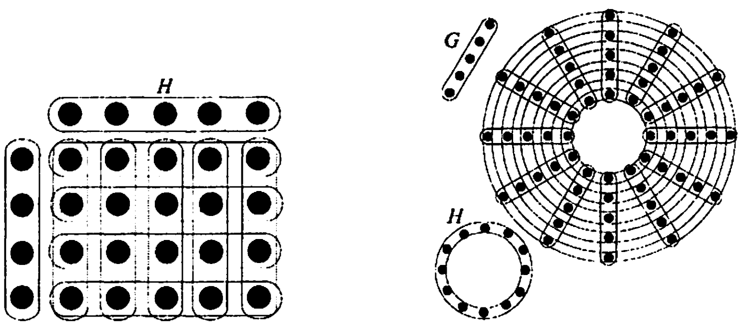

Figure 3: Two examples of Cartesian product of two hypergraphs G and H.

3 The Product of Fuzzy hypergraphs

In this section, we defined the operations of Cartesian product of two fuzzy hypergraphs, the strong product, normal product and lexicographic product of two strong fuzzy r-uniform hypergraphs.

Let $\mathcal{H}_i = (\sigma_i, \mu_i)$ be a fuzzy hypergraph, $i = 1, 2$. And we suppose its elementary hypergraphs H_1 contains m vertices.

Definition 3.1 Let $\mathcal{H}_1 = (\sigma_1, \mu_1)$ and $\mathcal{H}_2 = (\sigma_2, \mu_2)$ be two fuzzy hypergraphs, respectively, where σ_1 and σ_2 be fuzzy subsets of V_1 and V_2, and μ_1 and μ_2 be fuzzy subsets of E_1 and E_2. Then we denote the Cartesian product of two fuzzy hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\mathcal{H} = \mathcal{H}_1 \Box \mathcal{H}_2$ and define as follows:

$\forall (v_1, v_2) \in V$, $(\sigma_1 \times \sigma_2)(v_1, v_2) = \wedge\{\sigma_1(v_1), \sigma_2(v_2)\}$,

$\forall v_1 \in V_1, \forall e_2 \in E_2$, $\mu_1\mu_2(\{v_1\} \times e_2) = \wedge\{\sigma_1(v_1), \mu_2(e_2)\}$,

$\forall e_1 \in E_1, \forall v_2 \in V_2,\ \mu_1\mu_2(e_1 \times \{v_2\}) = \wedge\{\mu_1(e_1), \sigma_2(v_2)\}$.

Theorem 3.2 If $\mathcal{H}_1$ and $\mathcal{H}_2$ are the fuzzy hypergraphs, then $\mathcal{H}_1 \Box \mathcal{H}_2$ is a fuzzy hypergraph.

Proof. Let $v_1 \in V_1$, $e_1 \in E_1$, suppose e_1 contains p vertices, where $1 \leq p \leq m$ and $v_2 \in V_2$, $e_2 \in E_2$, suppose e_2 contains q vertices, where $1 \leq q \leq n$. Then we have

$$(\mu_1\mu_2)(\{v_1\} \times e_2)$$

$$= \wedge[\sigma_1(v_1), \mu_2(e_2)]$$

$$\leq \wedge[\sigma_1(v_1), \wedge_{v_2 \in e_2} \sigma_2(v_2)]$$

$$= \wedge\{\sigma_1(v_1), \wedge[\sigma_2(v_{21}), \sigma_2(v_{22}), \ldots, \sigma_2(v_{2q})]\}$$

$$= \wedge\{\wedge[\sigma_1(v_1), \sigma_2(v_{21})], \wedge[\sigma_1(v_2), \sigma_2(v_{22})], \ldots, \wedge[\sigma_1(v_q), \sigma_2(v_{2q})]\}$$

$$= \wedge[(\sigma_1 \times \sigma_2)(v_1, v_{21}), (\sigma_1 \times \sigma_2)(v_1, v_{22}), \ldots, (\sigma_1 \times \sigma_2)(v_1, v_{2q})]$$

$$= \wedge_{v_1 \in e_1, v_2 \in e_2} (\sigma_1 \times \sigma_2)(v_1, v_2),$$

$$(\mu_1\mu_2)(e_1 \times \{v_2\})$$

$$= \wedge[\mu_1(e_1), \sigma_2(v_1)]$$

$$\leq \wedge[\wedge_{v_1 \in e_1} \sigma_1(v_1), \sigma_2(v_2)]$$

$$= \wedge\{\wedge[\sigma_1(v_{11}), \sigma_1(v_{12}), \ldots, \sigma_1(v_{1p})], \sigma_2(v_2)\}$$

$$= \wedge\{\wedge[\sigma_1(v_{11}), \sigma_2(v_2)], \wedge[\sigma_1(v_{12}), \sigma_2(v_2)], \ldots, \wedge[\sigma_1(v_{1p}), \sigma_2(v_2)]\}$$

$$= \wedge[(\sigma_1 \times \sigma_2)(v_{11}, v_2), (\sigma_1 \times \sigma_2)(v_{12}, v_2), \ldots, (\sigma_1 \times \sigma_2)(v_{1p}, v_2)]$$

$$= \wedge_{v_1 \in e_1, v_2 \in e_2} ((\sigma_1 \times \sigma_2)(v_1, v_2)). \ \Box$$

