Density of integral sets with missing differences

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Abstract

Motzkin posed the problem of finding the maximal density $\mu(M)$ of sets of integers in which the differences given by a set M do not occur. The problem is already settled when $|M| \leq 2$ or M is a finite arithmetic progression. In this paper, we determine $\mu(M)$ when M has some other structure. For example, we determine $\mu(M)$ when M is a finite geometric progression.

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1 Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For a positive real number x and $S \subseteq \mathbb{N}$, we denote by $S(x)$ the number of elements $n \in S$ such that $n \leq x$. The upper and lower densities of S, denoted by $\overline{\delta}(S)$ and $\underline{\delta}(S)$ respectively, are given by

$$\overline{\delta}(S) := \limsup_{x \to \infty} \frac{S(x)}{x}, \quad \underline{\delta}(S) := \liminf_{x \to \infty} \frac{S(x)}{x}.$$

If $\overline{\delta}(S) = \underline{\delta}(S)$, we denote the common value by $\delta(S)$, and say that S has density $\delta(S)$.

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Given a set of positive integers M, we call a set $S \subseteq \mathbb{N}$ is an M*-set* if $a \in S, b \in S$ implies $a - b \notin M$. In an unpublished problem collection, Motzkin [9] posed the problem of determining the quantity

$$\mu(M) := \sup_S \overline{\delta}(S),$$

where the supremum is taken over all M-sets S. In [2], Cantor and Gordon proved that if $|M| = 1$, then $\mu(M) = 1/2$ and that if $M = \{m_1, m_2\}$, then $\mu(M) = [(m_1 + m_2)/2]/(m_1 + m_2)$. The following result is also proved.

Theorem A. *Let* $M_1 = \{m_1, m_2, \ldots\}$ *and* $M_2 = \{dm_1, dm_2, \ldots\}$, *where* d *is a positive integer. Then* $\mu(M_1) = \mu(M_2)$.

By Theorem A, we may assume that $\gcd(m_1, m_2, \ldots) = 1$ for the purpose of determining $\mu(M)$. Later, Haralambis [7] determined $\mu(M)$ for most members of the families $\{1, j, k\}$ and $\{1, 2, j, k\}$. In 1999, Gupta and Tripathi [6] completely determined $\mu(M)$ when M is a finite arithmetic progression.

Theorem B. *If* $M = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$ *with* $\gcd(a, d) = 1$ *and* $n > 1$, *then*

$$\mu(M) = \begin{cases} \frac{2a+(n-1)(d-1)}{2\{2a+(n-1)d\}} & \text{if } d \text{ is odd;} \\ \frac{1}{2} & \text{if } d \text{ is even.} \end{cases}$$

In 2011, Pandey and Tripathi [12] investigated this quantity when M is related to arithmetic progressions. For related results, one may refer to [5], [10] and [11].

Motzkin's problem has connections with some other problems, such as the T-*colouring problem*, problems related to *the fractional chromatic number of distance graphs* and the *Lonely Runner Conjecture*. One may refer to [1], [3], [13].

Recently, Chen and Yang [4], Khovanova and Konyagin [8] studied the upper density among sets of nonnegative integers in which no two elements have quotient belonging to M.

In this paper, the following results are proved.

Theorem 1. *Let* $M = \{1, q, m_3, m_4, \ldots, m_s\}$, *where* $1 < q < m_3 < m_4 < \cdots < m_s$. *If* $m_i \equiv \pm 1 \pmod{q+1}$ *for all integers* $i \in \{3, 4, \ldots, s\}$, *then we have*

$$\mu(M) = \begin{cases} 1/2 & \text{if } q \text{ is odd;} \\ \frac{q}{2(q+1)} & \text{if } q \text{ is even.} \end{cases}$$

From Theorem A and Theorem 1 we obtain the following corollary.

Corollary 1. *If* $M = \{a, aq, \ldots, aq^n\}$, *where* a, q, n *are positive integers with* $q \geq 2$, *then* $\mu(M) = 1/2$ *if* q *is odd, and* $\mu(M) = \frac{q}{2(q+1)}$ *if* q *is even.*

In the next theorems, we shall consider some other sets M with special structure.

