# Locating and differentiating-total dominating sets in unicyclic graphs *

**Wenjie NING[1†] Mei LU[2‡] Jia GUO[2§]**

[1]College of Science, China University of Petroleum (East China), Qingdao 266580, China

[2]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

**Abstract**

Given a graph $G = (V, E)$ with no isolated vertex, a subset $S$ of $V$ is a total dominating set of $G$ if every vertex in $V$ is adjacent to a vertex in $S$. A total dominating set $S$ of $G$ is a locating-total dominating set if for every pair of distinct vertices $u$ and $v$ in $V - S$, we have $N(u) \cap S \neq N(v) \cap S$, and $S$ is a differentiating-total dominating set if for every pair of distinct vertices $u$ and $v$ in $V$, we have $N[u] \cap S \neq N[v] \cap S$. The locating-total domination number (or the differentiating-total domination number) of $G$, denoted by $\gamma_t^L(G)$ (or $\gamma_t^D(G)$), is the minimum cardinality of a locating-total dominating set (or a differentiating-total dominating set) of $G$. In this paper, we investigate the bounds of locating and differentiating-total domination numbers of unicyclic graphs.

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**Keywords:** locating-total dominating set, differentiating-total dominating set, locating-total domination number, differentiating-total domination number, unicyclic graph

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†email: ningwenjie-0501@163.com

‡email: mlu@math.tsinghua.edu.cn

§email: guojia199011@163.com

# 1 Introduction

The concept of a locating-total dominating set and a differentiating dominating set in a graph was introduced in [6, 7]. It has been studied in [1]-[4] and elsewhere. The problem of placing monitoring devices in a system such that every site (including the monitors themselves) in the system is adjacent to a monitor can be modelled by total domination in graphs. Applications where it is also important that if there is a problem in a device, its location can be uniquely identified by the set of monitors, can be modelled by the combination of total domination and locating in graphs. In this paper, we consider two different variations of this combination.

Loops and parallel edges are admissible in a graph here. Given a graph $G = (V, E)$, the *degree* of $v$ in $G$, denoted by $d(v)$, is the number of edges incident with $v$, each loop counting as two edges. The *maximum degree* of $G$, denoted by $\Delta(G)$ or $\Delta$, is equal to $\max\{d(v) \mid v \in V\}$. A vertex of degree one is a *leaf* and the edge incident with a leaf is a *pendent edge.* A *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a support vertex adjacent to at least two leaves. We denote by $L(G)$ the set of leaves of $G$, $S(G)$ the set of support vertices of $G$ and $S_1(G)$ the set of strong support vertices of $G$, respectively. $G$ is *simple*, if $G$ has neither loops nor parallel edges. $G$ is *connected*, if for any two vertices $x$ and $y$, there is an $xy$-path in $G$. For a subset $S$ of $V$, we use $G[S]$ to denote the subgraph induced by $S$. If $A$ and $B$ are two disjoint subsets of $V$, then $[A, B] = \{uv \in E(G) | u \in A, v \in B\}$. Let $G$ and $H$ be two disjoint graphs. The *disjoint union* of $G$ and $H$, denoted by $G+H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G_1 \cong \cdots \cong G_k$, we write $kG_1$ for $G_1 + \cdots + G_k$.

Let $G = (V, E)$ be a simple graph on $n$ vertices. For a vertex $v$ in $G$, the set $N(v) = \{u \in V \mid uv \in E\}$ is called the *open neighborhood* of $v$ and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of $v$. For a subset $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ is the *open neighborhood* of $S$ and $N[S] = N(S) \cup S$ is the *closed neighborhood* of $S$. A subset $S$ of $V$ is a *dominating set*

($DS$) of $G$ if $N[S] = V$ and $S$ is a *total dominating set* ($TDS$) of $G$ if $N(S) = V$. A TDS $S$ is a *locating-total dominating set* ($LTDS$) if for every pair of distinct vertices $u$ and $v$ in $V - S$, $N(u) \cap S \neq N(v) \cap S$, and $S$ is a *differentiating-total dominating set* ($DTDS$) if for every pair of distinct vertices $u$ and $v$ in $V$, $N[u] \cap S \neq N[v] \cap S$. The *locating-total domination number* (or *differentiating-total domination number*) of $G$, denoted by $\gamma_t^L(G)$ (or $\gamma_t^D(G)$), is the minimum cardinality of a $LTDS$ (or $DTDS$) of $G$. A $LTDS$ (or $DTDS$) of cardinality $\gamma_t^L(G)$ (or $\gamma_t^D(G)$) is a *$\gamma_t^L(G)$-set* (or *$\gamma_t^D(G)$-set*).

A path of order $n$ is $P_n$. A star of order $n$ is denoted by $S_n$. A tree is called a *double star* $S(p,q)$, if it is obtained from $S_{p+2}$ and $S_{q+1}$ by identifying a leaf of $S_{p+2}$ with the center of $S_{q+1}$, where $p, q \geq 1$.

A connected graph of order $n$ and size $n$ is a *unicyclic graph*. By definition, a unicyclic graph might have loops and parallel edges. Given a graph $G = (V, E)$, the *corona*, *cor(G)*, is a graph obtained from $G$ by adding a pendent edge to each vertex of $G$.

Locating and differentiating-total dominating sets in trees and other graphs have been studied in several papers, see [2]-[5], [7] and [8]. In this paper, we investigate the bounds of locating and differentiating-total domination numbers of unicyclic graphs.

