Exact Wirelength of Circulant Networks into Cycle-of-ladders

Micheal Arockiaraj[1], L. Packiaraj[2], and R. Sundara Rajan[3]
[1]Department of Mathematics, Loyola College, Chennai, India
[2]Department of Mathematics, St. Joseph's College, Trichy, India
[3]School of Advanced Sciences, VIT University, Chennai, India
hilpackiaraj@gmail.com

Abstract

Graph embedding is an important factor to evaluate the quality of an interconnection network. It is also a powerful tool for implementation of parallel algorithms and simulation of different interconnection networks. In this paper, we compute the exact wirelength of embedding circulant networks into cycle-of-ladders.

Keywords: Embedding; Congestion; wirelength; circulant networks; cycle-of-ladders.

1 Introduction

The interconnection networks play a major role in the performance of distributed-memory multiprocessors and the one primary concern for choosing an appropriate interconnection network is the graph embedding ability. Recently, many interconnection networks and their properties have been studied in the literature [2, 3, 6, 11, 17, 19, 25]. Graph embedding is a technique in interconnection networks that maps a guest graph into a host graph (usually an interconnection network). Many applications, such as architecture simulations and processor allocations, can be modeled as graph embeddings [1, 2, 4, 5, 13, 14, 17, 19, 28]. The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the *dilation*. The dilation of an embedding is defined as the maximum distance between a pair of vertices of H that are images of adjacent vertices of G. It is a measure for the communication time needed when simulation one network on another [15].

Another important cost criteria is the *wirelength*. The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [16, 18]. Grid embedding plays an important role in computer architecture. VLSI layout problem [21], crossing number problem [22], edge embedding problem [23], are all a part of grid embedding. Embedding problems have been considered for binary trees into paths [16], complete binary trees into hypercubes [24], tori and grids into twisted cubes [27], meshes into locally twisted cubes [29], paths into twisted cubes [30], cycles into twisted cubes [31], meshes into faulty crossed cubes [32], star graph into path [33], snarks into torus [34], generalized ladders into hypercubes [35], grids into grids [36], binary trees into grids [37], hypercubes into cycles [38], generalized wheels into arbitrary trees [26], and hypercubes into grids [39]. Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [38, 40]. The embedding discussed in this paper produce exact wirelengths.

In this paper, we produce the exact wirelength of circulant networks into cycle-of-ladders. The rest of the paper is organized as follows. Section 2 gives definitions and other preliminaries. Section 3 gives basic results of circulant networks. Section 4 establishes the main results. Finally, Section 5 concludes the paper.

2 Preliminaries

Definition 2.1. (See [46].) *A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.*

In this paper we consider the network embedding and defined as follows. Let G and H be finite graphs with n vertices. An *embedding* f of G into H is defined [40] as follows:

1. f is a bijective map from $V(G) \to V(H)$
2. f is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.

The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G, H(e))$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f(G, H(e)) = |\{(u,v) \in E(G) : e \in P_f(u,v)\}|$$

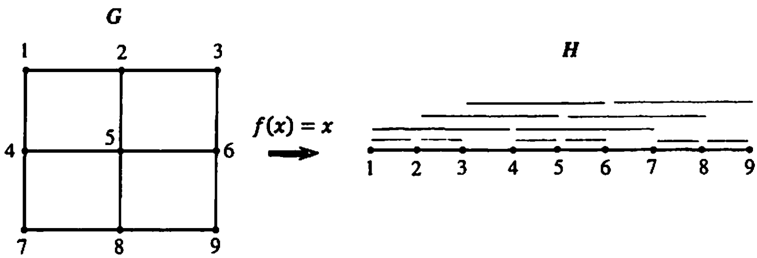

Figure 1: Wiring diagram of a grid G into a path H with $WL_f(G, H) = 24$.

where $P_f(u, v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f. If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G, H)$ is the minimum, over all embeddings $f : V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [42].

The Wirelength Problem The wirelength of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} d_H(f(u), f(v)) = \sum_{e\in E(H)} EC_f(G, H(e))$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u, v)$ in H. See Figure 1. Then, the *wirelength* of G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. The wirelength problem [26, 37, 38, 39, 40, 43], of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$.

