# The (distance) signless Laplacian spectral radii of digraphs with given dichromatic number*

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**Abstract** In this paper, we characterize the extremal digraph with the maximal signless Laplacian spectral radius and the minimal distance signless Laplacian spectral radius among all simple connected digraphs with given dichromatic number, respectively.

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## 1 Introduction

We begin by recalling some definitions. Let $M$ be an $n \times n$ matrix, $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $M$. It is obvious that the eigenvalues may be complex numbers since $M$ is not symmetric in general. We usually assume that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. The spectral radius of $M$ is defined as $\rho(M) = |\lambda_1|$, i.e., it is the largest modulus of the eigenvalues of $M$. If $M$ is a nonnegative matrix, it follows from the Perron-Frobenius theorem that the spectral radius $\rho(M)$ is a eigenvalue of $M$. If $M$ is a nonnegative irreducible matrix, it follows from the Perron-Frobenius theorem that $\rho(M) = \lambda_1$ is simple.

Let $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ be a digraph, where $V(\overrightarrow{G})$ and $E(\overrightarrow{G})$ are the vertex set and arc set of $\overrightarrow{G}$, respectively. A digraph $\overrightarrow{G}$ is simple if it has no loops and multiple arcs. A digraph $\overrightarrow{G}$ is strongly connected if for every pair of vertices $v_i, v_j \in V(\overrightarrow{G})$, there are directed paths from $v_i$ to $v_j$ and

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from $v_j$ to $v_i$. In this paper, we consider finite, simple strongly connected digraphs.

Let $\overrightarrow{G}$ be a digraph. If two vertices are connected by an arc, then they are called adjacent. For $e = (v_i, v_j) \in E(\overrightarrow{G})$, $v_i$ is the tail (the initial vertex) of $e$, $v_j$ is the head (the terminal vertex) of $e$.

Let $N^-_{\overrightarrow{G}}(v_i) = \{v_j \in V(\overrightarrow{G}) | (v_j, v_i) \in E(\overrightarrow{G})\}$ and $N^+_{\overrightarrow{G}}(v_i) = \{v_j \in V(\overrightarrow{G}) | (v_i, v_j) \in E(\overrightarrow{G})\}$ denote the in-neighbors and out-neighbors of $v_i$, respectively. Let $d_i^- = |N^-_{\overrightarrow{G}}(v_i)|$ denote the indegree of the vertex $v_i$ and $d_i^+ = |N^+_{\overrightarrow{G}}(v_i)|$ denote the outdegree of the vertex $v_i$ in $\overrightarrow{G}$.

For a digraph $\overrightarrow{G}$, let $A(\overrightarrow{G}) = (a_{ij})$ denote the adjacency matrix of $\overrightarrow{G}$, where $a_{ij}$ is equal to the number of arcs $(v_i, v_j)$. The spectral radius of $A(\overrightarrow{G})$, denoted by $\rho(\overrightarrow{G})$, is called the spectral radius of $\overrightarrow{G}$.

Let $diag(\overrightarrow{G}) = diag(d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal matrix with outdegree of the vertices of $\overrightarrow{G}$ and $Q(\overrightarrow{G}) = diag(\overrightarrow{G}) + A(\overrightarrow{G})$ be the signless Laplacian matrix of $\overrightarrow{G}$. The spectral radius of $Q(\overrightarrow{G})$, $\rho(Q(\overrightarrow{G}))$, denoted by $q(\overrightarrow{G})$, is called the signless Laplacian spectral radius of $\overrightarrow{G}$.

For $u, v \in V(\overrightarrow{G})$, the distance from $u$ to $v$, denoted by $d_{\overrightarrow{G}}(u, v)$ or $d_{uv}$, is the length of the shortest directed path from $u$ to $v$ in $\overrightarrow{G}$. For $u \in V(\overrightarrow{G})$, the transmission of vertex $u$ in $\overrightarrow{G}$ is the sum of distances from $u$ to all other vertices of $\overrightarrow{G}$, denoted by $Tr_{\overrightarrow{G}}(u)$.