## 2 Locating-total domination number of a unicyclic graph

Throughout this section, let $G = (V, E)$ be a simple unicyclic graph. Let $L(G) = L, S(G) = S, S_1(G) = S_1, S - S_1 = S_2$ and $A$ be a $\gamma_t^L(G)$-set of $G$ that contains a minimum number of leaves. Then we have $S \subseteq A$. If $S \neq \emptyset$, then for every $v \in S$, exactly one leaf adjacent to $v$ is not in $A$. Let $B = \{v \notin A \mid |N(v) \cap A| = 1\}$ and $C = \{v \notin A \mid |N(v) \cap A| \geq 2\}$. Let $L_1 = L \cap A$, $Q_1 = A - (L_1 \cup S)$, $L_2 = L - L_1$ and $Q_2 = B - L_2$. Then $A = L_1 \cup S \cup Q_1$, $B = L_2 \cup Q_2$, $V = A \cup B \cup C$ and we have the following lemma.

**Lemma 2.1** *Let $|L| = l$, $|S| = s$ and $|S_1| = s_1$. Suppose $s \neq 0$. Then*

*(1) $|[A, B \cup C]| \geq |B| + 2|C| = 2n - 2|A| - |B|$ and the equality holds if and only if $|N(v) \cap A| = 2$ for every vertex $v \in C$;*

*(2) $|[A, B \cup C]| = n - |E(G[A])| - |E(G[Q_2 \cup C])|$;*

*(3) $|L_1| = l - s$, $|L_2| = s$, $|Q_1| = |A| - l$, $|Q_2| = |B| - s$;*

*(4) $|Q_2| \leq |Q_1|$, $|B| \leq |A| - l + s$ and $|B| = |A| - l + s$ if and only if $|Q_1| = |Q_2|$;*

*(5) $|E(G[Q_2 \cup C])| \geq \frac{|Q_2|}{2}$ and the equality holds if and only if $G[Q_2 \cup C] \cong \frac{|Q_2|}{2} K_2 + |C| K_1$ and $C$ is an independent set in $G[Q_2 \cup C]$;*

*(6) $|E(G[S \cup Q_1])| \geq \frac{1}{2}(s - s_1 + |A| - l)$ and the equality holds if and only if $G[S \cup Q_1] \cong s_1 K_1 + \frac{|S_2 \cup Q_1|}{2} K_2$ and $S_1$ is an independent set in $G[S \cup Q_1]$;*

*(7) $|E(G[A])| \geq \frac{|A|}{2}$ and the equality holds if and only if $G[A] \cong \frac{|A|}{2} K_2$;*

*(8) $|E(G[A])| \geq l - s$ and the equality holds if and only if $|Q_1| = 0$, $|E(G[S])| = 0$ and every component of $G[A]$ is a star.*

**Proof.** (1)-(3) are obvious.

(4) As $A$ is an LTDS of $G$, for every $v \in Q_2$, we have $N(v) \cap A \subseteq Q_1$ and $N(u) \cap A \neq N(v) \cap A$ for every $u, v \in Q_2$ with $u \neq v$. Then $|Q_2| \leq |Q_1|$. By (3), $|B| = |Q_2| + |L_2| \leq |Q_1| + |L_2| = |A| - l + s$.

(5) Since $d(v) \geq 2$ for any $v \in Q_2 \subseteq B$, $N(v) \cap (C \cup Q_2) \neq \emptyset$. Thus,

$$|E(G[Q_2 \cup C])| = \frac{1}{2} \sum_{v \in Q_2 \cup C} d_{G[Q_2 \cup C]}(v) \geq \frac{1}{2} \sum_{v \in Q_2} d_{G[Q_2 \cup C]}(v) \geq \frac{|Q_2|}{2},$$

and the equality holds if and only if $G[Q_2 \cup C] \cong \frac{|Q_2|}{2} K_2 + |C| K_1$ and $C$ is an independent set in $G[Q_2 \cup C]$.

(6) For every $v \in S_2 \cup Q_1$, $N(v) \cap (S \cup Q_1) \neq \emptyset$ by the definition of LTDS. Thus,

$$|E(G[S \cup Q_1])| \geq \frac{1}{2} \sum_{v \in S_2 \cup Q_1} d_{G[S \cup Q_1]}(v) \geq \frac{1}{2}|S_2 \cup Q_1| = \frac{1}{2}(s - s_1 + |A| - l),$$

and the equality holds if and only if $G[S \cup Q_1] \cong s_1 K_1 + \frac{|S_2 \cup Q_1|}{2} K_2$ and $S_1$ is an independent set in $G[S \cup Q_1]$.

(7) By the definition of LTDS, we have $|E(G[A])| = \frac{1}{2} \sum_{v \in A} d_{G[A]}(v) \geq \frac{|A|}{2}$ and the equality holds if and only if $G[A] \cong \frac{|A|}{2} K_2$.

(8) Firstly, we have $|E(G[A])| \geq | E(G[L_1 \cup S_1])| \geq | L_1|$. By (3), $|E(G[A])| \geq l - s$. By the definition of an LTDS, the equality holds if and only if $|Q_1| = 0$, $|E(G[S])| = 0$ and every component of $G[A]$ is a star.

■

In [8], the following result was stated.