The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature [43], and are NP-complete [23].

Problem 1 (Minimum Cut Problem): Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V,\ |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u, v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $\theta_G(m) = |\theta_G(A)|$.

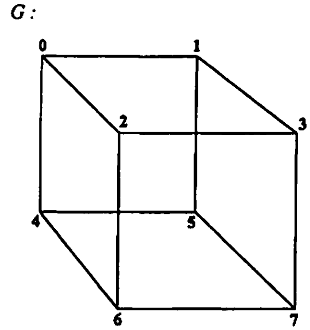

Figure 2: A={0,1,2,3} is a maximum subgraph of G on 4 vertices whereas B={ 0,1,6,7} is not a maximum subgraph of G.

Problem 2 (Maximum Subgraph Problem): Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A \subseteq V,\ |A|=m} |I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$. See Figure 2.

For a given m, where $m = 1, 2, \ldots, n$, we consider the problem of finding a subset A of vertices of G such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal. We say that optimal subsets are nested if there exists a total order $\mathcal{O}$ on the set V such that for any $m = 1, 2, \ldots, n$, the first m vertices in this order is an optimal subset. In this case we call the order $\mathcal{O}$ an optimal order [41, 43]. This implies that $WL(G, P_n) = \sum_{m=0}^{n} \theta_G(m)$.

Further, if a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. But, it is not true for Problem 2 in general. However for regular graphs a subset of vertices S is optimal with respect to Problem 1 if and only if S is optimal for Problem 2

Notation: For convenience we write $EC_f(e)$ instead of $EC_f(G, H(e)$ in the sequel.
For any set S of edges of H, $EC_f(S) = \sum_{e \in S} EC_f(e)$.

Lemma 2.2. (Congestion Lemma) (See [39].) *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) *For every edge$(a, b) \in G_i$, $i = 1, 2$, $P_f(a, b)$ has no edges in S.*

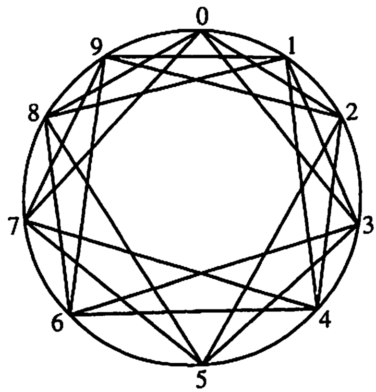

Figure 3: Circulant graph $G(10;\pm\{1,2,3\})$.

(ii) *For every edge (a,b) in G with $a \in G_1$ and $b \in G_2$, $P_f(a,b)$ has exactly one edge in S.*

(iii) G_1 is a maximum subgraph on k vertices where $k = |V(G_1)|$.

Then $EC_f(S)$ is minimum,that is, $EC_f(S) \leq EC_g(S)$ for any other embedding g of G into H, and $EC_f(S) = rk - 2|E(G_1)| = \theta_G(k)$ [43].

Lemma 2.3. (Partition Lemma) (See [39].) *Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$

3 Circulant networks

The circulant is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [7]. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [8]. It is also used in VLSI design and distributed computation [9, 10, 20]. The term circulant comes from the nature of its adjacency matrix. A matrix is circulant if all its rows are periodic rotations of the first one. Circulant matrices have been employed for designing binary codes[12]. Theoretical properties of circulant graphs have been studied extensively and surveyed in [9]. Every circulant graph is a vertex transitive graph and a Cayley graph [18]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [8, 9].

Definition 3.1. (See [18].) *A circulant undirected graph* $G(n;\pm S)$, *where* $S \subseteq \{1,2,\ldots,\lfloor n/2\rfloor\}$, $n \geq 3$ *is defined as a graph consisting of the vertex set* $V = \{0,1,\ldots,n-1\}$ *and the edge set* $E = \{(i,j) : |j-i| \equiv s(mod\ n), s \in S\}$.

The circulant graph shown in Figure 3 is $G(10;\pm\{1,2,3\})$. It is clear that $G(n;\pm 1)$ is the undirected cycle C_n and $G(n;\pm\{1,2,\ldots,\lfloor n/2\rfloor\})$ is the complete graph K_n. Further $G(n;\pm\{1,2,\ldots,\lfloor j\rfloor\})$, $1 \leq j < \lfloor n/2\rfloor$, $n \geq 3$ is a $2j$-regular graph.