Definition 3.3 Let $\mathcal{H}_1 = (\sigma_1, \mu_1)$ and $\mathcal{H}_2 = (\sigma_2, \mu_2)$ be two fuzzy $r-$uniform hypergraphs, respectively, where σ_1 and σ_2 be fuzzy subsets of V_1 and V_2, and μ_1 and μ_2 be fuzzy subsets of E_1 and E_2. Then we denote the *strong product* of two strong fuzzy $r-$uniform hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\mathcal{H} = \mathcal{H}_1 \widehat{\boxtimes} \mathcal{H}_2$ and define as follows:

$\forall (v_1, v_2) \in V,\ (\sigma_1 \times \sigma_2)(v_1, v_2) = \wedge\{\sigma_1(v_1), \sigma_2(v_2)\}$,

$\forall e_1 \in E_1, \forall e_2 \in E_2,\ \mu_1\mu_2(e_1 \times e_2) = \wedge\{\mu_1(e_1), \mu_2(e_2)\}$.

Theorem 3.4 If $\mathcal{H}_1$ and $\mathcal{H}_2$ are the strong fuzzy $r-$uniform hypergraphs, then $\mathcal{H}_1 \widehat{\boxtimes} \mathcal{H}_2$ is a strong fuzzy hypergraph.

Proof. Let $e_1 \in E_1$, $e_2 \in E_2$, then we have

$$(\mu_1\mu_2)(e_1 \times e_2)$$

$$= \wedge[\mu_1(e_1), \mu_2(e_2)]$$

$$= \wedge[\bigwedge_{v_1 \in e_1} \sigma_1(v_1), \bigwedge_{v_2 \in e_2} \sigma_2(v_2)]$$

$$= \wedge\{\wedge[\sigma_1(v_{11}), \sigma_2(v_{21})], \wedge[\sigma_1(v_{12}), \sigma_2(v_{22})], \ldots, \wedge[\sigma_1(v_{1r}), \sigma_2(v_{2r})]\}$$

$$= \wedge[(\sigma_1 \times \sigma_2)(v_{11}, v_{21}), (\sigma_1 \times \sigma_2)(v_{12}, v_{22}), \ldots, (\sigma_1 \times \sigma_2)(v_{1r}, v_{2r})]$$

$$= \bigwedge_{v_1 \in e_1, v_2 \in e_2} (\sigma_1 \times \sigma_2)(v_1, v_2). \ \square$$

Definition 3.5 Let $\mathcal{H}_1 = (\sigma_1, \mu_1)$ and $\mathcal{H}_2 = (\sigma_2, \mu_2)$ be two strong fuzzy $r-$uniform hypergraphs, respectively, where σ_1 and σ_2 be fuzzy subsets of V_1 and V_2, and μ_1 and μ_2 be fuzzy subsets of E_1 and E_2. Then we could know, for two strong fuzzy $r-$uniform hypergraphs, the *normal product* $\mathcal{H} = \mathcal{H}_1 \breve{\boxtimes} \mathcal{H}_2$ is the same as the *strong product* of two strong fuzzy $r-$uniform hypergraphs.

Obviously, according to representation of normal product, strong product and r-uniformity of $\mathcal{H}_1$ and $\mathcal{H}_2$, $|e| = \max_{i=1,2}\{|p_i(e)|\}$ is the same as $|e| = |p_i(e)| = \min_{j=1,2}\{|e_j|\}$. Furthermore, limited by $|e| = \max_{i=1,2}\{|p_i(e)|\}$ and $|e| = |p_i(e)| = \min_{j=1,2}\{|e_j|\}$, the edges $p_i(e) \in E(H_i)$ is the same as $p_i(e) \subseteq e_i \in E(H_i)$ for $i = 1, 2$. Hence, the *normal product* $\mathcal{H} = \mathcal{H}_1 \breve{\boxtimes} \mathcal{H}_2$ is the same as the *strong product* of two strong fuzzy $r-$uniform hypergraphs.

Theorem 3.6 If $\mathcal{H}_1$ and $\mathcal{H}_2$ are the strong fuzzy $r-$uniform hypergraphs, then $\mathcal{H}_1 \breve{\boxtimes} \mathcal{H}_2$ is a strong fuzzy hypergraph.
Proof. The proof is the same as Theorem 3.4.

