# On the monophonic sets of the strong product graphs

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**Abstract**

For vertices $u, v$ in a connected graph $G$, a $u-v$ chordless path in $G$ is a $u-v$ monophonic path. The monophonic interval $J_G[u, v]$ consists of all vertices lying on some $u-v$ monophonic path in $G$. For $S \subseteq V(G)$, the set $J_G[S]$ is the union of all sets $J_G[u, v]$ for $u, v \in S$. A set $S \subseteq V(G)$ is a monophonic set of $G$ if $J_G[S] = V(G)$. The cardinality of a minimum monophonic set of $G$ is the monophonic number of $G$, denoted by $mn(G)$. In this paper, bounds for the monophonic number of the strong product graphs are obtained and for several classes improved bounds and exact values are obtained.

**Key Words:** monophonic path, monophonic set, monophonic number, extreme monophonic graph, open monophonic number.
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# 1 Introduction

By a graph $G = (V(G), E(G))$ we mean a finite undirected connected graph without loops or multiple edges. The *distance* $d_G(u, v)$ between two vertices

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$u$ and $v$ in a connected graph $G$ is the length of a shortest $u$ - $v$ path in $G$. A $u$ - $v$ path of length $d_G(u,v)$ is called a $u$ - $v$ *geodesic*. A *chord* of a path $P: u_0, u_1, \ldots, u_n$ is an edge $u_iu_j$, with $j \geq i+2$. Any chordless path connecting $u$ and $v$ is a *u-v monophonic path* or an *m-path*. The *geodesic closed interval* $I_G[u,v]$ is the set of vertices of all $u-v$ geodesics. Similarly, the *monophonic closed interval* $J_G[u,v]$ is the set of vertices of all $u-v$ monophonic paths. The monophonic open interval is the set $J_G(u,v) = J_G[u,v] - \{u,v\}$. For $S \subseteq V(G)$, the *geodetic closure* $I_G[S]$ of $S$ is defined as the union of all geodesic closed intervals $I_G[u,v]$ over all pairs $u,v \in S$. The *monophonic closure* $J_G[S]$ is the set formed by the union of all monophonic closed intervals $J_G[u,v]$ with $u,v \in S$. A vertex $x$ in $S$ is a *monophonic interior vertex* of $S$ if $x \in J_G[S - \{x\}]$. The set of all monophonic interior vertices of $S$ is denoted by $S^o$. A set $S$ of vertices of $G$ is a *geodetic set* of $G$ if $I_G[S] = V(G)$ and $S$ is a *monophonic set* if $J_G[S] = V(G)$. The *monophonic number* $mn(G)$ (*geodetic number* $g(G)$, *respectively*) of $G$ is the minimum cardinality of a monophonic (geodetic, respectively) set in $G$. Since every geodetic set is a monophonic set, $mn(G) \leq g(G)$. The geodetic number of a graph was introduced and studied in [2, 3, 4, 5, 7, 8]. The geodetic number of Cartesian product graphs was discussed in [1]. In [13], the behaviour of intervals and the characterization of convex sets are given for the strong product of graphs. Also, recently Santhakumaran and Ullas Chandran [15] studied the geodetic number for strong product of graphs.

For a non-empty set $W \subseteq V(G)$, a connected subgraph of $G$ with the minimum number of edges that contains all of $W$ is necessarily a tree. Such a tree is called a *Steiner W-tree*. The *Steiner interval* $S(W)$ of $W$ consists of all vertices that lie on some Steiner $W$-tree of $G$. If $S(W) = V(G)$, then $W$ is called a *Steiner set* of $G$. The Steiner number of a graph was introduced by Chartrand and Zhang [6]. Steiner sets in a graph $G$ could be understood as a generalization of geodetic sets in $G$. Nevertheless, its relationship is not exactly obvious. In [6], the authors tried to show that every Steiner set in $G$ is also geodetic. Unfortunately, this particular result turned out to be wrong and was disproved by Pelayo [14]. However, in [9], the authors proved that every Steiner set is monophonic. This motivated us to study further monophonic sets in graphs. The concepts of monophonic sets and the monophonic number of a graph have not yet been fully explored and investigated. These concepts appeared in [3] and were studied by Pelayo et al. in [9, 10, 17]. Recently, the monophonic numbers of join and composition of graphs were discussed in [12]. The monophonic number of the Cartesian product graphs was studied in [16].