Let $\overrightarrow{G}$ be a connected digraph with vertex set $V(\overrightarrow{G}) = \{v_1, v_2, \ldots, v_n\}$. The distance matrix of $\overrightarrow{G}$ is the $n \times n$ matrix $\mathcal{D}(\overrightarrow{G}) = (d_{ij})$ where $d_{ij} = d_{\overrightarrow{G}}(v_i, v_j)$. The distance spectral radius of $\overrightarrow{G}$, denoted by $\rho^{\mathcal{D}}(\overrightarrow{G})$, is the spectral radius of $\mathcal{D}(\overrightarrow{G})$.

In fact, for $1 \le i \le n$, the transmission of vertex $v_i$, $Tr_{\overrightarrow{G}}(v_i)$ is just the $i$-th row sum of $\mathcal{D}(\overrightarrow{G})$. Let $Tr(\overrightarrow{G}) = diag(Tr_{\overrightarrow{G}}(v_1), Tr_{\overrightarrow{G}}(v_2), \ldots, Tr_{\overrightarrow{G}}(v_n))$ be the diagonal matrix of vertex transmission of $\overrightarrow{G}$. The distance signless Laplacian matrix of $\overrightarrow{G}$ is the $n \times n$ matrix defined similar to the undirected graph by Aouchiche and Hansen ([1]) as $\mathcal{Q}(\overrightarrow{G}) = Tr(\overrightarrow{G}) + \mathcal{D}(\overrightarrow{G})$. The distance signless Laplacian spectral radius of $\overrightarrow{G}$, $\rho(\mathcal{Q}(\overrightarrow{G}))$, denoted by $q^{\mathcal{D}}(\overrightarrow{G})$, is the spectral radius of $\mathcal{Q}(\overrightarrow{G})$.

Since $\overrightarrow{G}$ is a simple strongly connected digraph, then $Q(\overrightarrow{G})$ and $\mathcal{Q}(\overrightarrow{G})$ are nonnegative irreducible matrices. It follows from the Perron-Frobenius Theorem that $q(\overrightarrow{G})$ and $q^{\mathcal{D}}(\overrightarrow{G})$ are positive real numbers and there is a positive unit eigenvector (which is called the Perron vector) corresponding to $q(\overrightarrow{G})$ and $q^{\mathcal{D}}(\overrightarrow{G})$, respectively.

Let $\overrightarrow{G}$ be a simple, strongly connected digraph. A vertex set $F \subseteq V(\overrightarrow{G})$

is acyclic if the induced subdigraph $\overrightarrow{G}[F]$ is acyclic. A partition of $V(\overrightarrow{G})$ into $k$ acyclic sets is called a $k$-coloring of $\overrightarrow{G}$. The minimum integer $k$ for which there exists a $k$-coloring of $\overrightarrow{G}$ is the dichromatic number $\chi(\overrightarrow{G})$ of the digraph $\overrightarrow{G}$. The definition of the dichromatic number of a digraph was first introduced by Neumann-Lara ([24]). The same notation was independently introduced later by B. Mohar when considering the circular chromatic number of weighted (directed or undirected) graphs ([22]), and the dichromatic number of digraphs was further investigated in ([4]). Clearly, if $G$ is an undirected graph, and $\overrightarrow{G}$ is the digraph obtained from $G$ by replacing each edge with the pair of oppositely directed arcs joining the same pair of vertices, then $\chi(\overrightarrow{G})$ is the same as the usual chromatic number of the undirected graph $G$ since any two adjacent vertices in $\overrightarrow{G}$ induce a directed cycle of length 2.

Other definitions, terminology and notations not in the article can be found in [2, 3, 5, 6, 13, 21].