**Theorem 2.2** [8] *For* $n \geq 3$, $\gamma_t^L(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Let $\xi_1$ be the family of simple unicyclic graphs that can be obtained from any simple unicyclic graph $G'$ by first attaching at least two leaves to each vertex of $G'$, and then subdividing each edge of $G'$ exactly once.

**Theorem 2.3** *If $G$ is a simple unicyclic graph of order $n$, $|L(G)| = l$ and $|S(G)| = s$, then $\gamma_t^L(G) \geq (n + 2(l - s))/3$ and the equality holds if and only if $G \in \xi_1$.*

**Proof.** If $G = C_n$, then $\gamma_t^L(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor > n/3$ by Theorem 2.2. Now suppose $s \neq 0$. From Lemma 2.1 (1) and (4), we obtain $|[A, B \cup C]| \geq 2n - 3|A| + l - s$. By Lemma 2.1 (2) and (8), we have $|[A, B \cup C]| = n - |E(G[A])| - | E(G[Q_2 \cup C]) \leq n - |E(G[A])| \leq n - l + s$. Thus, $\gamma_t^L(G) \geq (n + 2(l - s))/3$.

The equality $\gamma_t^L(G) = (n + 2(l - s))/3$ holds if and only if $|Q_2| = |Q_1|$, $|N(v) \cap A| = 2$ for every vertex $v \in C$, $|E(G[Q_2 \cup C])| = 0$ and $|E(G[A])| = l - s$. By Lemma 2.1 (5) and (8), $C$ is an independent set in $G[Q_2 \cup C]$, $|Q_2| = |Q_1| = 0$, $|E(G[S])| = 0$ and every component of $G[A]$ is a star. Thus, every connected component of $G[A \cup B]$ is a star on at least three vertices and then $G \in \xi_1$. ■

Let us define a set of trees $\eta_1 = \{P_4\} \cup \{S_a | a \geq 3\}$. Let $\xi_2$ be the family of simple unicyclic graphs that can be obtained from $r$ disjoint copies of several trees in $\eta_1$ by first adding $r$ edges so that they are only incident with support vertices, each pair of them are non-parallel and the resulting graph is connected and has no loops, and then subdividing each new edge exactly once.

**Theorem 2.4** *Suppose $G$ is a simple unicyclic graph of order $n$, $|L(G)| = l$, $|S(G)| = s$ and $|S_1(G)| = s_1$. Then*

$$\gamma_t^L(G) \geq (2n + 3(l - s) - s_1)/5,$$

*with equality if and only if $G \in \xi_2$.*

**Proof.** If $G = C_n$, then $\gamma_t^L(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor > 2n/5$ by Theorem 2.2. Now suppose $s \neq 0$. From Lemma 2.1 (1) and (4), we obtain $|[A, B \cup C]| \geq 2n - 3|A| + l - s$. By Lemma 2.1 (2) and (6), $|[A, B \cup C]| \leq n - |E(G[A])| = n - |L_1| - |E(G[S \cup Q_1])| \leq n - (l - s) - \frac{1}{2}(s - s_1 + |A| - l)$. Thus $\gamma_t^L(G) = |A| \geq (2n + 3(l - s) - s_1))/5$.

The equality $\gamma_t^L(G) = (2n + 3(l - s) - s_1))/5$ holds if and only if $|E(G[Q_2 \cup C])| = 0$, $|N(v) \cap A| = 2$ for every vertex $v \in C$, $|Q_1| = |Q_2|$, $G[S \cup Q_1] \cong s_1 K_1 + \frac{|S_2 \cup Q_1|}{2} K_2$ and $S_1$ is an independent set in $G[S \cup Q_1]$. By the same argument as that of Theorem 2.3, we have that $0 = |Q_2| = |Q_1|$ and $A = L_1 \cup S$. Hence $G[S] \cong s_1 K_1 + \frac{s - s_1}{2} K_2$. Consequently, every connected component of $G[A \cup B]$ is either a $P_4$, or a $S_a$, where $a \geq 3$. Thus, $G \in \xi_2$. ∎

Let us define a set of trees $\eta_2 = \{S_a | a \geq 3\} \cup \{P_b | \ b \equiv 0 \pmod 4\}$ and a set of cycles $\eta_3 = \{C_{n'} | \ n' \equiv 0 \pmod 4\}$. For every tree $T \in \eta_2$, if $T = S_a$ for some $a \geq 3$, we define $D_T = S(S_a)$; if $T = P_b = v_1 v_2 \cdots v_b \in \eta_2$ for some $b \equiv 0 \pmod 4$, then we define $D_T = \cup_{i=1}^{b/4} \{v_{4i-2}, v_{4i-1}\}$. For every cycle $C_{n'} = v_1 v_1 \cdots v_{n'} \in \eta_3$, we define $D_{C_{n'}} = \cup_{i=1}^{n'/4} \{v_{4i-2}, v_{4i-1}\}$.

Let $\xi_3$ be the family of simple unicyclic graphs that can be obtained from $r$ disjoint copies of several trees in $\eta_2$ by first adding $r$ edges so that they are only incident with vertices in $\cup_{T \in \eta_2} D_T$, each pair of them are non-parallel and the resulting graph is connected and has no loops, and then subdividing each new edge exactly once.