In this paper, we study monophonic sets and the monophonic number in the strong product graphs.

The *strong product* of graphs $G$ and $H$, denoted by $G \boxtimes H$, has a vertex set $V(G) \times V(H)$, where two distinct vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if, (a) $x_1 = x_2$ and $y_1y_2 \in E(H)$, or (b) $y_1 = y_2$ and $x_1x_2 \in E(G)$, or (c) $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. The mappings $\pi_G : (x, y) \mapsto x$ and $\pi_H : (x, y) \mapsto y$ from $V(G \boxtimes H)$ onto $G$ and $H$, respectively, are called *projections.* For a set $S \subseteq V(G \boxtimes H)$, we define the *G-projection* of $S$ on $G$ as $\pi_G(S) = \{x \in V(G) : (x, y) \in S$ for some $y \in V(H)\}$, and the *H-projection* $\pi_H(S) = \{y \in V(H) : (x, y) \in S$ for some $x \in V(G)\}$. For a path $P : (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in $G \boxtimes H$, we define the *G-projection* $\pi_G(P)$ of $P$ as a sequence that is obtained from $(x_1, x_2, \ldots, x_n)$ by changing each constant subsequence with its unique element. The $H$-projection $\pi_H(P)$ is defined similarly. It is clear from the defintion of the strong product that for any path $P$ in $G \boxtimes H$, both $\pi_G(P)$ and $\pi_H(P)$ are walks in the factor graphs $G$ and $H$, respectively. For a path $P : u = u_0, u_1, \ldots, u_n = u'$ in $G$ and $y \in V(H)$, we use $P_y$ to denote the path $P_y : (u, y) = (u_0, y), (u_1, y), \ldots, (u_n, y) = (u', y)$ in $G \boxtimes H$. Similarly, we can define $Q_x$, where $Q$ is a path in $H$ and $x \in V(G)$. For a vertex $v$ in $G$, $N(v)$ denotes the set of all neighbors of $v$, and $N[v] = N(v) \cup \{v\}$. A vertex $v$ in $G$ is an *extreme vertex* if the subgraph induced by $N(v)$ is complete. The set of all extreme vertices is denoted by $Ext(G)$ and $e(G) = |Ext(G)|$. A graph $G$ is an *extreme monophonic graph* if all its extreme vertices form a monophonic set. Given a path $P$ in a graph and two vertices $x, y$ on $P$, we use $P[x, y]$ to denote the portion of $P$ between $x$ and $y$, inclusive of $x$ and $y$. For basic graph theoretic terminology, we refer to [7]. We also refer to [3] for results on distance in graphs and to [11] for metric structures in strong product graphs. Throughout the following $G$ denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

**Theorem 1.1.** [10] *Each extreme vertex of a connected graph $G$ belongs to every monophonic set of $G$.*

**Theorem 1.2.** [15] *Let $G$ and $H$ be connected graphs. Then $Ext(G \boxtimes H) = Ext(G) \times Ext(H)$.*

# 2 Bounds for the monophonic number of the strong product graphs

In this section we determine possible bounds for the monophonic number of the strong product of two connected graphs. The following lemma is used to prove an upper bound for the monophonic number of the strong product graphs.