Theorem 3.2. (See [45].) *A set of* k *consecutive vertices of* $G(n;\pm 1)$, $1 \leq k \leq n$ *induces a maximum subgraph of* $G(n;\pm S)$, *where* $S = \{1,2,\ldots,j\}$, $1 \leq j < \lfloor n/2\rfloor$, $n \geq 3$.

Theorem 3.3. (See [45].) *The number of edges in a maximum subgraph on* k *vertices of* $G(n;\pm S)$, $S = \{1,2,\ldots,j\}$, $1 \leq j < \lfloor n/2\rfloor$, $1 \leq k \leq n$, $n \geq 3$ *is given by*

$$\xi = \begin{cases} k(k-1)/2 & ; \quad k \leq j+1 \\ kj - j(j+1)/2 & ; \quad j+1 < k \leq n-j \\ \frac{1}{2}\{(n-k)^2 + (4j+1)k - (2j+1)n\} & ; \quad n-j < k \leq n. \end{cases}$$

4 Wirelength of circulant networks into cycle-of-ladders

In this section, we compute the exact wirelength of circulant networks into cycle-of-ladders.

Fang [44], propounds a kind of a new cycle-embedding aspect called the bipancycle-connectivity and a new graph called the cycle-of-ladders. By presenting algorithms to embed the cycle-of-ladders graphs into the hypercube and by generating the bipanconnected cycles in the cycle-of-ladders graphs, the hypercube has proved to be a bipancycle connected graph.

The work will help engineers to develop corresponding applications on the multiprocessor systems that employ the hypercube as the interconnection network. It will also help further investigations on the hypercube, for example, to find a fault tolerant algorithm to generate the bi-pan connected cycles on the hypercube appears interesting.

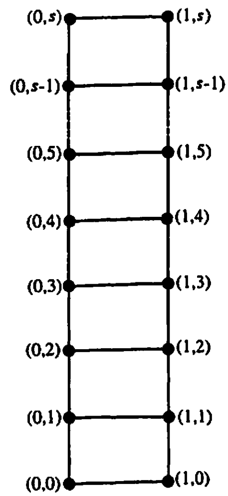

Figure 4: The structure of a ladder $L(s)$.

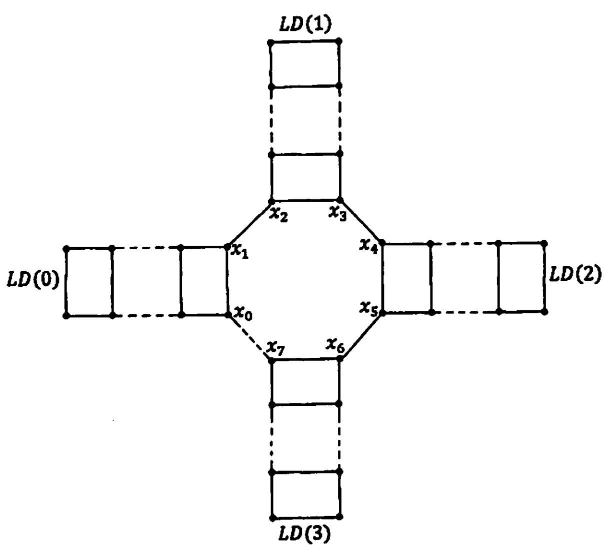

Figure 5: Cycle-of-ladders.

A path of length s is denoted by $P(s)$ and a cycle of length s is denoted by $C(s)$. A ladder of length s, denoted by an $L(s)$, is a $P(s) \times K_2$. Each vertex of an $L(s)$ is labelled by (b_0, b_1), where $b_0 = 0$ or $b_0 = 1$, and $0 \le b_1 \le s$. Each edge $((0, b_1), (1, b_1))$ is called a rung of the ladder $L(s)$, where $0 \le b_1 \le s$. Specifically, it is called the b_1^{th} rung. The 0^{th} rung is called the bottom rung of the ladder. The two paths $((0,0),(0,1),\ldots,(0,s))$ and $((1,0),(1,1),\ldots,(1,s))$ are called the bands of the $L(s)$. Specifically, the former is called the 0^{th} band and the latter is called the 1^{st} band. Clearly $L(s)$ contains $2(s+1)$ vertices and $3s+1$ edges. See Figure 4.