Let $\xi_4$ be the family of simple unicyclic graphs that can be obtained from a $C_{n'} \in \eta_3$ and $r(r \geq 0)$ disjoint copies of several trees in $\eta_2$ by first adding $r$ edges so that they are only incident with vertices in $(\cup_{T\in\eta_2} D_T) \cup D_{C_{n'}}$ and the resulting graph is connected, and then subdividing each new edge exactly once.

**Theorem 2.5** *Suppose $G$ is a simple unicyclic graph of order $n$, $|L(G)| = l$, $|S(G)| = s$ and $|S_1(G)| = s_1$. Then*

$$\gamma_t^L(G) \geq (n + l - s)/2 - (s + s_1)/4,$$

*with equality if and only if $G \in \xi_3 \cup \xi_4$.*

**Proof.** If $G = C_n$, then $\gamma_t^L(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor \geq n/2$ by Theorem 2.2 and $\gamma_t^L(C_n) = n/2$ if and only if $n \equiv 0 \pmod 4$, i.e., $C_n \in \xi_4$. Now suppose $s \neq 0$. From Lemma 2.1 (1), we obtain $|[A, B\cup C]| \geq 2n - 2|A| - |B|$. On the other hand,

$$\begin{array}{rlr} & |[A, B \cup C]| = n - |E(G[A])| - |E(G[Q_2 \cup C])| & \text{Lemma 2.1 (2)} \\ \leq & n - |E(G[A])| - \frac{|Q_2|}{2} & \text{Lemma 2.1 (5)} \\ = & n - \frac{|B|-s}{2} - (|L_1| + |E(G[S \cup Q_1])|) & \text{Lemma 2.1 (3)} \\ \leq & n - \frac{|B|-s}{2} - ((l - s) + \frac{1}{2}(s - s_1 + |A| - l)) & \text{Lemma 2.1 (6).} \end{array}$$

Combining with $|[A, B \cup C]| \geq 2n - 2|A| - |B|$, we have

$$\frac{3}{2}|A| \geq n - \frac{|B|}{2} - s + \frac{l}{2} - \frac{s_1}{2}.$$

By Lemma 2.1 (4), we have $2|A| \geq n + l - s - \frac{s+s_1}{2}$, which implies $\gamma_t^L(G) = |A| \geq (n + l - s)/2 - (s + s_1)/4$.

The equality $\gamma_t^L(G) = (n + l - s)/2 - (s + s_1)/4$ holds if and only if $|Q_1| = |Q_2|$, $G[Q_2 \cup C] \cong \frac{|Q_2|}{2}K_2 + |C|K_1$ and $C$ is an independent set in $G[Q_2 \cup C]$, $G[S \cup Q_1] \cong s_1K_1 + \frac{|S_2 \cup Q_1|}{2}K_2$ and $S_1$ is an independent set in $G[S \cup Q_1]$, and $|N(u_2) \cap A| = 2$ for every $u_2 \in C$. For every $u_1 \in Q_2 \subseteq B$, $N(u_1) \cap A \subseteq Q_1$ by the definition of an LTDS and $|N(u_1) \cap Q_1| = 1$.

If $|Q_2| = 0$, then $G \in \xi_2 \subset \xi_3$ by the same argument as that of Theorem 2.4, where $\xi_2$ is defined as in Theorem 2.4.

Now we consider the case $|Q_1| = |Q_2| \neq 0$. Note that $|Q_1| = |Q_2|$, every vertex in $Q_2$ is adjacent to exactly one vertex in $Q_1$, $G[S_2 \cup Q_1] \cong \frac{|S_2 \cup Q_1|}{2}K_2$

and $G[Q_2] \cong \frac{|Q_2|}{2} K_2$. Thus, every connected component of $G[Q_1 \cup Q_2 \cup S_2]$ is either a cycle of order $n'$, where $n' \equiv 0 \pmod 4$, or a path of order $a_i$, where $a_i \equiv 2 \pmod 4$. Consequently, every component of $G[A \cup B]$ is either in $\eta_3$, or in $\eta_2$.

Suppose $X_1, X_2, \cdots, X_{\omega_1}$ are the components of $G[A \cup B]$. For every $X_j$, if $X_j = P_{b_j} = v_1 v_2 \cdots v_{b_j}$ for some $b_j$ with $b_j \equiv 0 \pmod 4$, then we define $D_{X_j} = \cup_{i=1}^{b_j/4} \{v_{4i-2}, v_{4i-1}\}$; if $X_j = S_{a_j}$ for some $a_j \geq 3$, then we define $D_{X_j} = S(X_j)$; if $X_j = C_{n'} = v_1 v_2 \cdots v_{n'}$ for some $n'$ with $n' \equiv 0 \pmod 4$, then we define $D_{X_j} = \cup_{i=1}^{n'/4} \{v_{4i-2}, v_{4i-1}\}$. Thus, we have $S \cup Q_1 = \cup_{j=1}^{\omega_1} D_{X_j}$ by relabelling the vertices of $C_{n'}$ if possible. Note that for every vertex $u \in C$, $|N(u) \cap A| = |N(u) \cap (S \cup Q_1)| = 2$. If $G[A \cup B]$ contains a cycle, then $G \in \xi_4$; otherwise, $G \in \xi_3$. Thus, $G \in \xi_3 \cup \xi_4$. ■

**Remark.** If $n \geq l - s + 3s_1$, the lower bound in Theorem 2.4 is better than that given in Theorem 2.3. If $n \geq (2l + 3s + s_1)/2$, the lower bound in Theorem 2.5 is better than that given in Theorem 2.4.