**Lemma 2.1.** *Let $G$ and $H$ be connected graphs. Then for $u, u' \in V(G)$ and $v, v' \in V(H)$, $J_G[u, u'] \times J_H(v, v') \subseteq J_{G \boxtimes H}((u, v), (u', v'))$.*

**Proof.** Let $i \in \{0, 1, \ldots, n\}$ and $P : u = u_0, u_1, \ldots, u_i = x, \ldots, u_n = u'$ a $u - u'$ $m$-path in $G$ containing the vertex $x$; and for $j \in \{1, 2, \ldots, m-1\}$ let $Q : v = v_0, v_1, \ldots, v_j = y, \ldots, v_m = v'$ be a $v - v'$ $m$-path in $H$ containing the vertex $y$. We consider two cases

**Case 1.** $1 \leq i \leq n-1$. Since $(u_{i-1}, v)(x, v_1), (x, v_{m-1})(u_{i+1}, v') \in E(G \boxtimes H)$, it follows that $R = P_v[(u_0, v), (u_{i-1}, v)] \cup \{(u_{i-1}, v), (x, v_1)\} \cup Q_x[(x, v_1), (x, v_{m-1})] \cup \{(x, v_{m-1}), (u_{i+1}, v')\} \cup P_{v'}[(u_{i+1}, v'), (u_n, v')]$ is a $(u, v) - (u', v')$ path in $G \boxtimes H$ containing the vertex $(x, y)$. We show that $R$ is a $(u, v) - (u', v')$ $m$-path in $G \boxtimes H$. Since $P$ and $Q$ are $m$-paths, we see that $P_v[(u_0, v), (u_{i-1}, v)], Q_x[(x, v_1), (x, v_{m-1})]$ and $P_{v'}[(u_{i+1}, v'), (u_n, v')]$ are $m$-paths in $G \boxtimes H$. Now, since $x \neq u_k$ are non-adjacent for $k = 0, 1, \ldots, i-2 (i \geq 2)$; and $v \neq v_l$ are non-adjacent for $l = 2, 3, \ldots, m-1$, there is no chord between the vertices of $P_v[(u_0, v), (u_{i-1}, v)]$ and $Q_x[(x, v_1), (x, v_{m-1})]$. Note that if $i = 1$, then $P_v[(u_0, v), (u_{i-1}, v)]$ is the single point $(u_0, v)$ and so there is no chord between the vertices of $P_v[(u_0, v), (u_{i-1}, v)]$ and $Q_x[(x, v_1), (x, v_{m-1})]$. Similarly, we can show that there is no chord between the vertices of $Q_x[(x, v_1), (x, v_{m-1})]$ and $P_{v'}[(u_{i+1}, v'), (u_n, v')]$. Also, since $v$ and $v'$ are distinct and non-adjacent, there is no chord between the vertices of $P_v[(u_0, v), (u_{i-1}, v)]$ and $P_{v'}[(u_{i+1}, v'), (u_n, v')]$. Hence it follows that $R$ is a $(u, v) - (u', v')$ $m$-path in $G \boxtimes H$ containing the vertex $(x, y)$. Since $(x, y) \neq (u, v), (u', v')$, we have $(x, y) \in J_{G \boxtimes H}((u, v), (u', v'))$.

**Case 2.** $i = 0$ or $i = n$. Without loss of generality assume that $i = 0$ and so $x = u = u_0$. Since $(u, v_{m-1})$ and $(u_1, v')$ are adjacent, $R = Q_u[(u, v_0), (u, v_{m-1})] \cup \{(u, v_{m-1}), (u_1, v')\} \cup P_{v'}[(u_1, v'), (u_n, v')]$ is a $(u, v) - (u', v')$ path in $G \boxtimes H$ containing the vertex $(x, y)$. Now, since $u \neq u_k$ are non-adjacent for $k = 2, 3, \ldots, n$; and $v' \neq v_l$ are non-adjacent for $l = 0, 1, \ldots, m-2$, there is no chord between the vertices of $Q_u[(u, v_0), (u, v_{m-1})]$ and $P_{v'}[(u_1, v'), (u_n, v')]$. Hence $R$ is a $(u, v) - (u', v')$ $m$-path in $G \boxtimes H$ containing the vertex $(x, y)$. Since $y \neq v, v'$, we have $(x, y) \neq (u, v), (u', v')$. Thus $(x, y) \in J_{G \boxtimes H}((u, v), (u', v'))$. ∎

**Proposition 2.2.** *Let $G$ and $H$ be connected graphs. Let $S$ and $T$ be monophonic sets of $G$ and $H$, respectively. Then $S \times T$ is a monophonic set of $G \boxtimes H$.*