The spectral radius, the signless Laplacian spectral radius, the distance spectral radius and the distance signless Laplacian spectral radius of undirected graphs are well treated, but there is not much known about digraphs. In 2010, R.A. Brualdi wrote a stimulating survey on the spectra of digraphs ([7]). Furthermore, some upper or lower bounds on the spectral radius or the signless Laplacian spectral radius of digraphs were obtained (see [8, 10, 12, 15, 16] and so on), and some extremal digraphs which attain the maximum or minimum spectral radius and the distance spectral radius of digraphs with given parameters, such as given connectivity, given arc connectivity, given dichromatic number, given clique number, given girth and so on, were characterized, see e.g. [9, 14, 17, 18, 19, 20].

On the other hand, about the given chromatic number or given dichromatic number, Feng ([11]) proved that the Turán graph has the maximal spectral radius among all graphs with given chromatic number in 2007, B. Mohar ([23]) gave a lower bound on the spectral radius for digraphs with given dichromatic number in 2010, Lin-Shu ([14]) characterized the unique digraph with the maximal spectral radius with given dichromatic number in 2011, later, in 2013, Lin-Shu ([19]) determined the extremal digraph with the minimal distance spectral radius with given dichromatic number.

In the following paper, we will characterize the extremal digraph having the maximal signless Laplacian spectral radius and the minimal distance signless Laplacian spectral radius among all simple connected digraphs with given dichromatic number in Subsection 3, and Subsection 4, respectively. The technique used in this section is motivated by [14, 19] et al.

## 2 Some notations and lemmas

In this section, we give some notations and lemmas.

**Definition 2.1.** ([3], Chapter 2) Let $A = (a_{ij}), B = (b_{ij})$ be $n \times n$ matrices. If $a_{ij} \leq b_{ij}$ for all $i$ and $j$, then $A \leq B$. If $A \leq B$ and $A \neq B$, then $A < B$. If $a_{ij} < b_{ij}$ for all $i$ and $j$, then $A \ll B$.

**Lemma 2.1.** ([3], Chapter 2) *Let $A, B$ be $n \times n$ nonnegative matrices with the spectral radius $\rho(A)$ and $\rho(B)$. If $A \leq B$, then $\rho(A) \leq \rho(B)$. Furthermore, if $A < B$ and $B$ is irreducible, then $\rho(A) < \rho(B)$.*

By Lemma 2.1 and the definitions of $Q(\overrightarrow{G})$, $q(\overrightarrow{G})$, $\mathcal{Q}(\overrightarrow{G})$ and $q^{\mathcal{D}}(\overrightarrow{G})$, we have the following results in terms of digraphs.

**Corollary 2.1.** *Let $\overrightarrow{G}$ be a digraph and $\overrightarrow{H}$ be a spaning subdigraph of $\overrightarrow{G}$. Then*

(i) $q(\overrightarrow{H}) \leq q(\overrightarrow{G})$.

(ii) *If $\overrightarrow{G}$ is strongly connected, and $\overrightarrow{H}$ is a proper subdigraph of $\overrightarrow{G}$, then $q(\overrightarrow{H}) < q(\overrightarrow{G})$.*

(iii) *If $\overrightarrow{G}$ and $\overrightarrow{H}$ are strongly connected, then $q^{\mathcal{D}}(\overrightarrow{H}) \geq q^{\mathcal{D}}(\overrightarrow{G})$.*

(iv) *If $\overrightarrow{G}$ and $\overrightarrow{H}$ are strongly connected, and $\overrightarrow{H}$ is a proper subdigraph of $\overrightarrow{G}$, then $q^{\mathcal{D}}(\overrightarrow{H}) > q^{\mathcal{D}}(\overrightarrow{G})$.*

**Lemma 2.2.** ([3], Chapter 2) *Let $A$ be a nonnegative matrix, $x \in R^n$ and $x > 0$, $\alpha, \beta \geq 0$. Then*

(i) *If $\alpha x \leq Ax$ and $x > 0$, then $\alpha \leq \rho(A)$.*

(ii) *If $Ax \leq \beta x$ and $x \gg 0$, then $\rho(A) \leq \beta$.*

(iii) *If $A$ is irreducible, $\alpha x < Ax < \beta x$ and $x > 0$, then $\alpha < \rho(A) < \beta$ and $x \gg 0$.*

Let $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ be a connected digraph, for any two vertex sets $U_1, U_2 \subseteq V(\overrightarrow{G})$ with $U_1 \cap U_2 = \emptyset$, let $[U_1, U_2] = \{(u, v), (v, u) | u \in U_1, v \in U_2\}$ denote the arcs between $U_1$ and $U_2$, and $\delta^-(\overrightarrow{G})$ denote the minimum in-degree of $\overrightarrow{G}$.