Definition 3.7 Let $\mathcal{H}_1 = (\sigma_1, \mu_1)$ and $\mathcal{H}_2 = (\sigma_2, \mu_2)$ be two strong fuzzy $r-$uniform hypergraphs, respectively, where σ_1 and σ_2 be fuzzy subsets of V_1 and V_2, and μ_1 and μ_2 be fuzzy subsets of E_1 and E_2. Then we denote the *lexicographic product* of two strong fuzzy $r-$uniform hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\mathcal{H} = \mathcal{H}_1 \circ \mathcal{H}_2$ and define as follows:
$\forall (v_1, v_2) \in V$, $(\sigma_1 \times \sigma_2)(v_1, v_2) = \wedge\{\sigma_1(v_1), \sigma_2(v_2)\}$,

$\forall e_1 \in E_1, \forall e_2 \in E_2$, if $e_1 \in E_1, p(e_2) \subseteq V_2, |p_2(e)| \leq| e|$, then $\mu_1\mu_2(e_1 \times e_2) = \wedge\{\mu_1(e_1), \mu_2(e_2)\}$,

$\forall v_1 \in E_1, \forall e_2 \in E_2$, $\mu_1\mu_2(\{v_1\} \times e_2) = \wedge\{\sigma_1(v_1), \mu_2(e_2)\}$.

Theorem 3.8 If $\mathcal{H}_1$ and $\mathcal{H}_2$ are the strong $r-$uniform fuzzy hypergraphs, then $\mathcal{H}_1 \circ \mathcal{H}_2$ is a strong fuzzy hypergraph.
Proof. Suppose $e_1 \in E_1, p(e_2) \subseteq V_2, |p_2(e)| \leq| e|$, then we have

$$(\mu_1\mu_2)(e_1 \times e_2)$$

$$= \wedge[\mu_1(e_1), \mu_2(e_2)]$$

$$= \wedge[\bigwedge_{v_1 \in e_1} \sigma_1(v_1), \bigwedge_{v_2 \in e_2} \sigma_2(v_2)]$$

$$= \wedge\{\wedge[\sigma_1(v_{11}), \sigma_2(v_{21})], \wedge[\sigma_1(v_{12}), \sigma_2(v_{22})], \ldots, \wedge[\sigma_1(v_{1r}), \sigma_2(v_{2r})]\}$$

$$= \wedge[(\sigma_1 \times \sigma_2)(v_{11}, v_{21}), (\sigma_1 \times \sigma_2)(v_{12}, v_{22}), \ldots, (\sigma_1 \times \sigma_2)(v_{1r}, v_{2r})]$$

$$= \bigwedge_{v_1 \in e_1, v_2 \in e_2} (\sigma_1 \times \sigma_2)(v_1, v_2).$$

Suppose $v_1 \in E_1, e_2 \in E_2$, then we have

$$(\mu_1\mu_2)(\{v_1\} \times e_2)$$

$$= \wedge[\sigma_1(v_1), \mu_2(e_2)]$$

$$\leq \wedge[\sigma_1(v_1), \bigwedge_{v_2 \in e_2} \sigma_2(v_2)]$$

$$= \wedge\{\sigma_1(v_1), \wedge[\sigma_2(v_{21}), \sigma_2(v_{22}), \ldots, \sigma_2(v_{2q})]\}$$

$$= \wedge\{\wedge[\sigma_1(v_1), \sigma_2(v_{21})], \wedge[\sigma_1(v_2), \sigma_2(v_{22})], \ldots, \wedge[\sigma_1(v_r), \sigma_2(v_{2r})]\}$$

$$= \wedge[(\sigma_1 \times \sigma_2)(v_1, v_{21}), (\sigma_1 \times \sigma_2)(v_1, v_{22}), \ldots, (\sigma_1 \times \sigma_2)(v_1, v_{2r})]$$

$$= \bigwedge_{v_1 \in e_1, v_2 \in e_2} (\sigma_1 \times \sigma_2)(v_1, v_2). \ \square$$

4 Union and Join of fuzzy hypergraphs

In this section, we defined the operations of union and join of two fuzzy hypergraphs.