**Proof.** Let $(x, y) \in V(G \boxtimes H)$. Since $S$ and $T$ are monophonic sets of $G$ and $H$ respectively, we have $x \in J_G[u, u']$ and $y \in J_H[v, v']$ for some $u, u' \in S$ and $v, v' \in T$. If $x \in J_G(u, u')$ or $y \in J_H(v, v')$, then by Lemma 2.1, $(x, y) \in J_{G \boxtimes H}((u, v), (u', v'))$. Otherwise, $(x, y) \in S \times T$. Thus $S \times T$ is a monophonic set of $G \boxtimes H$. ∎

**Proposition 2.3.** *Let $G$ and $H$ be non trivial connected graphs. Then $max\{2, e(G)e(H)\} \leq mn(G \boxtimes H) \leq mn(G)mn(H)$.*

**Proof.** Let $S$ and $T$ be minimum monophonic sets of $G$ and $H$ respectively. By Proposition 2.2, $S \times T$ is a monophonic set of $G \boxtimes H$ and so $mn(G \boxtimes H) \leq |S \times T| = mn(G)mn(H)$. The other inequality follows from Theorems 1.1 and 1.2. ■

**Corollary 2.4.** *Let $G$ and $H$ be extreme monophonic graphs. Then $mn(G \boxtimes H) = mn(G)mn(H) = e(G)e(H)$.*

In view of Corollary 2.4, we leave the following problem as an open question.

**Problem 2.5.** Let $G$ and $H$ be connected graphs such that $G \boxtimes H$ is an extreme monophonic graph. Is it true that $G$ and $H$ are extreme monophonic graphs ?

In the following we introduce the concepts of open monophonic sets and open monophonic number of a graph and obtain an upper bound of $mn(G \boxtimes H)$ in terms of the open monophonic numbers of the factor graphs. Also, we find the exact value of the monophonic number of several classes of the strong product graphs.

A set $S \subseteq V(G)$ is an *open monophonic set* if for each vertex $v$, either (1) $v$ is an extreme vertex of $G$ and $v \in S$, or (2) $v$ lies as an internal vertex of an $x$-$y$ $m$-path for some $x, y \in S$. An open monophonic set of minimum cardinality is a *minimum open monophonic set* of $G$ and this cardinality is the *open monophonic number* $og(G)$. It is clear that a monophonic set $S$ of $G$ is an open monophonic set if and only if $S^o = S - Ext(G)$.

**Theorem 2.6.** *Let $G$ and $H$ be connected graphs. Let $S$ and $T$ be monophonic sets of $G$ and $H$, respectively. Then $mn(G \boxtimes H) \leq |S||T| - min\{|S|, |T^o|\}$.*

**Proof.** Let $S = \{g_1, g_2, \ldots, g_p\}$ and $T = \{h_1, h_2, \ldots, h_q\}$ be monophonic sets of $G$ and $H$ respectively. If $T^o = \phi$, then the result follows from Proposition 2.2. So, assume that $T^o \neq \phi$. Let $T^o = \{h_1, h_2, \ldots, h_m\}$, where $1 \leq m \leq q$.

**Case 1.** $p \geq m$. Let $W = S \times T - \bigcup_{i=1}^{m}\{(g_i, h_i)\}$. Then $|W| = pq - m$. We show that $W$ is a monophonic set of $G \boxtimes H$. Let $(x, y)$ be a vertex of $G \boxtimes H$. Since $S$ and $T$ are monophonic sets of $G$ and $H$ respectively, we have $x \in J_G[g_i, g_j]$ and $y \in J_H[h_k, h_l]$ for $0 \leq i < j \leq p$ and $0 \leq k < l \leq q$. We consider two subcases.