**Lemma 2.3.** ([5], Exercise 10.1.3) *Let $\overrightarrow{G}$ be a digraph with no dicycle. Then $\delta^-(\overrightarrow{G}) = 0$ and there exists an ordering $v_1, v_2, \ldots, v_n$ of $V(\overrightarrow{G})$ such that for $1 \leq i \leq n$, every arc of $\overrightarrow{G}$ with head $v_i$ has its tail in $\{v_1, v_2, \ldots, v_{i-1}\}$.*

**Definition 2.2.** ([6]) A digraph which an orientation of a complete graph is called a tournament; A digraph is acyclic if it has no dicycle; An acyclic tournament is a transitive tournament.

**Remark.** If a digraph $\overrightarrow{G}$ on $n$ vertices is a transitive tournament, then there exists an ordering $v_1, v_2, \ldots, v_n$ of $V(\overrightarrow{G})$ such that $E(\overrightarrow{G}) = \{(v_i, v_j) | 1 \leq i < j \leq n\}$.

Let $n, k$ be positive integers with $k \geq 2$, $\overrightarrow{G}_n$ be the set of all digraphs on $n$ vertices,

$$\overrightarrow{G}_{n,k} = \{\overrightarrow{G} | \overrightarrow{G} \in \overrightarrow{G}_n \text{ and } \chi(\overrightarrow{G}) = k\}.$$

From the definition of dichromatic number $\chi(\overrightarrow{G})$, we know that if $\overrightarrow{G} \in \overrightarrow{G}_{n,k}$, then $\overrightarrow{G}$ has $k$-color classes and each is an acyclic set. Suppose that the $k$-color classes of $\overrightarrow{G}$ are $V^1, V^2, \ldots, V^k$ having $n_1, n_2, \ldots, n_k$ vertices, respectively. Without loss of generality, we suppose that $n_1 \leq n_2 \leq \ldots \leq n_k$. Let $\mathbb{D}_{n,k} = \{\overrightarrow{G} \in \overrightarrow{G}_{n,k} | V(\overrightarrow{G}) = V^1 \cup V^2 \cup \ldots \cup V^k$, where $V^i$ is a transitive tournament for $i = 1, 2, \ldots, k$ and $[V^i, V^j] \subseteq E(\overrightarrow{G})$ for any $1 \leq i < j \leq k\}$. Let $T^*_{n,k} \in \mathbb{D}_{n,k}$ denote the digraph with $||V^i| - |V^j|| \leq 1$ for any $i, j \in \{1, \ldots, k\}$.

# 3 The maximal signless Laplacian spectral radius for digraphs with given dichromatic number

By (ii) of Corollary 2.1, we known that $q(\overrightarrow{G}+e) > q(\overrightarrow{G})$ for any strongly connected digraph $\overrightarrow{G} \in \overrightarrow{G}_n$ where $e \notin E(\overrightarrow{G})$, thus by the definition of dichromatic number, the strongly connected digraph on $n$ vertices with the dichromatic number $k$ maximizing signless Laplacian spectral radius must be in $\mathbb{D}_{n,k}$. In the following, we will show that the digraph $T^*_{n,k}$ is the unique digraph which achieves the maximal signless Laplacian spectral radius among all strongly connected digraphs on $n$ vertices with the dichromatic number $k$.