Definition 4.1 Consider the union $H = H_1 \cup H_2$ of two hypergraphs $H_1 = (V_1, E_1)$, $H_2 = (V_2, E_2)$. Let σ_i be a fuzzy subsets of V_i and μ_i be a fuzzy subset of E_i, $i = 1, 2$. Define the fuzzy subsets $\sigma_1 \cup \sigma_2$ of $V_1 \cup V_2$ and $\mu_1 \cup \mu_2$ of $X_1 \cup X_2$ as follows:

$$(\sigma_1 \cup \sigma_2)(v) = \sigma_1(v) \text{ if } v \in V_1 \text{ but } v \notin V_2,$$

$$(\sigma_1 \cup \sigma_2)(v) = \sigma_2(v) \text{ if } v \in V_2 \text{ but } v \notin V_1,$$

and $(\sigma_1 \cup \sigma_2)(v) = \vee[\sigma_1(v), \sigma_2(v)]$ if $v \in V_1 \cap V_2$;

$$(\mu_1 \cup \mu_2)(e) = \mu_1(e), \text{ if } e \in E_1 \text{ but } e \notin E_2,$$

$(\mu_1 \cup \mu_2)(e) = \mu_1(e)$, if $e \in E_2$ but $e \notin E_1$,

and $(\mu_1 \cup \mu_2)(e) = \vee[\mu_1(e), \mu_2(e)]$, if $e \in E_1 \cap E_2$.

Theorem 4.2 Let H be the union of the hypergraphs H_1 and H_2. Let (σ_i, μ_i) be a fuzzy hypergraph of H_i, $i = 1, 2$. Then $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a fuzzy hypergraph of H.

Proof. Suppose that $e \in E_1$ but $e \notin E_2$. Then

$$\mu_1 \cup \mu_2(e) = \mu_1(e_1) \leq \bigwedge_{v_1 \in e_1} \sigma_1(v_1)$$

$$= \wedge\{\sigma(v_{11}), \sigma(v_{12}), \ldots, \sigma(v_{1p})\}. \qquad (1)$$

If $v_{11}, v_{12}, \ldots, v_{1p} \in V_1$ but $v_{11}, v_{12}, \ldots, v_{1p} \notin V_2$, then

$$(1) = \wedge[(\sigma_1 \cup \sigma_2)(v_{11}), (\sigma_1 \cup \sigma_2)(v_{12}), \ldots, (\sigma_1 \cup \sigma_2)(v_{1p})].$$

If $v_{11}, v_{12}, \ldots, v_{1t} \in V_1 - V_2$, $v_{i1}, v_{i2}, \ldots, v_{is} \in V_1 \cap V_2$, where $i = 1, 2$, $t, s \geq 1$, and $t + s = p$, then

$$(1) = \wedge\{(\sigma_1 \cup \sigma_2)(v_{11}), (\sigma_1 \cup \sigma_2)(v_{12}), \ldots, (\sigma_1 \cup \sigma_2)(v_{1t}), \vee[\sigma_1(v_{i1}), \sigma_2(v_{i1})],$$

$$\vee[\sigma_1(v_{i2}), \sigma_2(v_{i2})], \vee[\sigma_1(v_{is}), \sigma_2(v_{is})]\}$$

$$= \wedge[(\sigma_1 \cup \sigma_2)(v_{11}), (\sigma_1 \cup \sigma_2)(v_{12}), \ldots, (\sigma_1 \cup \sigma_2)(v_{1t}), (\sigma_1 \cup \sigma_2)(v_{i1}),$$

$$(\sigma_1 \cup \sigma_2)(v_{i2}), \ldots, (\sigma_1 \cup \sigma_2)(v_{is})]$$

$$= \wedge[(\sigma_1 \cup \sigma_2)(v_{11}), (\sigma_1 \cup \sigma_2)(v_{12}), \ldots, (\sigma_1 \cup \sigma_2)(v_{1p})].$$

The last equation is due to $v_{i1}, v_{i2}, \ldots, v_{is} \in V_1 \cap V_2$ and there is no order of the vertices in the hyperedge, so we could look $v_{i1}, v_{i2}, \ldots, v_{is}$ as the $t+1$, $t+2$ to the p vertices in the hyperedge e_1.

If $v_{i1}, v_{i2}, \ldots, v_{ip} \in V_1 \cap V_2$, then

$$(1) = \wedge\{\vee[\sigma_1(v_{i1}), \sigma_2(v_{i1})], \vee[\sigma_1(v_{i2}), \sigma_2(v_{i2})], \ldots, \vee[\sigma_1(v_{ip}), \sigma_2(v_{ip})]\}$$

$$= \wedge[(\sigma_1 \cup \sigma_2)(v_{i1}), (\sigma_1 \cup \sigma_2)(v_{i2}), \ldots, (\sigma_1 \cup \sigma_2)(v_{ip})].$$

Suppose that $e \in E_2$ but $e \notin E_1$. Then

$$\mu_1 \cup \mu_2(e) = \mu_1(e_2) \leq \bigwedge_{v_2 \in e_{1}2} \sigma_2(v_2)$$

$$= \wedge[\sigma(v_{21}), \sigma(v_{22}), \ldots, \sigma(v_{2p})]. \qquad (2)$$

If $v_{21}, v_{22}, \ldots, v_{2p} \in V_2$ but $v_{21}, v_{22}, \ldots, v_{2p} \notin V_1$, then

$$(2) = \wedge[(\sigma_1 \cup \sigma_2)(v_{21}), (\sigma_1 \cup \sigma_2)(v_{22}), \ldots, (\sigma_1 \cup \sigma_2)(v_{2p})].$$