**Subcase 1.1.** $x \in J_G(g_i, g_j)$ or $y \in J_H(h_k, h_l)$. Suppose that $i = k$. Then $i \neq l$ and $j \neq k$ so that $(g_i, h_l), (g_j, h_k) \in W$. It follows from Lemma 2.1 that $(x, y) \in J_{G \boxtimes H}((g_i, h_l), (g_j, h_k)) \subseteq J_{G \boxtimes H}[W]$. Now, suppose that $i \neq k$. If $j = l$, then $i \neq l$ and $j \neq k$. Hence this is similar to the above case. If $j \neq l$, then $(g_j, h_l) \in W$. Since $i \neq k$, we have $(g_i, h_k) \in W$. Now, it follows from Lemma 2.1 that $(x, y) \in J_{G \boxtimes H}((g_i, h_k), (g_j, h_l))$. Hence

$(x,y) \in J_{G\boxtimes H}[W]$.
**Subcase 1.2** $x \in \{g_i, g_j\}$ and $y \in \{h_k, h_l\}$. Let $y = h_k$. If $k \geq m+1$, then it is clear that $(x,y) \in W \subseteq J_{G\boxtimes H}[W]$. If $k \leq m$, then $h_k \in T^o$ and so $h_k \in J_H(h_r, h_s)$ for $0 \leq r < s \leq q$. Hence we have $x \in J_G[g_i, g_j]$ and $y = h_k \in J_H(h_r, h_s)$. Then as in Subcase 1.1 we can prove that $(x,y) \in J_{G\boxtimes H}[W]$. If $y = h_l$, then we can prove similarly that $(x,y) \in J_{G\boxtimes H}[W]$. Hence $W$ is a monophonic set of $G \boxtimes H$.
**Case 2.** $p < m$. Let $W = S \times T - \bigcup_{i=1}^{p}\{(g_i, h_i)\}$. Then, as in Case 1, we can prove that $W$ is a monophonic set of $G \boxtimes H$. Hence the result follows. ∎

**Corollary 2.7.** *Let $G$ and $H$ be connected graphs. Then*

$$mn(G \boxtimes H) \leq mn(G)omn(H) - min\{mn(G), omn(H) - e(H)\}.$$

**Proof.** Let $S$ be a minimum monophonic set of $G$ and $T$ a minimum open monophonic set of $H$. Then $mn(G) = |S|$ and $omn(H) = |T|$ and $T^o = T - Ext(H)$. Hence the result follows from Theorem 2.6. ∎

**Proposition 2.8.** *Let $G$ be a connected graph. If $S$ is a monophonic set of $G \boxtimes K_n$, then $\pi_G(S)$ is a monophonic set of $G$.*

**Proof.** Let $x$ be a vertex of $G$ such that $x \notin \pi_G(S)$. Then $(x,y) \notin S$ for any $y \in V(K_n)$. Since $S$ is a monophonic set of $G \boxtimes K_n$, there exist $(u,v), (u',v') \in S$ such that $(x,y)$ lies on an $m$-path $P : (u,v) = (u_0,v_0), (u_1,v_1), \ldots, (u_k,v_k) = (x,y), \ldots, (u_m,v_m) = (u',v')$ with $1 \leq k \leq m-1$. We first claim that all the $u_i$'s ($1 \leq i \leq m$) are distinct. Suppose that $u_i = u_j$ for some $i < j$. If $j = i+1$, then it follows that either $(u_{i-1}, v_{i-1})$ is adjacent to $(u_j, v_j)$ or $(u_{j+1}, v_{j+1})$ is adjacent to $(u_i, v_i)$, which is a contradiction to the fact that $P$ is an $m$-path in $G \boxtimes K_n$. If $j \neq i+1$, then the edge $(u_i,v_i)(u_j,v_j)$ is a chord of the path $P$, which is also a contradiction. Thus all the $u_i$'s are distinct and it follows that $\pi_G(P) : u = u_0, u_1, \ldots, u_k = x, \ldots, u_m = u'$ is an $m$-path in $G$ with $u, u' \in \pi_G(S)$. Hence $\pi_G(S)$ is a monophonic set of $G$. ∎