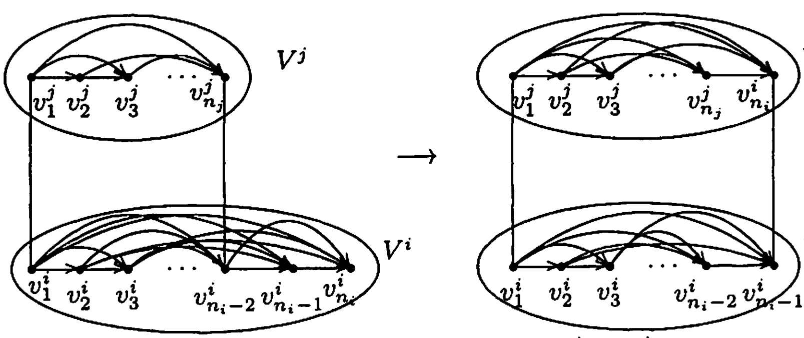

Figure 1. The transformation from $\overrightarrow{G}$ to $\overrightarrow{H}$

**Lemma 3.1.** *Let $\overrightarrow{G} \in \mathbb{D}_{n,k}$ be a strongly connected digraph with $V(\overrightarrow{G}) = V^1 \cup V^2 \cup \ldots \cup V^k$, where for each $i \in \{1, 2, \ldots, k\}$, $|V^i| = n_i$ with $n_1 \leq n_2 \leq \ldots \leq n_k$ and $n_1 + n_2 + \ldots + n_k = n$. If there exist $i, j \in \{1, 2, \ldots, k\}$ with $j < i$ and $n_i - n_j \geq 2$, let $\overrightarrow{H} = \overrightarrow{G} - \{(v^i_{n_i}, v^j_s) \mid s = 1, \ldots, n_j\} + \{(v^i_{n_i}, v^i_t) \mid t = 1, \ldots, n_i - 1\}$ (see Figure 1). Then $q(\overrightarrow{H}) > q(\overrightarrow{G})$.*

*Proof.* Since $\overrightarrow{G} \in \mathbb{D}_{n,k}$, each $V^p (p = 1, 2, \ldots, k)$ is transitive tournament, we can give a vertex ordering $v^p_1, v^p_2, \ldots, v^p_{n_p}$ such that $(v^p_s, v^p_t) \in E(\overrightarrow{G})$ for any $1 \leq s < t \leq n_p$. Let $x = (x^1_1, x^1_2, \ldots, x^1_{n_1}, x^2_1, x^2_2, \ldots, x^2_{n_2}, \ldots, x^k_1, x^k_2, \ldots, x^k_{n_k})$ be the Perron vector of $Q(\overrightarrow{G})$, where $x^p_s$ correspond to the vertex $v^p_s$ for each $1 \leq p \leq k$ and $1 \leq s \leq n_p$. Then $x \gg 0$. Now we show the following two claims.

**Claim 1.** $x^p_s = x^r_s$ for any $1 \leq p < r \leq k$ and $1 \leq s \leq n_p$.

Now we prove $x^p_s = x^r_s$ holds for any $1 \leq p < r \leq k$ and $1 \leq s \leq n_p$ by induction on $s$. When $s = 1$, we have

$$q(\overrightarrow{G})x^p_1 = (n-1)x^p_1 + \sum_{t=2}^{n_p} x^p_t + x^r_1 + \sum_{t=2}^{n_r} x^r_t + \sum_{m \neq p,r} \sum_{t=1}^{n_m} x^m_t,$$

$$q(\overrightarrow{G})x^r_1 = (n-1)x^r_1 + \sum_{t=2}^{n_r} x^r_t + x^p_1 + \sum_{t=2}^{n_p} x^p_t + \sum_{m \neq p,r} \sum_{t=1}^{n_m} x^m_t.$$

Then $(q(\overrightarrow{G}) + 2 - n)(x^p_1 - x^r_1) = 0$, which implies that $x^p_1 = x^r_1$ for all $1 \leq p < r \leq k$.