Theorem 2. *Suppose that* $M = \{m_1, m_2, \ldots, m_n\}$ *with* $m_1 < m_2 < \cdots < m_n$ *and satisfy the following two conditions:*

(i) $\{m_j - m_i : 2 \leq i < j \leq n\} \subseteq M$;

(ii) the set M *does not contain a multiple of* n.

Then we have $\mu(M) = 1/n$.

Theorem 3. *Let* $M = \{i + kj : 0 \leq k \leq n-1\} \cup \{j\}$ *with* $\gcd(i,j) = 1$ *and* $n \geq 1$. *Let* $i + nj \equiv r \pmod{n+1}$ *with* $0 \leq r \leq n$. *If* $\gcd(r, i+nj) = 1$, *then* $\mu(M) \geq \frac{i+nj-r}{(n+1)(i+nj)}$. *Furthermore, if* $r = n$, *then* $\mu(M) \geq \frac{i+nj-n}{(n+1)(i+nj)}$.

By Theorem 2 and Theorem 3, we have the following corollary.

Corollary 2. *If* $M = \{i, j, i+j, i+2j\}$ *with* $i < j$ *and* $\gcd(i,j) = 1$, *then*

$$\mu(M) \begin{cases} = 1/4 & \text{if } i + 3j \equiv 0 \text{ or } 2 \pmod{4}; \\ \geq \frac{i+3j-r}{4(i+3j)} & \text{if } i + 3j \equiv r \pmod{4}, \text{ where } r = 1 \text{ or } 3. \end{cases}$$

Theorem 4. *If* $M = \{i + k_1 j : 0 \leq k_1 \leq n-1\} \cup \{k_2 j : 1 \leq k_2 \leq n\}$ *with* $n \geq 1$, $\gcd(i,j) = 1$ *and* $i + nj \equiv 1 \pmod{n+1}$, *then*

$$\mu(M) = \frac{i + nj - 1}{(n+1)(i+nj)}.$$

2 Preliminary Lemmas

In this section, we state two useful lemmas which give the lower and upper bound for $\mu(M)$.

Lemma 1. *(See [2, Theorem 1].) Let* $M = \{m_1, m_2, m_3, \ldots\}$*, and let* c *and* m *be positive integers such that* $\gcd(c, m) = 1$*. Put*

$$d = \min_k |cm_k|_m,$$

where $|x|_m$ *denotes the absolute value of the absolutely least remainder of* $x \pmod m$*. Then* $\mu(M) \geq d/m$*.*

Lemma 2. *(See [7, Lemma 1].) Let* M *be a given set of positive integers,* α *a real number in the interval* $[0, 1]$*, and suppose that for any* M*-set* S *with* $0 \in S$ *there exists a positive integer* k *(possibly dependent on* S*) such that* $S(k) \leq (k+1)\alpha$*. Then* $\mu(M) \leq \alpha$*.*

3 Proof of Theorem 1

Suppose that q is odd. For any M-set S, by $1 \in M$, we know that it contains no two consecutive integers. Thus, $\mu(M) \leq 1/2$. On the other hand, by $m_i \equiv \pm 1 \pmod{q+1}$ and $2 \mid q+1$, we know that M consists of only odd numbers. Hence, the example $S = \{0, 2, 4, \ldots\}$ shows that equality can hold, and so $\mu(M) = 1/2$.

Now we consider the case in which q is even. If $\mu(M) > \frac{q}{2(q+1)}$, then there exists an M-set S and an interval $[c, c+q]$ such that $|S \cap [c, c+q]| > q/2$. That is, $|S \cap [c, c+q]| \geq q/2 + 1$. Noting that $1 \in M$, we know that S does not contain consecutive integers, and so $S \cap [c, c+q] = \{c, c+2, \ldots, c+q\}$. It follows that $q \in S - S$, a contradiction. Hence, $\mu(M) \leq \frac{q}{2(q+1)}$.