Definition 4.1. (See [44].) *A cycle-of-ladders is a graph unified by a bone cycle BC and k ladders $LD(0)$,*
$LD(1), \ldots, LD(k-1)$ with $BR(0), BR(1), \ldots, BR(k-1)$ as the bottom rungs, respectively, such that each $BR(i)$ is contained in the BC where $0 \le i \le k-1$.

The structure of a cycle-of-ladders graph is shown in Figure 5, where $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ is the bone cycle and $(x_0, x_1), (x_2, x_3), (x_4, x_5)$ and (x_6, x_7) are the BR(0), BR(1), BR(2) and BR(3), respectively.

In this paper we assume that each ladder $LD(i)$ $0 \le i \le k-1$, is of length s. Clearly this type of cycle-of-ladders contains $2k(s+1)$ vertices and it is denoted by $COL(k, s)$.

Embedding Algorithm

Input : A circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j < \lfloor n/2 \rfloor$ and a cycle-of-ladders $COL(k, s)$ where $n = 2k(s+1)$.

Algorithm : Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$, as $0, 1, \ldots, n-1$ in the clockwise sense. Label the vertices of $COL(k, s)$ as follows: Label the 0^{th} band vertices of $LD(0)$ from top to bottom as $0, 1, \ldots, s$. For $1 \le i \le k-1$, label the 1^{st} band vertices of $LD(i)$ from bottom to top as $(2i-1)(s+1), (2i-1)(s+1)+1, \ldots, 2i(s+1)-1$ and the 0^{th} band vertices from top to bottom as $2i(s+1), 2i(s+1)+1, \ldots, 2i(s+1)+s$. Label the 1^{st} band vertices of $LD(0)$ from bottom to top as $(2k-1)(s+1), (2k-1)(s+1)+1, \ldots, 2k(s+1)-1$.

Output : An embedding f of $G(n; \pm\{1, 2, \ldots, j\})$ into $COL(k, s)$ given by $f(x) = x$ with minimum wirelength.

Proof of correctness : We assume that the labels represent the vertices to which they are assigned.

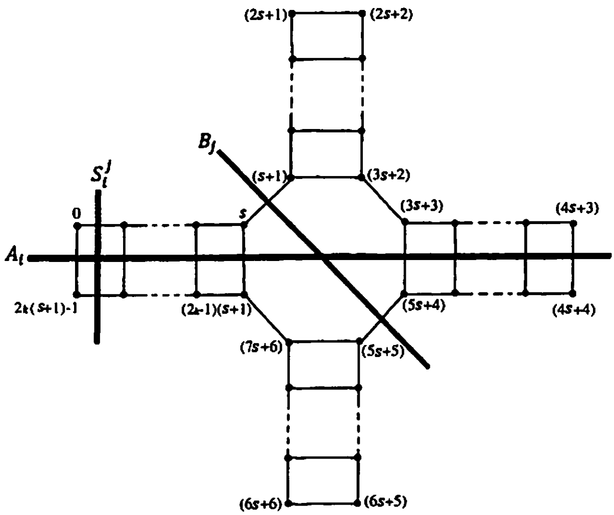

Figure 6: The edge cuts of $COL(4, s)$.

Case 1 (k even):
For $1 \leq i \leq \frac{k}{2}$, let A_i be the set of edges which contains all the rungs of $LD(i-1)$ and $LD(\frac{k}{2}+i-1)$. For $1 \leq j \leq \frac{k}{2}$, let B_j be the set of edges which contains the edge between $LD(i-1)$ and $LD(i)$ and the edge between $LD(\frac{k}{2}+i-1)$ and $LD(\frac{k}{2}+i)$. For $1 \leq i \leq k, 1 \leq j \leq s$, let S_i^j be the set of edges in $LD(i-1)$ which contains the edges between $(s-j+1)^{th}$ rung and $(s-j)^{th}$ rung. See Figure 6. Then $\{A_i : 1 \leq i \leq \frac{k}{2}\} \cup \{B_j : 1 \leq j \leq \frac{k}{2}\} \cup \{S_i^j : 1 \leq i \leq k, 1 \leq j \leq s\}$ is a partition of $E(COL(k, s))$.