If $v_{21}, v_{22}, \ldots, v_{2t} \in V_2 - V_1$, $v_{i1}, v_{i2}, \ldots, v_{is} \in V_1 \cap V_2$, where $i = 1, 2$, $t, s \geq 1$, and $t + s = p$, then

$$(2) = \wedge\{(\sigma_1 \cup \sigma_2)(v_{21}), (\sigma_1 \cup \sigma_2)(v_{22}), \ldots, (\sigma_1 \cup \sigma_2)(v_{2t}), \vee[\sigma_1(v_{i1}), \sigma_2(v_{i1})],$$

$$\vee[\sigma_1(v_{i2}), \sigma_2(v_{i2})], \vee[\sigma_1(v_{is}), \sigma_2(v_{is})]\}$$

$$= \wedge[(\sigma_1 \cup \sigma_2)(v_{21}), (\sigma_1 \cup \sigma_2)(v_{22}), \ldots, (\sigma_1 \cup \sigma_2)(v_{2t}), (\sigma_1 \cup \sigma_2)(v_{i1}),$$

$$(\sigma_1 \cup \sigma_2)(v_{i2}), \ldots, (\sigma_1 \cup \sigma_2)(v_{is})]$$

$$= \wedge[(\sigma_1 \cup \sigma_2)(v_{21}), (\sigma_1 \cup \sigma_2)(v_{22}), \ldots, (\sigma_1 \cup \sigma_2)(v_{2p})].$$

The last equation is due to $v_{i1}, v_{i2}, \ldots, v_{is} \in V_1 \cap V_2$ and there is no order of the vertices in the hyperedge, so we could look $v_{i1}, v_{i2}, \ldots, v_{is}$ as the $t+1$, $t+2$ to the p vertices in the hyperedge e_1.

If $v_{i1}, v_{i2}, \ldots, v_{ip} \in V_1 \cap V_2$, then

$$(2) = \wedge\{\vee[\sigma_1(v_{i1}), \sigma_2(v_{i1})], \vee[\sigma_1(v_{i2}), \sigma_2(v_{i2})], \ldots, \vee[\sigma_1(v_{ip}), \sigma_2(v_{ip})]\}$$

$$= \wedge[(\sigma_1 \cup \sigma_2)(v_{i1}), (\sigma_1 \cup \sigma_2)(v_{i2}), \ldots, (\sigma_1 \cup \sigma_2)(v_{ip})].$$

Suppose that $e \in E_2 \cap E_1$. Then

$$(\mu_1 \cup \mu_2)(e)$$

$$= \vee[\mu_1(e), \mu_2(e)]$$

$$\leq \vee\{\wedge[\sigma_1(v_{i1}), \sigma_1(v_{i2}), \ldots, \sigma_1(v_{ip})], \wedge[\sigma_2(v_{i1}), \sigma_2(v_{i2}), \ldots, \sigma_2(v_{ip})]\}$$

$$\leq \wedge\{\vee[\sigma_1(v_{i1}), \sigma_2(v_{i1})], \vee[\sigma_1(v_{i2}), \sigma_2(v_{i2})], \ldots, \vee[\sigma_1(v_{ip}), \sigma_2(v_{ip})]\}$$

$$= \wedge[(\sigma_1 \cup \sigma_2)(v_{i1}), (\sigma_1 \cup \sigma_2)(v_{i2}), \ldots, (\sigma_1 \cup \sigma_2)(v_{ip})]. \square$$

Example 4.3 Let $V_1 = \{a, b, c, d, e\}$, $V_2 = \{a, c, d, e, f\}$ and $E_1 = \{e_{11}, e_{12}\}$, $E_2 = \{e_{21}, e_{22}\}$, where $e_{11} = \{a, b, c\}$, $e_{12} = \{c, d, e\}$, $e_{21} = \{c, d, e\}$, $e_{22} = \{e, f, a\}$. Define the fuzzy subsets σ_1, σ_2, μ_1, μ_2 of V_1 V_2, E_1, E_2 respectively as follows:

$\sigma_1(a) = 0.3$, $\sigma_1(b) = 0.4$, $\sigma_1(c) = 0.3$, $\sigma_1(d) = 0.5$, $\sigma_1(e) = 0.2$, $\mu_1(e_{11}) = 0.3$, $\mu_1(e_{12}) = 0.5$, $\sigma_2(c) = 0.3$, $\sigma_2(d) = 0.1$, $\sigma_2(e) = 0.4$, $\sigma_2(f) = 0.2$, $\sigma_2(a) = 0.5$, $\mu_2(e_{21}) = 0.1$, $\mu_2(e_{22}) = 0.2$.