# 3 Differentiating-total domination number of a unicyclic graph

Note that not every unicyclic graph has a *DTDS*. In Theorem 3.1, we characterize the simple unicyclic graphs having a *DTDS*.

Let $\Omega$ be the family of simple unicyclic graphs $G$ in which the unique cycle $C$ is of length three and $C$ has at least two vertices of degree 2.

**Theorem 3.1** *Suppose $G$ is a simple unicyclic graph. Then $G$ has a DTDS if and only if $G \notin \Omega$.*

**Proof.** It suffices to show $G$ has no *DTDS* if and only if $G \in \Omega$.

If $G \in \Omega$, then we may assume the unique cycle $C = uvwu$ in $G$ satisfies $d(u) = d(v) = 2$. Then $N[u] = N[v] = \{u, v, w\}$. For every subset $S$ of $V$, we have $N[u] \cap S = N[v] \cap S$. Thus, $G$ has no *DTDS*.

On the other hand, if $G$ has no *DTDS*, then $V$ is not a *DTDS* of $G$. Since $V$ is a *TDS* of $G$, by the definition of a *DTDS*, there exists a pair of distinct vertices $u$ and $v$ in $V$ such that $N[u] = N[v]$. As $G$ is a simple unicyclic graph, we have $N[u] = N[v] = \{u, v, w\}$ for some $w \in V$ and $d(u) = d(v) = 2$. Thus, $G \in \Omega$. ∎

In the following, let $G$ be a simple unicyclic graph with $G \notin \Omega$. Let $L(G) = L, S(G) = S$ and $A$ be a $\gamma_t^D(G)$-set of $G$. Then $S \subseteq A$. Let $B = \{v \notin A \mid |N(v) \cap A| = 1\}$ and $C = \{v \notin A \mid |N(v) \cap A| \geq 2\}$. Let $L_1 = L \cap A$, $Q_1 = A - (L_1 \cup S)$, $L_2 = L - L_1$, $Q_2 = B - L_2$ and $\omega$ be the number of components of $G[A]$. Then $A = L_1 \cup S \cup Q_1$, $B = L_2 \cup Q_2$ and $V = A \cup B \cup C$. We have the following lemma.

**Lemma 3.2** *Let $|L| = l$, $|S| = s$. Suppose $s \neq 0$. Then*

*(1) $|[A, B \cup C]| \geq |B| + 2|C| = 2n - 2|A| - |B|$ and the equality holds if and only if $|N(v) \cap A| = 2$ for every $v \in C$;*

*(2) $|[A, B \cup C]| = n - |E(G[A])| - |E(G[Q_2 \cup C])|$;*

*(3) $|L_1| \geq l - s$ and $|L_2| \leq s$;*

*(4) $|Q_2| \leq |A| - |L_1| - |L_2|$, i.e., $|B| \leq |A| - |L_1|$;*

*(5) $|E(G[Q_2 \cup C])| \geq \frac{|Q_2|}{2}$ and the equality holds if and only if $G[Q_2 \cup C] \cong |C|K_1 + \frac{|Q_2|}{2}K_2$ and $C$ is an independent set in $G[Q_2 \cup C]$;*

*(6) $\omega \leq \frac{|A|}{3}$, $|E(G[A])| \geq \frac{2|A|}{3}$ and the equality holds if and only if $G[A] \cong \frac{|A|}{3}P_3$;*

*(7) $|E(G[A])| \geq |L_1|$ and the equality holds if and only if $|Q_1| = 0$, $|E(G[S])| = 0$ and every component of $G[A]$ is a star on at least three vertices.*

**Proof.** From the fact that $A$ is a $\gamma_t^D(G)$-set of $G$, (1)-(2) hold.

(3) Since $N[u] \cap A \neq N[v] \cap A$ for every $u, v \in L_2$ with $u \neq v$, $|L_2| \leq s$. Note that $|L_1| + |L_2| = l$. Thus, $|L_1| \geq l - s$.

(4) Since for every $u \in L_2$ and $v \in Q_2$, $N[u] \cap A \neq N[v] \cap A$ and $N(v) \cap L_1 = \emptyset$, $|Q_2| \leq |A| - |L_1| - |L_2|$. $|B| \leq |A| - |L_1|$ holds by $B = Q_2 \cup L_2$.

(5) By a similar proof as in Lemma 2.1 (5), (5) holds.

(6) By the definition of the DTDS, every component of $G[A]$ has at least 3 vertices. Thus, $\omega \leq \frac{|A|}{3}$. Note that either $|E(G[A])| = |A| - \omega$ or

$|E(G[A])| = |A| - \omega + 1$. Hence $|E(G[A])| \geq |A| - \frac{|A|}{3} = \frac{2|A|}{3}$. Moreover, the equality $|E(G[A])| = \frac{2|A|}{3}$ holds if and only if $G[A] \cong \frac{|A|}{3}P_3$.

(7) Obviously, we have $|E(G[A])| \geq |E(G[L_1 \cup S])| \geq |L_1|$. By the definition of a DTDS, the equality holds if and only if $|Q_1| = 0$, $|E(G[S])| = 0$ and every component of $G[A]$ is a star on at least three vertices. ∎

In [7], the following result was stated.