For each i, $1 \leq i \leq \frac{k}{2}$, $E(COL(k, s)) \setminus A_i$ has two components H_{i1} and H_{i2} where

$$V(H_{i1}) = \{2(i-1)(s+1), 2(i-1)(s+1)+1, \ldots, 2(i-1)(s+1) + k(s+1)-1\}.$$

Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.3, G_{i1}, is an optimal set, and each A_i satisfies conditions (i), (ii) and (iii) of the congestion lemma. Therefore $EC_f(A_i)$ is minimum.

For each j, $1 \leq j \leq \frac{k}{2}$, $E(COL(k, s)) \setminus B_j$ has two components H_{j1} and H_{j2} where

$$V(H_{j1}) = \{(2j-1)(s+1), (2j-1)(s+1)+1, \ldots, (2j-1)(s+1) + k(s+1)-1\}.$$

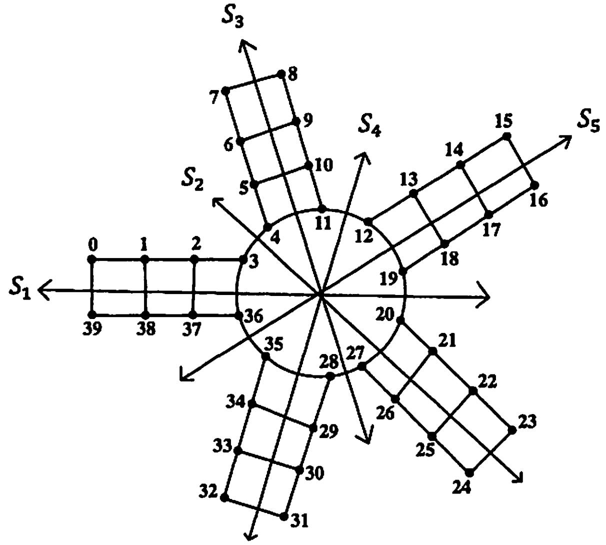

Figure 7: The edge cuts of $COL(5,3)$.

Let $G_{j1} = f^{-1}(H_{j1})$ and $G_{j2} = f^{-1}(H_{j2})$. By Theorem 3.3, G_{j1}, is an optimal set, and each B_j satisfies conditions (i), (ii) and (iii) of the congestion lemma. Therefore $EC_f(B_j)$ is minimum.

For each i, j, $1 \leq i \leq k$, $1 \leq j \leq s$, $E(COL(k,s))\backslash S_i^j$ has two components H_{i1}^j and H_{i2}^j where

$$V(H_{i1}^j) = \begin{cases} \{0,1,\ldots,j-1\} \cup \{2k(s+1)-1, \\ 2k(s+1)-2,\ldots,2k(s+1)-j\} \quad ; \quad if \quad i=0 \\ \{2i(s+1)-1, 2i(s+1)-2,\ldots,2i(s+1)-j\} \cup \{2i(s+1), \\ 2i(s+1)+1,\ldots,2i(s+1)+j-1\} \quad ; \quad if \quad i \neq 0 \end{cases}$$

Let $G_{i1}^j = f^{-1}(H_{i1}^j)$ and $G_{i2}^j = f^{-1}(H_{i2}^j)$. Since G_{i1}^j, is an optimal set, each S_i^j satisfies conditions (i), (ii) and (iii) of the congestion lemma. Therefore $EC_f(S_i^j)$ is minimum. The Partition Lemma implies that the wirelength is minimum.