From Definition 4.1 and Theorem 4.2 we could get $H = H_1 \cup H_2$ with the set of vertices $V(H) = \{a, b, c, d, e, f\}$ and the set of edges $E(H) = \{e'_1, e'_2, e'_3\}$, where $e'_1 = \{a, b, c\}$, $e'_2 = \{c, d, e\}$, $e'_3 = \{e, f, a\}$ and $\sigma(a) = 0.5$, $\sigma(b) = 0.4$, $\sigma(c) = 0.3$, $\sigma(d) = 0.5$, $\sigma(e) = 0.4$, $\sigma(f) = 0.2$, $\mu(e'_1) = 0.3$, $\mu(e'_2) = 0.3$, $\mu(e'_3) = 0.2$. See Fig 4.

Definition 4.4 Consider the join $H = H_1 + H_2 = (V_1 + V_2, E_1 + E_2 + E')$ of two hypergraphs $H_1 = (V_1, E_1)$, $H_2 = (V_2, E_2)$. Let σ_i be a fuzzy subsets of V_i and μ_i be a fuzzy subset of E_i, $i = 1, 2$. Define the fuzzy subsets $\sigma_1 + \sigma_2$ of $V_1 + V_2$ and $\mu_1 + \mu_2$ of $E_1 + E_2 + E'$ as follows:

$$(\sigma_1 + \sigma_2)(v) = (\sigma_1 \cup \sigma_2)(v),$$

$$(\mu_1 + \mu_2)(e) = (\mu_1 \cup \mu_2)(e) \text{ if } e \in E_1 \cup E_2,$$

and $(\mu_1 + \mu_2)(e) = \wedge[\sigma_1(v_{11}), \sigma_1(v_{12}), \ldots, \sigma_1(v_{1s}), \sigma_2(v_{21}), \sigma_2(v_{22}), \ldots, \sigma_2(v_{2t})]$, where $e = \{v_{11}, v_{12}, \ldots, v_{1s}, v_{21}, v_{22}, \ldots, v_{2t}\} \in E'$.

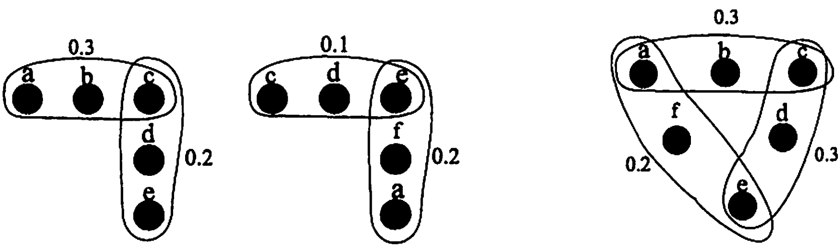

Figure 4: An example of the union of two fuzzy hypergraphs.

Theorem 4.5 Let H be the join of two hypergraphs H_1 and H_2. Let (σ_i, μ_i) be a fuzzy hypergraph of H_i, $i = 1, 2$. Then $(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ is a fuzzy hypergraph of H.

Proof. Suppose that $e \in E_1 \cup E_2$. Then the desired result follows from Theorem 4.2. Suppose $e \in X'$, then

$(\mu_1 + \mu_2)(e)$

$= \wedge[\sigma_1(v_{11}), \sigma_1(v_{12}), \ldots, \sigma_1(v_{1s}), \sigma_2(v_{21}), \sigma_2(v_{22}), \ldots, \sigma_2(v_{2t})]$

$= \wedge[(\sigma_1 \cup \sigma_2)(v_{11}), (\sigma_1 \cup \sigma_2)(v_{12}), \ldots, (\sigma_1 \cup \sigma_2)(v_{1s}), (\sigma_1 \cup \sigma_2)(v_{21}), (\sigma_1 \cup \sigma_2)(v_{22}), \ldots, (\sigma_1 \cup \sigma_2)(v_{2t})]$

$= \wedge[(\sigma_1 + \sigma_2)(v_{11}), (\sigma_1 + \sigma_2)(v_{12}), \ldots, (\sigma_1 + \sigma_2)(v_{1s}), (\sigma_1 + \sigma_2)(v_{21}), (\sigma_1 + \sigma_2)(v_{22}), \ldots, (\sigma_1 + \sigma_2)(v_{2t})]$. □

The fuzzy subhypergraph $(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ of Theorem 4.5 is called the join of (σ_1, μ_1) and (σ_2, μ_2).

Definition 4.6 Let (σ, μ) be a fuzzy hypergraph of a hypergraph $H = (V, E)$. Then (σ, μ) is called a strong fuzzy hypergraph of H if and only if $\mu(e) = \bigwedge_{v \in e} \sigma(v)$.