Now we suppose that $x^p_s = x^r_s$ holds for any $s$ where $1 \leq s < N \leq n_p$, that is, $x^p_s = x^r_s$ for each $s \in \{1, 2, \ldots, N-1\}$. Now we consider the case $s = N$. We have

$$q(\overrightarrow{G})x^p_N = (n-N)x^p_N + \sum_{t=N+1}^{n_p} x^p_t + \sum_{t=1}^{N-1} x^r_t + x^r_N + \sum_{t=N+1}^{n_r} x^r_t + \sum_{m \neq p,r} \sum_{t=1}^{n_m} x^m_t,$$

$$q(\overrightarrow{G})x^r_N = (n-N)x^r_N + \sum_{t=N+1}^{n_r} x^r_t + \sum_{t=1}^{N-1} x^p_t + x^p_N + \sum_{t=N+1}^{n_p} x^p_t + \sum_{m \neq p,r} \sum_{t=1}^{n_m} x^m_t.$$

Since $x^p_t = x^r_t$ for $t = 1, 2, ..., N-1$, then we have $(q(\overrightarrow{G}) + 1 + N - n)(x^p_N - x^r_N) = 0$. Thus $x^p_N = x^r_N$ for all $1 \leq p < r \leq k$.

Combining the above arguments, we have $x^p_s = x^r_s$ for all $1 \leq p < r \leq k$ and $1 \leq s \leq n_p$. Claim 1 holds.

**Claim 2.** $Q(\overrightarrow{H})x > Q(\overrightarrow{G})x = q(\overrightarrow{G})x$.

We assume $Q(\overrightarrow{G}) = (q_{ij})$ and $Q(\overrightarrow{H}) = (h_{ij})$, then $(Q(\overrightarrow{G})x)_t = \sum\limits_{k=1}^{n} q_{tk}x_k$ and $(Q(\overrightarrow{H})x)_t = \sum\limits_{k=1}^{n} h_{tk}x_k$, thus

**Case 1:** $v_t \neq v^i_{n_i}$.

We have $(Q(\overrightarrow{G})x)_t = (Q(\overrightarrow{H})x)_t$.

**Case 2:** $v_t = v^i_{n_i}$.

Noting that

$$(Q(\overrightarrow{G})x)_t = \sum_{m\neq i,j} \sum_{t=1}^{n_m} x_t^m + \sum_{t=1}^{n_j} x_t^j + (n - n_i)x^i_{n_i},$$

and

$$(Q(\overrightarrow{H})x)_t = \sum_{m\neq i,j} \sum_{t=1}^{n_m} x_t^m + \sum_{t=1}^{n_i-1} x_t^i + (n - n_j - 1)x^i_{n_i},$$

then $(Q(\overrightarrow{H})x)_t - (Q(\overrightarrow{G})x)_t = (n_i - n_j - 1)x^i_{n_i} + \sum\limits_{t=1}^{n_i-1} x_t^i - \sum\limits_{t=1}^{n_j} x_t^j > 0$ by Claim 1 and $n_j \leq n_i - 2$, therefore $Q(\overrightarrow{H})x > q(\overrightarrow{G})x = Q(\overrightarrow{G})x$ by Definition 2.1, thus Claim 2 holds.

By Claim 2 and Lemma 2.2, we have $q(\overrightarrow{H}) > q(\overrightarrow{G})$. □

Note that the strongly connected digraph on $n$ vertices with the dichromatic number $k$ maximizing the signless Laplacian spectral radius must be in $\mathbb{D}_{n,k}$ by Corollary 2.1, and $T^*_{n,k}$ can be obtained from any strongly connected digraph in $\mathbb{D}_{n,k}$ by several steps of the transformations of Lemma 3.1, thus we obtain the main result of this section as follows.

**Theorem 3.1.** *The digraph $T^*_{n,k}$ is the unique digraph with the maximal signless Laplacian spectral radius among all strongly connected digraphs on $n$ vertices with the dichromatic number $k$.*

# 4 The minimal distance signless Laplacian spectral radius for strongly connected digraphs with given dichromatic number

By (iv) of Corollary 2.1, we known that $q^{\mathcal{D}}(\overrightarrow{G} + e) < q^{\mathcal{D}}(\overrightarrow{G})$ for any strongly connected digraph $\overrightarrow{G} \in \overrightarrow{G}_n$ where $e \notin E(\overrightarrow{G})$, thus by the definition of dichromatic number, the strongly connected digraph on $n$ vertices with the dichromatic number $k$ minimizing the distance signless Laplacian spectral radius must be in $\mathbb{D}_{n,k}$. In the following, we will show that the digraph $T^*_{n,k}$ is the unique digraph which achieves the minimal distance signless Laplacian spectral radius among all strongly connected digraphs on $n$ vertices with the dichromatic number $k$.