Next we shall prove that $\mu(M) \geq \frac{q}{2(q+1)}$. Since

$$\frac{q}{2} \times m_i \equiv \pm\frac{q}{2} \pmod{q+1}$$

for all integers $i \in \{3,4,\ldots,s\}$ and $1 \cdot \frac{q}{2} \equiv \frac{q}{2} \pmod{q+1}$, $q \cdot \frac{q}{2} \equiv -\frac{q}{2} \pmod{q+1}$, by Lemma 1, taking $c = q/2$ and $m = q+1$, we have $\gcd(c,m) = 1$ and $d = q/2$. Thus, $\mu(M) \geq \frac{q}{2(q+1)}$.

Therefore, $\mu(M) = \frac{q}{2(q+1)}$ if q is even.

4 Proof of Theorem 2

For any positive integer x and an M-set $S \subseteq [0,x]$, we shall prove that $|S| \leq (x + m_n + 1)/n$.

First, we prove $|S + M| \geq (n-1)|S|$ by induction on $|S|$. Clearly, it is true for $|S| = 1$. Now suppose that $|S' + M| \geq (n-1)|S'|$ holds for all M-sets S' with $|S'| < |S|$. Write $S = \{b_1, b_2, \ldots, b_{|S|}\}$. By $\{m_j - m_i : 2 \leq i < j \leq n\} \subseteq M$, it follows that

$$(b_{|S|} + \{m_2, m_3, \ldots, m_n\}) \cap (\{b_1, b_2, \ldots, b_{|S|-1}\} + M) = \emptyset.$$

Otherwise, there exist three integers i, j, k with $2 \leq i < j \leq n$ and $1 \leq k \leq |S| - 1$ such that $b_{|S|} + m_i = b_k + m_j$, and then $b_{|S|} - b_k = m_j - m_i \in M$, a contradiction. Hence, by the induction hypothesis, we have

$$|\{b_1, \ldots, b_{|S|}\} + M| \geq (n-1) + |\{b_1, \ldots, b_{|S|-1}\} + M| \geq (n-1)|S|.$$

By $S \cap (S + M) = \emptyset$ and $S \cup (S + M) \subseteq [0, x + m_n]$, it follows that $n|S| \leq |S| + |S + M| \leq x + m_n + 1$, and so $|S| \leq (x + m_n + 1)/n$.

Hence, $\mu(M) = \sup_S \overline{\delta}(S) \leq \lim_{x\to\infty}(x + m_n + 1)/(nx) = 1/n$.

On the other hand, since M does not contain a multiple of n, the set $\{0, n, 2n, \ldots\}$ is an M-set. So $\mu(M) \geq 1/n$.

Therefore, $\mu(M) = 1/n$.

5 Proof of Theorem 3

Let $t = i + nj$. Then $t \equiv r \pmod{n+1}$. We consider the following two cases.

Case 1: $\gcd(r,t)=1$. By $\gcd(i,j)=1$, we have $\gcd(j,t)=1$. Then there exists an integer x such that

$$xj \equiv \frac{t-r}{n+1} \pmod t.$$

Since $\gcd(r,t)=1$, it follows that $\gcd(\frac{t-r}{n+1},t)=1$, and then $\gcd(x,t)=1$. For such x we have

$$x(i+kj) \equiv \frac{(k+1)(t-r)}{n+1}+r \pmod t$$

for $k=0,1,\dots,n-1$. Noting that $\gcd(x,t)=1$ and

$$x(i+(n-1)j) \equiv \frac{n(t-r)}{n+1}+r \equiv -\frac{(t-r)}{n+1} \pmod t,$$

by Lemma 1, we have $\mu(M) \geq \frac{t-r}{(n+1)t} = \frac{i+nj-r}{(n+1)(i+nj)}$.

Case 2: $r=n$ and $\gcd(r,t)>1$. Then there exists an integer x' such that

$$x'j \equiv \frac{t+1}{n+1} \pmod t.$$

Since $\gcd(\frac{t+1}{n+1},t)=1$, we have $\gcd(x',t)=1$. For such x' we have

$$x'(i+kj) \equiv \frac{(k+1)(t-n)}{n+1}+k \pmod t$$

for $k=0,1,\dots,n-1$. Noting that $\gcd(x',t)=1$,

$$x'i \equiv \frac{t-n}{n+1} \pmod t,$$

and

$$x'(i+(n-1)j) \equiv \frac{n(t-n)}{n+1}+n-1 \equiv -\frac{(t+1)}{n+1} \pmod t,$$

by Lemma 1, we have $\mu(M) \geq \frac{t-n}{(n+1)t} = \frac{i+nj-n}{(n+1)(i+nj)}$.