**Theorem 3.3** [7] *For $n \geq 3$,*

$$\gamma_t^D(P_n) = \begin{cases} \lceil \frac{3n}{5} \rceil & \text{if } n \not\equiv 3 \pmod 5, \\ \lceil \frac{3n}{5} \rceil + 1 & \text{if } n \equiv 3 \pmod 5. \end{cases}$$

**Theorem 3.4** *For $n \geq 4$, we have $\gamma_t^D(C_n) = \gamma_t^D(P_n)$.*

**Proof.** Suppose $C_n = v_1v_2\cdots v_nv_1$. Let $D_1$ be a *DTDS* of $P_n = C_n - v_1v_2$. It is easy to check that $D_1$ is also a *DTDS* of $C_n$. Thus, we obtain $\gamma_t^D(P_n) = \gamma_t^D(C_n - v_1v_2) \geq \gamma_t^D(C_n)$.

On the other hand, suppose $D_2$ is a $\gamma_t^D(C_n)$-set of $C_n$. Since $\gamma_t^D(P_n) \geq \gamma_t^D(C_n) = |D_2|$, we obtain $|D_2| \leq n-1$ by Theorem 3.3. Thus, there is a vertex of $C_n$ which is not contained in $D_2$. Without loss of generality, we assume $v_1 \notin D_2$. By the definition of a *DTDS*, $\{v_2, v_n\} \cap D_2 \neq \emptyset$. We assume $v_n \in D_2$ and let $P_n = C_n - v_1v_2$. It is easy to check that $D_2$ is also a *DTDS* of $P_n$. Thus, we have $\gamma_t^D(C_n) = |D_2| \geq \gamma_t^D(P_n)$. ∎

Let $\xi_5$ be the family of simple unicyclic graphs that can be obtained from $r$ disjoint copies of a $cor(P_3)$, a $S_{2,1}$ and a star $S_4$ by first adding $r$ edges such that they are incident only with support vertices, each pair of them are non-parallel and the resulting graph is connected and has no loops, and then subdividing each new edge exactly once.

**Theorem 3.5** *Suppose $G$ is a simple unicyclic graph of order $n$ with $G \notin \Omega$, $|L(G)| = l$ and $|S(G)| = s$. Then $\gamma_t^D(G) \geq 3(n+l-s)/7$, with equality if and only if $G \in \xi_5$.*

**Proof.** If $G = C_n$, then $\gamma_t^D(C_n) \geq \lceil \frac{3n}{5} \rceil > \frac{3n}{7}$ by Theorems 3.4 and 3.3. Now suppose $s \neq 0$. From Lemma 3.2 (1), (4) and (3), we obtain $||[A, B \cup C]|| \geq 2n - 3|A| + l - s$. By Lemma 3.2 (2) and (6), $||[A, B \cup C]|| \leq$

$n - |E(G[A])| \leq n - \frac{2|A|}{3}$. Thus, $\gamma_t^D(G) = |A| \geq 3(n + l - s)/7$.

The equality $\gamma_t^D(G) = 3(n + l - s)/7$ holds if and only if $|E(G[Q_2 \cup C])| = 0$, $|N(v) \cap A| = 2$ for every vertex $v \in C$, $|L_1| = l - s$, $|L_2| = s$, $|Q_2| = |A| - |L_1| - |L_2|$ and $G[A] \cong \frac{|A|}{3}P_3$. The equality $|E(G[Q_2 \cup C])| = 0$ implies $|Q_2| = 0$ by Lemma 3.2 (5). Combining $|Q_1| = |A| - |L_1| - |S|$ and $|A| = |L_1| + |L_2|$ with $|L_2| = s$, we have $Q_1 = \emptyset$, which implies $A = L_1 \cup S$. For every component $P_3$ of $G[A]$, $|V(P_3) \cap L_1| \leq 2$. Thus, $G[A \cup B]$ is composed of several disjoint copies of a corona $cor(P_3)$, a double tree $S_{2,1}$ and a star $S_4$. Since $|N(v) \cap A| = 2$ for every vertex $v \in C$ and $G$ is a simple unicyclic graph, we get $G \in \xi_5$. ∎

Let $\xi_6$ be the family of simple unicyclic graphs that can be obtained from a simple unicyclic graph $G'$ by attaching at least three leaves to each vertex of $G'$, and then subdividing each edge of $G'$ exactly once.

**Theorem 3.6** *Suppose $G$ is a simple unicyclic graph of order $n$ with $G \notin \Omega$, $|L(G)| = l$ and $|S(G)| = s$. Then $\gamma_t^D(G) \geq (n + 2(l - s))/3$, with equality if and only if $G \in \xi_6$.*

**Proof.** If $G = C_n$, then $\gamma_t^D(C_n) \geq \lceil \frac{3n}{5} \rceil > \frac{n}{3}$ by Theorems 3.4 and 3.3. Now suppose $s \neq 0$. From Lemma 3.2 (1) and (4), we obtain $|[A, B \cup C]| \geq 2n - 3|A| + |L_1|$. By Lemma 3.2 (2) and (7), $|[A, B \cup C]| \leq n - |E(G[A])| \leq n - |L_1|$. Thus, we have $\gamma_t^D(G) = |A| \geq (n + 2|L_1|)/3 \geq (n + 2(l - s))/3$ by Lemma 3.2 (3).