**Lemma 4.1.** *Let $\overrightarrow{G} \in \mathbb{D}_{n,k}$ be a strongly connected digraph with $V(\overrightarrow{G}) = V^1 \cup V^2 \cup \ldots \cup V^k$, where for each $i \in \{1,2,\ldots,k\}$, $|V^i| = n_i$ with $n_1 \leq n_2 \leq \ldots \leq n_k$ and $n_1 + n_2 + \ldots + n_k = n$. If there exist $i, j \in \{1,2,\ldots,k\}$ with $j < i$ and $n_i - n_j \geq 2$, let $\overrightarrow{H} = \overrightarrow{G} - \{(v^i_{n_i}, v^j_s) \mid s = 1,\ldots,n_j\} + \{(v^i_{n_i}, v^i_t) \mid t = 1,\ldots,n_i - 1\}$ (see Figure 1). Then $q^{\mathcal{D}}(\overrightarrow{G}) > q^{\mathcal{D}}(\overrightarrow{H})$.*

*Proof.* Since $\overrightarrow{G} \in \mathbb{D}_{n,k}$, each $V^p (p = 1,2,\ldots,k)$ is transitive tournament, we can give a vertex ordering $v^p_1, v^p_2, \ldots, v^p_{n_p}$ such that $(v^p_s, v^p_t) \in E(\overrightarrow{G})$ for any $1 \leq s < t \leq n_p$. Let $x = (x^1_1, x^1_2, \ldots, x^1_{n_1}, x^2_1, x^2_2, \ldots, x^2_{n_2}, \ldots, x^k_1, x^k_2, \ldots, x^k_{n_k})$ be the Perron vector of $\mathcal{Q}(\overrightarrow{G})$, where $x^p_s$ correspond to the vertex $v^p_s$ for each $1 \leq p \leq k$ and $1 \leq s \leq n_p$. Then $x \gg 0$. Now we show the following two claims.

**Claim 1.** $x^p_s = x^r_s$ for any $1 \leq p < r \leq k$ and $1 \leq s \leq n_p$.

Now we prove $x^p_s = x^r_s$ holds for any $1 \leq p < r \leq k$ and $1 \leq s \leq n_p$ by induction on $s$. When $s = 1$, we have

$$q^{\mathcal{D}}(\overrightarrow{G})x^p_1 = (n-1)x^p_1 + \sum_{t=2}^{n_p} x^p_t + \sum_{t=1}^{n_r} x^r_t + \sum_{m \neq p,r} \sum_{t=1}^{n_m} x^m_t,$$

$$q^{\mathcal{D}}(\overrightarrow{G})x^r_1 = (n-1)x^r_1 + \sum_{t=2}^{n_r} x^r_t + \sum_{t=1}^{n_p} x^p_t + \sum_{m \neq p,r} \sum_{t=1}^{n_m} x^m_t.$$

Then $(q^{\mathcal{D}}(\overrightarrow{G}) + 2 - n)(x^p_1 - x^r_1) = 0$, which implies that $x^p_1 = x^r_1$ for all $1 \leq p < r \leq k$.