Case 2 (k Odd):
For $1 \leq i \leq k$, when i is odd, let S_i be the set of edges which contains all the rungs of $LD(i-1)$ and the edge between $LD(\frac{k}{2}+i-1)$ and $LD(\frac{k}{2}+i)$ a when i is even, let S_i be the set of edges which contains the edge between $LD(i-1)$ and $LD(i)$ and all the rung of $LD(\frac{k}{2}+i)$. For $1 \leq i \leq k, 1 \leq j \leq s$, let S_i^j be the set of edges in $LD(i-1)$ which contains the edges between

$(s-j+1)^{th}$ rung and $(s-j)^{th}$ rung. See Figure 7. Then $\{S_i : 1 \leq i \leq k\} \cup \{S_i^j : 1 \leq i \leq k, 1 \leq j \leq s\}$ is a partition of $E(COL(k,s))$.

As in Case 1, it is easy to prove that the wirelength is minimum.

Theorem 4.2. *The exact wirelength of circulant graph $G(n; \pm\{1,2,\ldots,j\})$, $1 \leq j < \lfloor n/2 \rfloor$ into $COL(k,s)$ is given by*

$$WL(G, COL(k,s)) = k\{\theta_G(m(s+1)) + \sum_{j=1}^{s} \theta_G(2j)\}.$$

Proof: Following the notations of the Embedding Algorithm, we divide the proof into two cases.

Case 1 (k even): *By congestion lemma,*

i) $EC_f(A_i) = \theta_G(m(s+1))$, $1 \leq i \leq \frac{k}{2}$

ii) $EC_f(B_j) = \theta_G(m(s+1))$, $1 \leq j \leq \frac{k}{2}$

iii) $EC_f(S_i^j) = \theta_G(2j)$, $1 \leq i \leq k$ *and* $1 \leq j \leq s$.

Then by partion lemma,

$$WL(G, COL(k,s)) = \sum_{i=1}^{\frac{k}{2}} EC_f(A_i) + \sum_{j=1}^{\frac{k}{2}} EC_f(B_j) + \sum_{i=1}^{k}\sum_{j=1}^{s} EC_f(S_i^j)$$

$$= \sum_{i=1}^{\frac{k}{2}} \theta_G(m(s+1)) + \sum_{j=1}^{\frac{k}{2}} \theta_G(m(s+1)) + \sum_{i=1}^{k}\sum_{j=1}^{s} \theta_G(2j)$$

$$= \frac{k}{2}\theta_G(m(s+1)) + \frac{k}{2}\theta_G(m(s+1)) + k\sum_{j=1}^{s} \theta_G(2j)$$

$$= k\{\theta_G(m(s+1)) + \sum_{j=1}^{s} \theta_G(2j)\}.$$

Case 2 (k Odd):

By congestion lemma,

i) $EC_f(S_i) = \theta_G(m(s+1))$, $1 \leq i \leq k$

ii) $EC_f(S_i^j) = \theta_G(2j)$, $1 \leq i \leq k$ *and* $1 \leq j \leq s$.

Then by partion lemma,

$$WL(G, COL(k,s)) = \sum_{i=1}^{k} EC_f(S_i) + \sum_{i=1}^{k}\sum_{j=1}^{s} EC_f(S_i^j)$$

$$= \sum_{i=1}^{k} \{\theta_G(m(s+1))\} + \sum_{i=1}^{k}\sum_{j=1}^{s} \theta_G(2j)$$

$$= k\{\theta_G(m(s+1)) + \sum_{j=1}^{s} \theta_G(2j)\}.$$

Hence the proof.

5 Conclusion

In this paper, we embed circulant networks into cycle-of-ladders to yield the minimum wirelength. In our opinion, the reduction technique developed in this paper is very powerful and may be applied to the fault-tolerant embeddings of cycle-of-ladders in other kinds of interconnection networks.

Acknowledgment

The authors would like to thank the anonymous referees for their comments and suggestions. These comments and suggestions were very helpful for improving the quality of this paper.

References

[1] L. Auletta, A.A. Rescigno, V. Scarano, *Embedding graphs onto the supercube*, IEEE Trans. Comput, Vol. 44, no. 4, 593 - 597, 1995.

[2] M.M. Bae, B. Bose, *Edge disjoint Hamiltonian cycles in k-ary n-cubes and hypercubes*, IEEE Trans. Comput, Vol. 52, no. 10, 1271 - 1284, 2003.

[3] A. Khonsari, M. Ould-Khaoua, *A performance model of compressionless routing in k-ary n-cube networks*, Perform. Evaluat, Vol. 63, no. 4 - 5, 423 - 440, 2006.