The equality $\gamma_t^D(G) = (n + 2(l - s))/3$ holds if and only if $|E(G[Q_2 \cup C])| = 0$, $|N(v) \cap A| = 2$ for every vertex $v \in C$, $|L_1| = l - s$, $|L_2| = s$, $|Q_2| = |A| - |L_1| - |L_2|$, $|Q_1| = 0$, $|E(G[S])| = 0$ and every component of $G[A]$ is a star on at least three vertices. The equality $|E(G[Q_2 \cup C])| = 0$ implies $|Q_2| = 0$ by Lemma 3.2 (5). Since $|N(v) \cap A| = 2$ for every vertex $v \in C$ and $G$ is a simple and unicyclic graph, we get $G \in \xi_6$. ∎

Before we present the next result, we define two families of simple unicyclic graphs, denoted by $\xi_7$ and $\xi_8$.

First let $\mathcal{G}_{\Delta \leq 3} = \{G |\ G$ is a tree of order $n \geq 2$ and $\Delta(G) \leq 3$, or $G$ is a unicyclic graph of order $n \geq 1$ and $Delta(G) \leq 3\}$. For every $G \in \mathcal{G}_{\Delta \leq 3}$, we construct a new graph $G'$ from $G$ as follows.

First for every edge $e = ab \in E(G)$, if $a \neq b$, we replace it with a path $P_4 = v_1v_2v_3v_4$ where $v_1 = a$ and $v_4 = b$; if $a = b$, we replace it with a cycle $C_3 = v_1v_2v_3v_1$ where $v_1 = a = b$. Then we obtain a new graph $G_1$. We perform the next three operations.

• Step 1. For every vertex $v$ in $V(G)$ with $d_G(v) = 3$, suppose $N_{G_1}(v) = \{u_1, u_2, u_3\}$. Replace $v$ by a path $P_3 = v_1v_2v_3$ and add a new set $M$ of edges, where $M = \{v_1u_i, v_2u_j, v_3u_k\}$ with $\{i, j, k\} = \{1, 2, 3\}$. Note that $|M| = 3$. Define $D_v = \{v_1, v_2, v_3\}$.

Perform this operation on every vertex $v \in V(G)$ with $d_G(v) = 3$ in $G_1$, then we obtain a graph $G_2$ and a set $D_2 = \cup_{v, d_G(v)=3} D_v$.

• Step 2. For every vertex $v$ in $V(G)$ with $d_G(v) = 2$, suppose $N_{G_2}(v) = \{u_1, u_2\}$. Then in $G_2$, replace $v$ by a path $P_4 = v_1v_2v_3v_4$ and add a new set $M$ of edges, where $M = \{v_1u_i, v_2u_j\}$ with $\{i, j\} = \{1, 2\}$. Note that $|M| = 2$. Define $D_v = \{v_1, v_2, v_3\}$.

Perform this operation on every vertex $v \in V(G)$ with $d_G(v) = 2$ in $G_2$, then we obtain a graph $G_3$ and a set $D_3 = \cup_{v, d_G(v)=2} D_v$.

• Step 3. For every vertex $v$ in $V(G)$ with $d_G(v) = 1$, suppose $N_{G_3}(v) = \{u\}$. We perform exactly one of the next two operations on $v$:

(1) Replace $v$ by a double star $S_{2,1}$ where $V(S_{2,1}) = \{u_1, u_2, u_3, u_4, u_5\}$, $u_3, u_4$ are support vertices and $N_{S_{2,1}}(u_3) = \{u_1, u_2\}$. Then add a new edge $uu_1$ and define $D_v = \{u_1, u_3, u_4\}$.

(2) Replace $v$ by a star $S_4$ where $V(S_4) = \{w_1w_2w_3w_4\}$ and $w_4$ is a support vertex of $S_4$. Add a new edge $uw_1$ and define $D_v = \{w_1, w_2, w_4\}$.

Perform either (1) or (2) on every vertex $v \in V(G)$ with $d_G(v) = 1$ in $G_3$, then we obtain a graph $G'$ and a set $D_4 = \cup_{v, d_G(v)=1} D_v$.

Let $D_{G'} = D_2 \cup D_3 \cup D_4$, $\mathcal{G}'_{\Delta\leq 3} = \{G' \mid G'$ is obtained from a graph $G \in \mathcal{G}_{\Delta\leq 3}$ as above$\}$ and $\mathcal{T}'_{\Delta\leq 3}$ be the set of trees in $\mathcal{G}'_{\Delta\leq 3}$. Define a set $D = S(cor(P_3)) \cup S(S_{2,1}) \cup S(S_4) \cup_{G' \in \mathcal{G}'_{\Delta\leq 3}} D_{G'}$ and a set of trees $\eta_4 = \{cor(P_3)\} \cup \{S_{2,1}\} \cup \{S_4\} \cup \mathcal{T}'_{\Delta\leq 3}$.

Let $\xi_7$ be the family of simple unicyclic graphs that can be obtained from $r$ disjoint copies of several trees in $\eta_4$ by first adding $r$ edges such that they are incident only with vertices in $D$, each pair of them are non-paralell and the resulting graph is connected and has no loops, and then subdividing each new edge exactly once.

Let $\xi_8$ be the family of simple unicyclic graphs that can be obtained from $r$ disjoint copies of several trees in $\eta_4$ and a unicyclic graph $G_0 \in \mathcal{G}'_{\Delta\leq 3}$ by first adding $r$ edges such that they are incident only with vertices in $D$ and the resulting graph is connected, and then subdividing each new edge exactly once.