**Corollary 2.9.** *Let $G$ be a connected graph. Then $mn(G) \leq mn(G \boxtimes K_n)$.*

**Proof.** Let $S$ be a minimum monophonic set of $G \boxtimes K_n$. Then by Proposition 2.8, $\pi_G(S)$ is a monophonic set of $G$. Hence $mn(G) \leq |\pi_G(S)| \leq |S| = mn(G \boxtimes K_n)$. ∎

# 3 Exact monophonic numbers

In this section we determine the exact values of the monophonic numbers of the strong product for several classes of graphs. $G \circ H$ denotes the composition of two graphs $G$ and $H$. It is proved in [12] that $mn(G \circ K_n) = min\{n|S| - (n-1)|S^o| : S \text{ is a monophonic set of } G\}$, where we observe that $G \circ K_n = G \boxtimes K_n$ and so the next theorem follows directly.

**Theorem 3.1.** *Let $G$ be a connected graph. Then*

$$mn(G \boxtimes K_n) = min\{n|S| - (n-1)|S^o| : S \text{ is a monophonic set of } G\}.$$

Using this result, further we attain bounds for the monophonic number of the strong product of a connected graph $G$ and the complete graph $K_n$ in terms of the monophonic number and the open monophonic number of $G$.

**Corollary 3.2.** *Let $G$ be a connected graph. Then*
$e(G)(n-1) + mn(G) \leq mn(G \boxtimes K_n) \leq e(G)(n-1) + omn(G)$.

**Proof.** Suppose that $mn(G \boxtimes K_n) < e(G)(n-1) + mn(G)$. Then, by Theorem 3.1, there exists a monophonic set $S$ of $G$ such that $n|S| - (n-1)|S^o| < e(G)(n-1) + mn(G)$. Thus, $n|S| < e(G)(n-1) + mn(G) + (n-1)|S^o|$. Since $S^o \subseteq S - Ext(G)$, we have $n|S| < e(G)(n-1) + mn(G) + (n-1)(|S| - e(G))$. This implies that $n|S| < mn(G) + (n-1)|S|$. Hence $|S| < mn(G)$, which is a contradiction. Thus $e(G)(n-1)+mn(G) \leq mn(G \boxtimes K_n)$. For the other inequality, let $S$ be a minimum open monophonic set of $G$. Then $omn(G) = |S|$. By Theorem 3.1, we have $mn(G \boxtimes K_n) \leq n|S| - (n-1)|S^o| = n|S| - (n-1)(|S| - e(G)) = e(G)(n-1) + omn(G)$. ■

Now, we proceed to characterize graphs $G$ for which $mn(G \boxtimes K_n) = e(G)(n-1) + mn(G)$.

**Corollary 3.3.** *Let $G$ be a connected graph. Then*
$mn(G \boxtimes K_n) = e(G)(n-1) + mn(G)$ *if and only if* $mn(G) = omn(G)$.

**Proof.** Suppose that $mn(G) = omn(G)$. Then the result follows from Corollary 3.2. Conversely, assume that $mn(G \boxtimes K_n) = e(G)(n-1)+mn(G)$. By Theorem 3.1, there exists a monophonic set $S$ of $G$ such that $mn(G \boxtimes K_n) = n|S| - (n-1)|S^o|$ and so $n|S| - (n-1)|S^o| = e(G)(n-1) + mn(G)$ $\Longrightarrow$(Equ. 1). Since $S^o \subseteq S - Ext(G)$, we have $e(G)(n-1)+mn(G) = n|S| - (n-1)|S^o| \geq n|S| - (n-1)(|S| - e(G)) = |S| + (n-1)e(G)$. Thus $|S| \leq mn(G)$ and so $S$ is a minimum monophonic set of $G$. Hence $mn(G) = |S|$. Now we claim that $S$ is an open monophonic set of $G$ From (Equ. 1), we have $(n-1)|S^o| = n|S| - e(G)(n-1) - mn(G) = n \cdot mn(G) - e(G)(n-1) - mn(G)$. Hence $(n-1)|S^o| = (n-1)(mn(G) - e(G))$ and so $|S^o| = mn(G) - e(G) = |S| - e(G)$. Therefore, $S^o = S - Ext(G)$ and hence $S$ is an open monophonic set of $G$. Thus $omn(G) \leq |S| = mn(G)$. Since $mn(G) \leq omn(G)$, the result follows. ■

Next, we proceed to find certain classes of graphs for which the upper bound of Corollary 3.2 is attained.