[4] J. Fan, X. Lin, X. Jia, *Optimal path embedding in crossed cubes*, IEEE Trans. Parallel Distrib. Syst, Vol. 16, no. 12, 1190 - 1200, 2005.

[5] W.-C. Fang, C.-C. Hsu, C.-M. Wang, *On the fault-tolerant embeddings of complete binary trees in the mesh interconnection networks*, Inform. Sci, Vol. 151, 51 - 70, 2003.

[6] S.M. Zhou, N. Du, B.X. Chen, *A new family of interconnection networks of odd fixed degrees*, Parallel Distrib. Comput, Vol. 66, no. 5, 698 - 704, 2006.

[7] G.K. Wong, D.A. Coppersmith, *A combinatorial problem related to multimodule memory organization*, J. Assoc. Comput. Machin, Vol. 21, no. 3, 392 - 401, 1974.

[8] F.T. Boesch, J. Wang, *Reliable circulant networks with minimum transmission delay*, IEEE Trans. Parallel Distrib. Syst, Vol. 32, no. 12, 1286 - 1291, 1985.

[9] J.C. Bermond, F. Comellas, D.F. Hsu, *Distributed loop computer networks*, A survey Journal of Parallel and Distributed Computing, Vol. 24, no. 1, 2 - 10, 1995.

[10] R. Beivide, E. Herrada, J.L. Balc azar, A. Arruabarrena, *Optimal Distance Networks of Low Degree for Parallel Computers*, IEEE Transactions on Computers, Vol. 40, no. 10, 1109 - 1124, 1991.

[11] P.-L. Lai, J.J.M. Tan, C.-P. Chang, L.-H. Hsu, *Conditional diagnosability measures for large multiprocessor systems*, IEEE Trans.Comput, Vol. 54, no. 2, 165 - 175, 2005.

[12] M. Karlin, *New binary coding results by circulants*, IEEE Transactions on Information Theory, Vol. 15, no. 1, 81 - 92, 1969.

[13] J.-S. Fu, *Hamiltonicity of the WK-recursive network with and without faulty nodes*, IEEE Trans. Parallel Distrib. Syst, Vol. 16, no. 9, 853 - 865, 2005.

[14] J. Park, H. Kim, H. Lim, *Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements*, IEEE Trans. Parallel Distrib. Syst, Vol. 17, no. 3, 227 - 240, 2006.

[15] T. Dvořák, *Dense sets and embedding binary trees into hypercubes*, Discrete Applied Mathematics, Vol. 155, no. 4, 506 - 514, 2007.

[16] Y.L. Lai, K. Williams, *A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs*, J. Graph Theory, Vol. 31, 75 - 94, 1999.

[17] M.-C. Yang, T.-K. Li, J.J.M. Tan, L.-H. Hsu, *On embedding cycles into faulty twisted cubes*, Inform. Sci, Vol. 176, 676 - 690, 2006.

[18] J-M Xu, *Topological structure and analysis of interconnection networks*, Kluwer Academic, Amsterdam, 2001.

[19] J. Fan, X. Jia, X. Lin, *Complete path embeddings in crossed cubes*, Inform. Sci, Vol. 176, 3332 - 3346, 2006.

[20] R.S. Wilkov, *Analysis and Design of Reliable Computer Networks*, IEEE Trans. Communications, Vol. 20, no. 3, 660 - 678, 1972.

[21] S.N Bhatt, F.T Leighton, *A framework for solving VLSI graph layout problems*, J Comput Syst Sci, Vol. 28, 300 - 343, 1984.

[22] H.N Djidjev, I. Vrto, *Crossing numbers and cutwidths*, J Graph Algorithms Appl, Vol. 7, 245 - 251, 2003.

[23] M.R Garey, D.S Johnson DS, *Computers and intractability, a guide to the theory of NP-completeness*, Freeman, San Francisco, 1979.

[24] S.L Bezrukov, *Embedding complete trees into the hypercube*, Discrete Appl Math, Vol. 110, 101 - 119, 2001.