**Theorem 3.7** *Suppose $G$ is a simple unicyclic graph of order $n$ with $G \notin \Omega$, $|L(G)| = l$, $|S(G)| = s$. Then $\gamma_t^D(G) \geq \frac{6}{11}(n + \frac{l}{2} - s)$, with equality if and only if $G \in \xi_7 \cup \xi_8$.*

**Proof.** If $G = C_n$, then $\gamma_t^D(C_n) \geq \lceil \frac{3n}{5} \rceil > \frac{6n}{11}$ by Theorems 3.4 and 3.3. Now suppose $s \neq 0$. Let $A$, $B$, $C$, $L$, $S$, $L_1$, $L_2$, $Q_1$, $Q_2$ and $\omega$ be defined as above. From Lemma 3.2 (1), we obtain $|[A, B \cup C]| \geq 2n - 2|A| - |B| = 2n - 2|A| - (|Q_2| + |L_2|)$. By Lemma 3.2 (2), (5) and (6), $|[A, B \cup C]| = n - |E(G[A])| - |E(G[Q_2 \cup C])| \leq n - \frac{2|A|}{3} - \frac{|Q_2|}{2}$. Thus, we have $\frac{4|A|}{3} \geq n - \frac{|Q_2|}{2} - |L_2|$. By Lemma 3.2 (4), $(\frac{4}{3} + \frac{1}{2})|A| \geq n + \frac{|L_1| - |L_2|}{2}$. From Lemma 3.2 (3), we have $|L_1| - |L_2| \geq l - 2s$. Thus, $\gamma_t^D(G) = |A| \geq \frac{6}{11}(n + \frac{l}{2} - s)$.

The equality $\gamma_t^D(G) = \frac{6}{11}(n + \frac{l}{2} - s)$ holds if and only if equalities (1)-(6) in Lemma 3.2 all hold. As $|Q_1| = |A| - |L_1| - |S| = |A| - |L_1| - |L_2|$, we obtain $|Q_1| = |Q_2|$. Note that $G[Q_2 \cup C] \cong |C|K_1 + \frac{|Q_2|}{2}K_2$ and $C$ is an independent set in $G[Q_2 \cup C]$, $G[A] \cong \frac{|A|}{3}P_3$ and every vertex in $C$ is adjacent to exactly two vertices in $A$. On the other hand, for every component $P_3$ in $G[A]$, $|V(P_3) \cap L_1| \leq 2$ and $V(P_3) - L_1 \subseteq S \cup Q_1$.

If $|Q_2| = |Q_1| = 0$, then $G \in \xi_5 \subset \xi_7$ by a similar proof as in Theorem 3.5.

Now we consider the case $|Q_2| = |Q_1| \neq 0$. In this case, every component $P_3 = v_1v_2v_3$ of $G[A]$ is in one of the next seven situations (up to switching the role of $v_1$ and $v_3$):

1. $v_1, v_3 \in L_1$ and $v_2 \in S$.
2. $v_1 \in L_1$ and $v_2, v_3 \in S$.
3. $v_1, v_2, v_3 \in S$.
4. $v_1 \in L_1$, $v_2 \in S$ and $v_3 \in Q_1$.
5. $v_1, v_2 \in S$ and $v_3 \in Q_1$.
6. $v_1 \in S$ and $v_2, v_3 \in Q_1$.
7. $v_1, v_2, v_3 \in Q_1$.

Let $G_1, G_2, \cdots, G_{\omega_1}$ be the components of $G[A \cup B]$. For a component $G_i$, if $V(G_i) \cap Q_2 = \emptyset$, then $G_i \in \{cor(P_3), S_{2,1}, S_4\}$. If $V(G_i) \cap Q_2 \neq \emptyset$, then we construct a graph $G_i'$ from $G_i$ by contracting every edge $e$ of $G_i$ with $e \notin [Q_1, Q_2] \cup E(G[Q_2])$. Then we can see $G_i'$ is obtained from a graph $G_i'' \in \mathcal{G}_{\Delta \leq 3}$ by replacing every edge $e = ab \in E(G_i'')$ with either a path $P_4 = v_1v_2v_3v_4$ when $a \neq b$, where $v_1 = a$ and $v_4 = b$, or a cycle $C_3 = v_1v_2v_3v_1$ when $a = b$, where $v_1 = a = b$. Because every component of $G[A]$ is a path $P_3$ in one of the above seven situations and every vertex of $Q_2$ is adjacent to exactly one vertex in $Q_1$, we have $G_i \in \mathcal{G}'_{\Delta \leq 3}$, where $\mathcal{G}'_{\Delta \leq 3}$ is defined as in the construction of $\xi_8$. Thus, for $i = 1, 2, \cdots, \omega_1$, either $G_i \in \eta_4$ or $G_i$ is a unicyclic graph in $\mathcal{G}'_{\Delta \leq 3}$. Note that every vertex in $C$ is adjacent to exactly two vertices in $A$, thus $G \in \xi_7 \cup \xi_8$. ■

**Remark.** If $2n \geq 5(l - s)$, the lower bound in Theorem 3.5 is better than that given in Theorem 3.6. If $3n \geq 4l + 3s$, the lower bound in Theorem 3.7 is better than that given in Theorem 3.5.

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