**Corollary 3.4.** If $G$ is a connected graph such that $omn(G) = mn(G)+1$, then $mn(G \boxtimes K_n) = e(G)(n-1) + mn(G) + 1$.

**Proof.** This follows from Corollaries 3.2 and 3.3. ∎

**Theorem 3.5.** *Let $G$ be a connected graph. Then $max\{2, e(G)\} \leq mn(G) \leq omn(G) \leq 3mn(G) - 2e(G)$.*

**Proof.** The lower bound follows from Theorem 1.1. To prove the upper bound of $omn(G)$, let $mn(G) = p$. If $p = e(G)$, then $omn(G) = p$ so that $omn(G) = 3mn(G) - 2e(G)$. So, assume that $e(G) < p$. Let $S$ be a minimum monophonic set of $G$. Then $Ext(G) \subseteq S$. Let $S - Ext(G) = \{v_1, v_2, \ldots, v_{p-e(G)}\}$. For each $j$ with $1 \leq j \leq p - e(G)$, let $v_{j,1}$ and $v_{j,2}$ be two non-adjacent neighbors of $v_j$. Then $v_j$ lies on the $m$-path $P$ : $v_{j,1}, v_j, v_{j,2}$. Let $T = S \cup \{v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, \ldots, v_{p-e(G),1}, v_{p-e(G),2}\}$. Then $|T| \leq p + 2(p - e(G)) = 3mn(G) - 2e(G)$. We show that $T$ is an open monophonic set of $G$. Let $x \in T - Ext(G)$. If $x \notin S$, then $x \in J_G(y, z)$ for some $y, z \in S$ and so $x \in T^o$. If $x \in S$, then $x = v_j$ for some $j$ with $1 \leq j \leq p - e(G)$ and so $x \in J_G(v_{j,1}, v_{j,2})$. Hence $x \in T^o$ and so $T^o = T - Ext(G)$. This shows that $T$ is an open monophonic set of $G$. Thus $omn(G) \leq |T| = 3mn(G) - 2e(G)$. ∎

**Theorem 3.6.** *Let $G$ be a connected graph and $n \geq 2(mn(G) - e(G)) + 1$. Then $mn(G \boxtimes K_n) = e(G)(n-1) + omn(G)$.*

**Proof.** Let $S$ be a monophonic set of $G$. If $S$ is an open monophonic set, then $S^o = S - Ext(G)$. Hence $n|S| - (n-1)|S^o| = n|S| - (n-1)(|S| - e(G)) = e(G)(n-1) + |S| \geq e(G)(n-1) + omn(G)$. Now, if $S$ is not an open monophonic set of $G$, then $S^o \subsetneq S - Ext(G)$. Thus, $n|S| - (n-1)|S^o| \geq n|S| - (n-1)(|S| - e(G) - 1) = (n-1)e(G) + |S| + (n-1) \geq (n-1)e(G) + mn(G) + 2(mn(G) - e(G)) = (n-1)e(G) + 3mn(G) - 2e(G)$. By Theorem 3.5, we have $n|S| - (n-1)|S^o| \geq (n-1)e(G) + omn(G)$. Hence the result follows from Corollary 3.2. ∎

**Remark 3.7.** The converse of Theorem 3.6 is not true. For the graph $G = C_6$, we have that $mn(C_6 \boxtimes K_3) = 3$ (see Theorem 3.9). However, $n = 3 < 5 = 2(mn(G) - e(G)) + 1$.

In view of Theorem 3.6, we leave the following problem as an open question.

**Problem 3.8.** Characterize the class of graphs $G$ for which $mn(G \boxtimes K_n) = (n-1)e(G) + omn(G)$.

In the following, we obtain the exact values of the monophonic numbers of some standard classes of the strong product graphs.