Now we suppose that $x^p_s = x^r_s$ holds for any $s$ where $1 \leq s < N \leq n_p$, that is, $x^p_s = x^r_s$ for each $s \in \{1,2,\ldots,N-1\}$. Now we consider the case $s = N$. Noting that

$$q^{\mathcal{D}}(\overrightarrow{G})x^p_N = (N+n-2)x^p_N + 2\sum_{t=1}^{N-1} x^p_t + \sum_{t=N+1}^{n_p} x^p_t + \sum_{m \neq p} \sum_{t=1}^{n_m} x^m_t,$$

and

$$q^{\mathcal{D}}(\overrightarrow{G})x^r_N = (N+n-2)x^r_N + 2\sum_{t=1}^{N-1} x^r_t + \sum_{t=N+1}^{n_r} x^r_t + \sum_{m \neq r} \sum_{t=1}^{n_m} x^m_t,$$

since $x_t^p = x_t^r$ for $t = 1, 2, ..., N-1$, then we have

$$(q^{\mathcal{D}}(\overrightarrow{G}) + 3 - N - n)(x_N^p - x_N^r) = \sum_{t=1}^{N-1} x_t^p - \sum_{t=1}^{N-1} x_t^r = 0.$$

Thus $x_N^p = x_N^r$ for all $1 \le p < r \le k$.

Combining the above arguments, we have $x_s^p = x_s^r$ for all $1 \le p < r \le k$ and $1 \le s \le n_p$. Claim 1 holds.

**Claim 2.** $\mathcal{Q}(\overrightarrow{G})x = q^{\mathcal{D}}(\overrightarrow{G})x > \mathcal{Q}(\overrightarrow{H})x$.

We assume $\mathcal{Q}(\overrightarrow{G}) = (q_{ij})$ and $\mathcal{Q}(\overrightarrow{H}) = (h_{ij})$, then $(\mathcal{Q}(\overrightarrow{G})x)_t = \sum_{k=1}^{n} q_{tk}x_k$ and $(\mathcal{Q}(\overrightarrow{H})x)_t = \sum_{k=1}^{n} h_{tk}x_k$, thus

**Case 1:** $v_t \neq v_{n_i}^i$.

We have $(\mathcal{Q}(\overrightarrow{G})x)_t = (\mathcal{Q}(\overrightarrow{H})x)_t$.

**Case 2:** $v_t = v_{n_i}^i$.

Noting that

$$(\mathcal{Q}(\overrightarrow{G})x)_t = \sum_{m\neq i,j} \sum_{t=1}^{n_m} x_t^m + \sum_{t=1}^{n_j} x_t^j + 2\sum_{t=1}^{n_i-1} x_t^i + (n + n_i - 2)x_{n_i}^i,$$

and

$$(\mathcal{Q}(\overrightarrow{H})x)_t = \sum_{m\neq i,j} \sum_{t=1}^{n_m} x_t^m + 2\sum_{t=1}^{n_j} x_t^j + \sum_{t=1}^{n_i-1} x_t^i + (n + n_j - 1)x_{n_i}^i,$$

then $(\mathcal{Q}(\overrightarrow{G})x)_t - (\mathcal{Q}(\overrightarrow{H})x)_t = (n_i - n_j - 1)x_{n_i}^i + \sum_{t=1}^{n_i-1} x_t^i - \sum_{t=1}^{n_j} x_t^j > 0$ by Claim 1 and $n_j \le n_i - 2$, therefore $q^{\mathcal{D}}(\overrightarrow{G})x = \mathcal{Q}(\overrightarrow{G})x > \mathcal{Q}(\overrightarrow{H})x$ by Definition 2.1, thus Claim 2 holds.

By Claim 2 and Lemma 2.2, we have $q^{\mathcal{D}}(\overrightarrow{G}) > q^{\mathcal{D}}(\overrightarrow{H})$. □

Noting that the strongly connected digraph on $n$ vertices with the dichromatic number $k$ minimizing the distance signless Laplacian spectral radius must be in $\mathbb{D}_{n,k}$ by Corollary 2.1, and $T_{n,k}^*$ can be obtained from any strongly connected digraph in $\mathbb{D}_{n,k}$ by several steps of the transformations of Lemma 4.1, thus we obtain the main result of this section as follows.

**Theorem 4.1.** *The digraph $T_{n,k}^*$ is the unique digraph with the minimal* *distance signless Laplacian spectral radius among all strongly connected digraphs on n vertices with the dichromatic number k.*

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