Theorem 4.7 If H is the join of two hypergraphs H_1 and H_2, then every strong fuzzy hypergraph (σ, μ) of H is a join of a strong fuzzy hypergraph of H_1 and a strong fuzzy hypergraph of H_2.
Proof. Define the fuzzy subsets σ_1, σ_2, μ_1 and μ_2 of V_1, V_2, E_1 and E_2 as follows: $\sigma_i(v) = \sigma(v)$ if $v \in V_i$ and $\mu_i(e) = \mu(e)$ if $e \in E_i$, $i = 1, 2$. Then (σ_i, μ_i) is a fuzzy hypergraph of H_i, $i = 1, 2$, and $\sigma = \sigma_1 + \sigma_2$ as in the proof of Theorem 4.2.

If $e \in E_1 \cup E_2$, then $\mu(e) = (\mu_1 + \mu_2)(e)$ as in the proof of Theorem 4.2.

Suppose that $e \in E'$, where $v_{11}, v_{12}, \ldots, v_{1s} \in E_1$ and $v_{21}, v_{22}, \ldots, v_{2t} \in V_2$. Then

$$\begin{aligned}(\mu_1 + \mu_2)(v) &= \wedge[\sigma_1(v_{11}), \sigma_1(v_{12}), \ldots, \sigma_1(v_{1s}), \sigma_2(v_{21}), \sigma_2(v_{22}), \ldots, \sigma_2(v_{2t})] \\ &= \wedge[\sigma(v_{11}), \sigma(v_{12}), \ldots, \sigma(v_{1s}), \sigma(v_{21}), \sigma(v_{22}), \ldots, \sigma(v_{2t})] \\ &= \mu(e)\end{aligned}$$

where the latter equality holds because (σ, μ) is strong. □

Remark 4.8 Let σ_1, σ_2, μ_1, μ_2 be fuzzy subsets of V_1, V_2, E_1, E_2 respectively. Then $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a fuzzy hypergraph of $H_1 \cup H_2$, but (σ_i, μ_i) is not a fuzzy hypergraph of G_i, $i = 1, 2$.

Example 4.9 Let $V_1 = V_2 = \{a, b, c\}$ and $E_1 = E_2 = \{a, b, c\}$. Define the fuzzy subsets σ_1, σ_2, μ_1, μ_2 of V_1 V_2, E_1, E_2 respectively as follows:

$$\sigma_1(a) = \sigma_2(b) = 1,\ \sigma_1(b) = \sigma_2(c) = \tfrac{1}{4},\ \sigma_1(c) = \sigma_2(a) = \tfrac{1}{3},\ \mu_1(e) = \mu_2(e) = \tfrac{1}{2}.$$

Then (σ_i, μ_i) is not a fuzzy hypergraph of H_i, $i = 1, 2$. However,

$$\begin{aligned}(\mu_1 \cup \mu_2)(e) &= \vee\{\mu_1(e) \cup \mu_2(e)\}(e) = \tfrac{1}{2} < 1 \\ &= \wedge\{\vee\{\sigma_1(a), \sigma_2(a)\}, \vee\{\sigma_1(b), \sigma_2(b)\}, \vee\{\sigma_1(c), \sigma_2(c)\}\} \\ &= \wedge\{(\sigma_1 \cup \sigma_2)(a), (\sigma_1 \cup \sigma_2)(b), (\sigma_1 \cup \sigma_2)(c)\}.\end{aligned}$$

Thus $(\sigma_1 \cup \sigma_2), (\mu_1 \cup \mu_2)$ is a fuzzy hypergraph of $H_1 \cup H_2$.

Theorem 4.10 Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. Suppose that $V_1 \cap V_2 = \emptyset$. Let σ_1, σ_2, μ_1, μ_2 be fuzzy subsets of V_1, V_2, E_1, E_2 respectively. Then $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a strong fuzzy hypergraph of $H_1 \cup H_2$ if and only if (σ_1, μ_1) and (σ_2, μ_2) are strong fuzzy hypergraphs of H_1 and H_2, respectively.
Proof. Suppose that $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a strong fuzzy hypergraph of $H_1 \cup H_2$. Let $e \in E_1$, then $e \notin E_2$ and the vertices in e belong to V_1 but not belong to V_2, suppose $e = \{v_{11}, v_{12}, \ldots, v_{1s}\}$. Hence

$$\begin{aligned}\mu_1(e) &= (\mu_1 \cup \mu_2)(e) \\ &\leq \min[(\sigma_1 \cup \sigma_2)(v_{11}), (\sigma_1 \cup \sigma_2)(v_{12}), \ldots, (\sigma_1 \cup \sigma_2)(v_{1s})] \\ &= \min[\sigma_1(v_{11}), \sigma_1(v_{12}), \ldots, \sigma_1(v_{1s})].\end{aligned}$$

Thus (σ_1, μ_1) is a strong fuzzy hypergraph of H_1.

Similarly, (σ_2, μ_2) is a strong fuzzy hypergraph of H_2. The converse is Theorem 4.2.