**Theorem 3.9.** *For integers $m \geq 3$ and $n \geq 2$,*

$$mn(C_m \boxtimes K_n) = \begin{cases} 3n & \text{if } m = 3 \\ 4 & \text{if } m = 4, 5 \\ 3 & \text{if } m \geq 6 \end{cases}$$

**Proof.** If $m = 3$, then $C_m \boxtimes K_n = K_{3n}$ and so $mn(C_m \boxtimes K_n) = 3n$. So, assume that $m \geq 4$. Since any two non-adjacent vertices of $C_m$ form a monophonic set of $C_m$, we have $mn(C_m) = 2$. Let $m = 4$ and let $S$ be a monophonic set of $C_4$. If $|S| = 2$, then $|S^o| = 0$ and so $n|S| - (n-1)|S^o| = 2n \geq 4$. If $|S| = 3$, then $|S^o| = 1$ and so $n|S| - (n-1)|S^o| = 2n + 1 \geq 5$. If $|S| = 4$, then $S = V(C_4)$ so that $S = S^o$. This implies that $n|S| - (n-1)|S^o| = 4$. Hence it follows from Theorem 3.1 that $mn(C_4 \boxtimes K_n) = 4$. Let $m = 5$ and let $S$ be a monophonic set of $C_5$. If $|S| = 2$, then $|S^o| = 0$ and so $n|S| - (n-1)|S^o| = 2n \geq 4$. If $|S| = 3$, then we have $|S^o| = 1$ or $|S^o| = 2$. Hence $n|S| - (n-1)|S^o| = 2n + 1$ or $n|S| - (n-1)|S^o| = n + 2$, which is greater than or equal to 4. Also, if $|S| = 4$ or $|S| = 5$, then $S = S^o$ and so $n|S| - (n-1)|S^o| = |S|$. Hence it follows from Theorem 3.1 that $mn(C_5 \boxtimes K_n) = 4$ $(n \geq 2)$. Finally, let $m \geq 6$. Then $omn(C_m) = 3 = mn(C_m) + 1$ and so by Corollary 3.4, $mn(C_m \boxtimes K_n) = 3$. ■

**Theorem 3.10.** *For integers $2 \leq r \leq s$ and $n \geq 2$, $mn(K_{r,s} \boxtimes K_n) = 4$.*

**Proof.** If $r \geq 4$, then it is easily seen that $mn(K_{r,s}) = omn(K_{r,s}) = 4$ and so by Corollary 3.3, $mn(K_{r,s} \boxtimes K_n) = 4$. If $r = 3$, then $mn(K_{r,s}) = 3$ and $omn(K_{r,s}) = 4$. Hence, by Corollary 3.4, $mn(K_{r,s} \boxtimes K_n) = 4$. Now, let $r = 2$. If $s = 2$, then $K_{r,s} = C_4$ and so by Theorem 3.9, $mn(K_{r,s} \boxtimes K_n) = 4$. If $s \geq 3$, let $(X, Y)$ be the partite sets of $K_{2,s}$ with $|X| = 2$. Now, $X$ and $Y$ are monophonic sets of $K_{2,s}$. Let $S$ be any monophonic set of $K_{2,s}$. If $S = X$ or $Y$, then $S^o = \phi$ and so $n|S| - (n-1)|S^o| = n|S| \geq 4$. Assume that $S \neq X, Y$. Then $|S| \geq 3$. If $|S| = 3$, then $|S^o| = 1$ and so $n|S| - (n-1)|S^o| = 2n + 1 \geq 5$. If $|S| \geq 4$, then $S^o = S$ or $|S^o| = 1$. If $|S^o| = 1$, then $n|S| - (n-1)|S^o| \geq 3n + 1 \geq 7$. If $S^o = S$, then $n|S| - (n-1)|S^o| = |S|$. Now, let $S = \{x_1, x_2, y_1, y_2\}$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then $S$ is a monophonic set of $K_{2,s}$ with $S^o = S$. Hence it follows from Theorem 3.1 that $mn(K_{2,s} \boxtimes K_n) = 4$. ■

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