6 Proof of Corollary 2

We consider the following three cases.

Case 1: $i + 3j \equiv r \pmod 4$ with $r = 1$ or 3. By Theorem 3, we have $\mu(M) \geq \frac{i+3j-r}{4(i+3j)}$.

Case 2: $i + 3j \equiv 0 \pmod 4$, then we have $i \equiv j \equiv 1 \pmod 4$ or $i \equiv j \equiv 3 \pmod 4$. In this case, the set M does not contain a multiple of 4, thus by Theorem 2, we have $\mu(M) = 1/4$.

Case 3: $i + 3j \equiv 2 \pmod 4$, then we have $i \equiv 1 \pmod 4$, $j \equiv 3 \pmod 4$ or $i \equiv 3 \pmod 4$, $j \equiv 1 \pmod 4$. In this case, we know that the set M contains only one even number $i + j$. Let

$$S = \bigcup_{k=0}^{\infty} \left(2k(i+j) + \{0, 2, 4, \ldots, i+j-2\}\right).$$

Clearly, S is an M-set and $\delta(S) = 1/4$. Hence $\mu(M) \geq 1/4$. By the proof of Theorem 2, we have $\mu(M) \leq 1/4$. Therefore, $\mu(M) = 1/4$.

7 Proof of Theorem 4

First we follow the proof of Theorem 3 and show that $\mu(M) \geq \frac{i+nj-1}{(n+1)(i+nj)}$. Let $i + nj = t$. Then there exists an integer x such that

$$xj \equiv \frac{t-1}{n+1} \pmod t.$$

Clearly, $\gcd(x, t) = 1$. For such x we have

$$x(i + k_1 j) \equiv \frac{(k_1+1)(t-1)}{n+1} + 1 \pmod t$$

for $k_1 = 0, 1, \ldots, n-1$ and

$$x(k_2 j) \equiv \frac{k_2(t-1)}{n+1} \pmod t$$

for $k_2 = 1, 2, \ldots, n$. Noting that

$$x(i + (n-1)j) \equiv \frac{n(t-1)}{n+1} + 1 \equiv -\frac{t-1}{n+1} \pmod t$$

and

$$x(nj) \equiv \frac{n(t-1)}{n+1} \equiv -\frac{t-1}{n+1} - 1 \pmod t,$$

by Lemma 1 we have $\mu(M) \geq \frac{t-1}{(n+1)t}$.

Now we will prove $\mu(M) \leq \frac{t-1}{(n+1)t}$. Let S be any M-set with $0 \in S$. Then for $t = i + nj$,

$$\bigcup_{m=1}^{(t-n-2)/(n+1)} A_m \cup B$$

is a decomposition of $\{0, 1, 2, \ldots, t-1\}$ into disjoint sets, where

$$B = \{0, j\} \cup \{i + kj : 0 \leq k \leq n-1\},$$

$$A_m = \{(j-i)m + i + kj : 1 \leq k \leq n-1\} \cup \{(j-i)m, (j-i)m + j\}$$

with $1 \leq m \leq (t-n-2)/(n+1)$ and the elements of each set A_m are considered modulo t.

Since S is an M-set and $0 \in S$, it follows that $|A_m \cap S| \leq 1$ for each m and $|S \cap B| = 1$.

Hence, $S(t-1) \leq 1 + (t-n-2)/(n+1) = (t-1)/(n+1)$ for any M-set S. By Lemma 2, it follows that $\mu(M) \leq (t-1)/((n+1)t)$.

Therefore, we obtain

$$\mu(M) = \frac{t-1}{(n+1)t} = \frac{i + nj - 1}{(n+1)(i+nj)}.$$

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