Theorem 4.11 Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. Suppose that $V_1 \cap V_2 = \emptyset$. Let σ_1, σ_2, μ_1, μ_2 be fuzzy subsets of V_1, V_2, E_1, E_2 respectively.

Then $(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ is a fuzzy hypergraph of $H_1 + H_2$ if and only if (σ_1, μ_1) and (σ_2, μ_2) are fuzzy hypergraphs of H_1 and H_2, respectively.
Proof. The desired result follows from the proof of Theorem 4.7 and Theorem 4.4.

5 Application of fuzzy hypergraphs

Graph models find wide application in many areas of mathematics, computer science, the natural and social sciences, such as in [24], it introduces a hypernetworks in a directed hypergraph. In social science, the structural approach that is based on the study of interaction among social actors is called social network analysis. In [19], it introduces the concept of fuzzy social network. Social network analysts study the structure formed by the nodes (people or group) connected by the links (relationships or flow). Mostly social network analysis considers the links between its actors as binary (1 if present, or 0 if not). In reality, not all the actors are related with the same degree. For example, hyperlinks between the two websites belonging to the same college will exhibit strong ties while these same websites will form weak ties with the third website belonging to some other college. But, based on the current methods these social links between all the three websites will be considered with equal importance, as shown in figure 5, wherein actors A and B are supposed to form stronger ties since they belong to the same group. Thus, the ties between two actors cannot be effectively represented as binary relation because these relations are inherently fuzzy.

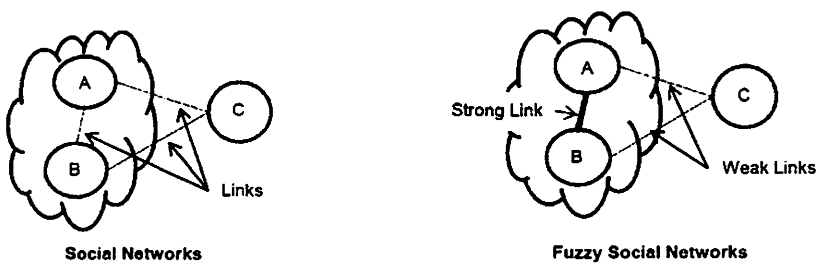

Figure 5: Social networks and fuzzy social networks.

The definition of undirected fuzzy social networks could see in Definition 5.1.
Definition 5.1[15]. Let $\tilde{G} = (V, \tilde{E})$ be fuzzy social network structure, $V = \{v_1, v_2, \ldots, v_n\}$ is the actors set,

$$\tilde{E} = \begin{bmatrix} \tilde{e}_{11} & \ldots & \tilde{e}_{1n} \\ \ldots & \ldots & \ldots \\ \tilde{e}_{n1} & \ldots & \tilde{e}_{nn} \end{bmatrix}$$

is the fuzzy relationship on V, then $\tilde{G}$ is a fuzzy social networks. If $\tilde{e}_{ij} = \tilde{e}_{ji}$, then $\tilde{G}$ is called undirected fuzzy social networks.

While this kind of graph-based model consider only pairwise relationships between two person (or group), and they neglect the relationship in higher order. Modeling the high-order relationship among person will significantly improve group behaviour performance. Unlike a graph that has an edge between two vertices, a set of vertices is connected by a hyperedge in a hypergraph. So we consider fuzzy social hypernetwork. Let $\tilde{H} = (V, \tilde{E})$ be fuzzy social hypernetwork structure, $V = \{v_1, v_2, \ldots, v_n\}$ is the actors set, $\sigma : V \to [0, 1]$ denotes the degree of actors take apart in this social hypernetwork, $\tilde{E} = \{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n\}$ denotes the set of fuzzy relationships, such as friendship, cooperation and trade relations, etc., and due to the definition of fuzzy hypergraph, the degree of these relations is decided by the membership degree of actors take apart in the relationship. Thus this model uses the full definition of fuzzy hypergraph.

6 Conclusions

Theoretical concepts of graphs and hypergraphs are highly utilized by computer science applications. Especially in research areas of computer science such as data mining, image segmentation, clustering, image capturing and networking. Because the complex graph can be obtained by the simple graph through the operations of graphs, so study the operation of graphs is meaningful. We used the definition of fuzzy hypergraphs which is proposed by Yu Bin in [25] to define some operation of fuzzy hypergraphs, such as Cartesian product, strong product, normal product and union, join and investigate some of their important properties. We also present applications of fuzzy hypergraphs. We plan to extend our research work to (1) Operations on intuitionistic fuzzy hypergraphs, (2) More generalized operations on fuzzy hypergraphs and intuitionistic fuzzy hypergraphs, (3) Applied models of fuzzy hypergraphs and intuitionistic fuzzy hypergraphs.

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