[25] G.-Y. Chang, G.-H. Chen, G.-J. Chang, *(t, k)-Diagnosis for matching composition networks*, IEEE Trans. Comput, Vol. 55, no. 1, 88 - 92, 2006.

[26] I.Rajasingh, J.Quadras, P.Manuel, A.William, *Embedding of cycles and wheels into arbitrary trees*, Networks, Vol. 44, 173 - 178, 2004.

[27] P-L Lai, C-H Tsai, *Embedding of tori and grids into twisted cubes*, Theor Comput Sci, Vol. 411, no. 40 - 42, 3763 - 3773, 2010.

[28] J. Park, H. Kim, H. Lim, *Embedding of rings and meshes onto faulty hypercubes using free dimensions*, IEEE Trans.Comput, Vol. 43, no. 5, 608 - 613, 1994.

[29] Y.Han J.Fan, S.Zhang, J.Yang, P.Qian, *Embedding meshes into locally twisted cubes*, Inf Sci, Vol. 180, no. 19, 3794 - 3805, 2010.

[30] J. Fan, X. Jia, X .Lin, *Optimal embeddings of paths with various lengths in twisted cubes*, IEEE Trans Parallel Distrib Syst, Vol. 18, no. 4, 511 - 521, 2007.

[31] J. Fan, X. Jia, X .Lin, *Embedding of cycles in twisted cubes with edge-pancyclic*, Algorithmica, Vol. 51, no. 3, 264 - 282, 2008.

[32] X.Yang, Q.Dong, Y.Y.Tang, *Embedding meshes/tori in faulty crossed cubes*, Inf Process Lett, Vol. 110, no. 14-15, 559 - 564, 2010.

[33] M-C.Yang, *Path embedding in star graphs*, Appl Math Comput, Vol. 207, no. 2, 283 - 291, 2009.

[34] A.Vodopivec, *On embeddings of snarks in the torus*, Discrete Math, Vol. 308, no. 10, 1847 - 1849, 2008.

[35] R.Caha, V. Koubek, *Optimal embeddings of generalized ladders into hypercubes*, Discrete Math, Vol. 233, 65 - 83, 2001.

[36] M.Rottger, U.P.Schroeder, *Efficient embeddings of grids into grids*, Discrete Appl Math, Vol. 108, no. 12, 143 - 173, 2001.

[37] J.Opatrny , D.Sotteau, *Embeddings of complete binary trees into grids and extended grids with total vertex-congestion 1*, Discrete Appl Math, Vol. 98, 237 - 254, 2000.

[38] J.D.Chavez, R.Trapp, *The cyclic cutwidth of trees*, Discrete Appl Math, Vol. 87, 25 - 32, 1998.

[39] P.Manuel, I.Rajasingh, B.Rajan, H. Mercy, *Exact wirelength of hypercube on a grid*, Discrete Appl Math, Vol. 157, no. 7, 1486 - 1495, 2009.

[40] S.L. Bezrukov, J.D. Chavez, L.H. Harper, M. Röttger and U.P. Schroeder, *Embedding of hypercubes into grids*, Mortar Fine Control System, 693 - 701, 1998.

[41] L.H. Harper, *Global methods for combinatorial isoperimetric problems*, Cambridge University Press,Cambridge 2004.

[42] S.L. Bezrukov, J.D. Chavez, L.H. Harper, M. Röttger, U.P. Schroeder, *The congestion of n-cube layout on a rectangular grid*, Discrete Mathematics,Vol. 213, 13 - 19, 2000.

[43] S.L. Bezrukov, S.K. Das, R. Elsässer, *An edge-isoperimetric problem for powers of the Petersen graph*, Annals of Combinatorics, Vol. 4, 153 - 169, 2000.

[44] Jywe-Fei Fang, *The bipancycle-connectivity of the hypercube*, Information Sciences, Vol. 178, 4679 - 4687, 2008.

[45] I. Rajasingh, P. Manuel, M. Arockiaraj B. Rajan, *Embeddings of circulant networks*, Journal of Combinatorial Optimization, DOI:10.1007/s10878-011-9443-x.

[46] Narsingh Deo, *Graph Theory with applications to engineering and computer science*, Prentice-Hall of India Private Limited New Delhi,90 - 91, 2004.